Solutions for Problem Sheet 6

Problem 1. Exactness all follow from continuity and local Lipschitz property for the coefficients.

(1) We have

$$A(t) = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix},\ a(t) = 0,\ \sigma(t) = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \mathrm{e}^{tA} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dB_s, \quad t \geqslant 0.$$

(2) We have

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix},\ a(t) = \begin{pmatrix} 0 \\ \frac{G(t)}{L} \end{pmatrix},\ \sigma(t) = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \mathrm{e}^{tA} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \frac{G(s)}{L} \end{pmatrix} ds + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix} dB_s.$$

(3) Let $\tau_0 \triangleq \inf\{t \geqslant 0 : \ X_t = 0\}$ and let e be the explosion time. Define $Y_t \triangleq \ln X_t$. According to Itô's formula, we have

$$dY_t = r(K - X_t)dt + \beta dB_t - \frac{\beta^2}{2}dt, \quad t < \tau_0 \wedge e.$$

Define $Z_t \triangleq \ln X_t + r\int_0^t X_s ds$. It follows that

$$dZ_t = \left(rK - \frac{\beta^2}{2}\right) dt + \beta dB_t.$$

Therefore,

$$\ln\left(\frac{X_t}{x}\right) + r\int_0^t X_s ds = \beta B_t + \left(rK - \frac{\beta^2}{2}\right)t,$$

and hence

$$X_t \cdot \exp\left(r\int_0^t X_s ds\right) = x\exp\left(\beta B_t + \left(rK - \frac{\beta^2}{2}\right)t\right).$$

Integrating with respect to dt, we have

$$\exp\left(r\int_0^t X_s ds\right) - 1 = rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right)s\right)ds.$$

Therefore,

$$\int_0^t X_s ds = \frac{1}{r}\cdot\ln\left(1 + rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right)s\right)ds\right).$$

Differentiating with respect to t, we arrive at

$$X_t = \frac{x\exp\left(\beta B_t + \left(rK - \frac{\beta^2}{2}\right)t\right)}{1 + rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right)s\right)ds}, \quad t < \tau_0 \wedge e. \tag{1}$$

This in particular implies that $\tau_0 = e = \infty$ almost surely, and (1) defines the global solution to the SDE.

Problem 2. (1) Since B_t is a Gaussian process, we know that X_t is also a Gaussian process. The mean function is $m(t) \triangleq \mathbb{E}[X_t] = 0$, and the covariance function is

$$\begin{aligned}
\rho(s,t) &\triangleq \mathbb{E}[X_s X_t] \\
&= \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] \\
&= s \wedge t - st - st + st \\
&= s \wedge t - st \\
&= \begin{cases} s(1-t), & s \leqslant t; \\ t(1-s), & s > t. \end{cases}
\end{aligned}$$

(2) The SDE is a linear SDE with $A(t) = -(1-t)^{-1}$, $a(t) = 0$ and $\sigma(t) = 1$. By the general formula for the solution, we have $\Phi(t) = 1 - t$, and

$$Y_t = (1-t)\int_0^t \frac{dB_s}{1-s}, \quad 0 \leqslant t < 1.$$

Since the integrand is deterministic, it is immediate that Y_t is a Gaussian process. The mean function is zero, and for $s < t < 1$, we have

$$\mathbb{E}[Y_s Y_t] = (1-s)(1-t)\mathbb{E}\left[\int_0^s \frac{du}{(1-u)^2}\right] = s(1-t).$$

In particular, Y_t has the same mean and covariance functions as X_t. Since they are both Gaussian processes, we conclude that

$$(X_t)_{0\leqslant t<1} \stackrel{\text{law}}{=} (Y_t)_{0\leqslant t<1}.$$

Moreover, since $X_1 = 0$, and by continuity, the probability $\mathbb{P}(\lim_{t\uparrow 1} X_t = 0)$ is determined by the distribution of $(X_t)_{0\leqslant t<1}$, therefore we conclude that

$$\mathbb{P}\left(\lim_{t\uparrow 1} Y_t = 0\right) = 1.$$

(3) Let B_t be a one dimensional Brownian motion, and let $S_t \triangleq \sup_{0\leqslant s\leqslant 1} B_s$. The joint density $f_{(S_1,B_1)}(x,y)$ of (S_1, B_1) is given by

$$\mathbb{P}(S_1 \in dx,\ B_1 \in dy) = \frac{2(2x-y)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(2x-y)^2}{2}} dxdy, \quad x \geqslant 0, x \geqslant y.$$

Now a crucial observation is, the process X_t has the same distribution as the Brownian motion B_t conditioned on $B_1 = 0$. More precisely, for $0 < t_1 < \cdots < t_n < 1$, the joint density of $(X_{t_1}, \cdots, X_{t_n})$ is the same as the conditional density of $(B_{t_1}, \cdots, B_{t_n})$ conditioned on $B_1 = 0$. Therefore, the desired probability is

$$\mathbb{P}\left(S_1 \geqslant x | B_1 = 0\right) = \int_x^\infty \frac{f_{(S_1,B_1)}(z,0)dz}{f_{B_1}(0)} = \int_x^\infty 4z \cdot \mathrm{e}^{-2z^2} dz = \mathrm{e}^{-2x^2}.$$

Problem 3. (1) The coefficients are given by $\sigma(x) = -2|x|^{3/2}$ and $b(x) = 3x^2$. They are continuous and locally Lipschitz. Therefore, the SDE is exact.

(2) The result follows from the comparison theorem, since the unique solution to the SDE

$$\begin{cases} dY_t = -2|Y_t|^{\frac{3}{2}} dB_t, & t \geqslant 0, \\ Y_0 = 0, \end{cases}$$

is the zero solution.

(3) Let $\tau_0 \triangleq \inf\{t \geqslant 0 : \ Y_t = 0\}$, and define $Z_t \triangleq Y_t^{-1/2}$. According to Itô's formula, we conclude that

$$dZ_t = dB_t, \quad t < \tau_0 \wedge e.$$

Therefore,

$$Y_t = \frac{1}{(1+B_t)^2}, \quad t < \tau_0 \wedge e.$$

This in particular implies that $\tau_0 = \infty$ almost surely (otherwise we have $1/(1+B_{\tau_0})^2 = 0$ which is absurd). In other words, Y_t never reaches zero and we have

$$Y_t = \frac{1}{(1+B_t)^2}, \quad t < e.$$

It follows that $e = \inf\{t \geqslant 0 : \ B_t = -1\}$. Therefore,

$$\mathbb{P}(e > t) = 1 - 2\mathbb{P}(B_t \geqslant 1) = 1 - 2\int_1^\infty \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{u^2}{2t}} du.$$

From the density formula for e, it is easy to see that

$$\mathbb{P}(e < \infty) = 1, \quad \mathbb{E}[e] = \infty.$$

Problem 4. (1) According to Tanaka's formula,

$$\begin{aligned}
|X_t - Y_t| &= \int_0^t \mathrm{sgn}(X_s - Y_s) d(X_s - Y_s) + L_t^0(X - Y) \\
&= \int_0^t \mathrm{sgn}(X_s - Y_s)\,(\sigma(s, X_s) - \sigma(s, Y_s))\, dB_s \\
&\quad + \int_0^t \mathrm{sgn}(X_s - Y_s)\,(b(s, X_s) - b(s, Y_s))\, ds + L_t^0(X - Y).
\end{aligned}$$

By using

$$X_t \vee Y_t = \frac{X_t + Y_t + |X_t - Y_t|}{2},$$

one easily see that

$$\begin{aligned}
&X_t \vee Y_t \\
&= X_0 + \frac{1}{2}\int_0^t (\sigma(s, X_s) + \sigma(s, Y_s) + \mathrm{sgn}(X_s - Y_s)\cdot(\sigma(s, X_s) - \sigma(s, Y_s)))\, dB_s \\
&\quad + \frac{1}{2}\int_0^t (b(s, X_s) + b(s, Y_s) + \mathrm{sgn}(X_s - Y_s)\cdot(b(s, X_s) - b(s, Y_s)))\, ds \\
&\quad + L_t^0(X - Y) \\
&= X_0 + \int_0^t \sigma(s, X_s \vee Y_s) dB_s + \int_0^t b(s, X_s \vee Y_s) ds + L_t^0(X - Y).
\end{aligned}$$

Therefore, $X_t \vee Y_t$ is a solution if and only if $L^0(X-Y) \equiv 0$.

(2) Let X_t and Y_t be two solutions defined on the same set-up with $X_0 = Y_0$ almost surely. According to Part (1), we know that $Z_t \triangleq X_t \vee Y_t$ is also a solution. By uniqueness in law, we conclude that $X \overset{\text{law}}{=} Z$. In particular, for each $t \geqslant 0$, $X_t \overset{\text{law}}{=} Z_t$. But we also know that $X_t \leqslant Z_t$ almost surely. This implies that $X_t = Z_t$ almost surely. Indeed, for each $n \geqslant 1$, define

$$X_t^n \triangleq \begin{cases} -n, & X_t < -n; \\ X_t, & -n \leqslant X_t \leqslant n; \\ n, & X_t > n, \end{cases}$$

and similarly for Z_t^n. Then $X_t^n \overset{\text{law}}{=} Z_t^n$ and also $X_t^n \leqslant Z_t^n$ almost surely. It follows that

$$\mathbb{E}[Z_t^n - X_t^n] = 0,$$

which implies that $X_t^n = Z_t^n$ almost surely. Since n is arbitrary, we conclude that $X_t = Z_t$.

Problem 5. (1) According to Itô's formula, we have

$$d\mathcal{E}_t^G = \mathcal{E}_t^G dG_t.$$

Therefore,

$$\begin{aligned} dZ_t &= \left(\int_0^t (\mathcal{E}_s^G)^{-1} dH_s\right) d\mathcal{E}_t^G + dH_t + d\mathcal{E}_t^G \cdot dH_t \\ &= Z_t dG_t + dH_t + \mathcal{E}_t^G d\langle G, H\rangle_t \\ &= Z_t dG_t + dH_t. \end{aligned}$$

(2)By the comparison theorem (c.f. Theorem 6.11 in the lecture notes), we may assume without loss of generality that $X_0^1 = X_0^2$. Now suppose that b^1 is Lipschitz continuous. Then

$$\begin{aligned} X_t^2 - X_t^1 &= \int_0^t \left(b^2(X_s^2) - b^1(X_s^1)\right) ds + \int_0^t \left(\sigma(s, X_s^2) - \sigma(s, X_s^1)\right) dB_s \\ &= \int_0^t \left(b^2(X_s^2) - b^1(X_s^2)\right) ds + \int_0^t \left(\sigma(s, X_s^2) - \sigma(s, X_s^1)\right) dB_s \\ &\quad + \int_0^t \left(b^1(X_s^2) - b^1(X_s^1)\right) ds \\ &= H_t + \int_0^t (X_s^2 - X_s^1) dG_s, \end{aligned}$$

where

$$
\begin{aligned}
H_t &\triangleq \int_0^t \left(b^2(X_s^2) - b^1(X_s^2)\right) ds, \\
G_t &\triangleq \int_0^t \mathbf{1}_{\{X_s^1 \neq X_s^2\}} (X_s^2 - X_s^1)^{-1} \left((\sigma(s, X_s^2) - \sigma(s, X_s^1))dB_s \right. \\
&\quad \left. + (b^1(X_s^2) - b^1(X_s^1))ds\right).
\end{aligned}
$$

Define $\mathcal{E}_t^G$ to be the stochastic exponential of G as before, and define

$$
Z_t \triangleq \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} dH_s.
$$

It follows from the first part that

$$
Z_t = H_t + \int_0^t Z_s dG_s.
$$

But we have already seen that $X_t^2 - X_t^1$ also satisfies the same (linear) equation. Apparently we have uniqueness in this context. Therefore,

$$
Z_t = X_t^2 - X_t^1 = \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} \left(b^2(X_s^2) - b^1(X_s^2)\right) ds.
$$

Since $\mathcal{E}_t^G > 0$ for all t, by the assumption $b^2 > b^1$, we conclude that with probability one, $X_t^2 > X_t^1$ for all $t > 0$.

The result does not hold if σ is not Lipschitz continuous. For example, consider the two SDEs

$$
dX_t^i = 2\sqrt{|X_t^i|} dB_t + \alpha^i dt,
$$

where $\alpha^1 = 0$ and $\alpha^2 = 1$. Suppose that $X_0^1 = X_0^2 = 0$, then $X_t^1 = 0$ and X_t^2 is the square of a one dimensional Brownian motion. Since the Brownian motion visits the origin infinitely often, we see that the result fails in this case.

Problem 6. (1) Existence. Let $\sigma(t)$ be the square root of $a(t)$. Define

$$
X_t = \begin{cases} x, & 0 \leqslant t \leqslant s; \\ x + \int_s^t \sigma(u) dB_u + \int_s^t b(u) du, & t \geqslant s. \end{cases}
$$

Then the law of X satisfies the desired properties.

Uniqueness. Let $\mathbb{P}^{(s,x)}$ be a probability measure on W^n which satisfies the given properties. The martingale characterization assures that under $\mathbb{P}^{(s,x)}$,

$$w_t = x + \int_s^t \sigma(u)dB_u + \int_s^t b(u)du$$

for some Brownian motion on some enlargement of path space. Therefore, the law of w, which is $\mathbb{P}^{(s,x)}$, must coincide with the one defined in the existence part.

For each cylinder set $\Lambda \triangleq \{w : (w_{u_1}, \cdots, w_{u_m}) \in \Gamma\}$, one can write down $\mathbb{P}^{(s,x)}(\Lambda)$ explicitly as it is essentially Gaussian. The measurability of $(s,x) \mapsto \mathbb{P}^{(s,x)}(\Lambda)$ then follows immediately. For general $\Lambda \in \mathcal{B}(W^n)$, we can simply use the monotone class theorem to obtain the desired measurability property.

(2) We may assume that

$$x_t = x + \int_s^t \sigma(u)dB_u + \int_s^t b(u)du, \quad t \geqslant s,$$

where B_u is a Brownian motion. Then

$$\begin{aligned}
&\mathbb{P}^{(s,x)}\left(x_{t_2} \in \Gamma | \mathcal{B}_{t_1}(W^n)\right) \\
&= \mathbb{P}^{(s,x)}\left(x_{t_1} + \int_{t_1}^{t_2} \sigma(u)dB_u + \int_{t_1}^{t_2} b(u)du \in \Gamma | \mathcal{B}_{t_1}(W^n)\right) \\
&= \mathbb{P}(t_1, x_{t_1}; t_2, \Gamma).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}(s, x; t_2, \Gamma) &= \mathbb{P}^{(s,x)}\left(x_{t_2} \in \Gamma\right) \\
&= \mathbb{E}^{(s,x)}\left[\mathbb{P}^{(s,x)}\left(x_{t_2} \in \Gamma | \mathcal{B}_{t_1}(W^n)\right)\right] \\
&= \mathbb{E}^{(s,x)}\left[\mathbb{P}(t_1, x_{t_1}; t_2, \Gamma)\right] \\
&= \int_{\mathbb{R}^n} \mathbb{P}(s, x; t_1, dy)\mathbb{P}(t_1, y; t_2, \Gamma).
\end{aligned}$$

(3) Let u be a solution to the Cauchy problem. For fixed $t > 0$ and $x \in \mathbb{R}^n$, let X_s^x be the solution to the SDE

$$\begin{cases} dX_s^x = \sigma(t-s)dB_s + b(t-s)ds, & 0 \leqslant s \leqslant t; \\ X_0^x = x. \end{cases}$$

By applying Itô's formula to the process $s \mapsto v(t-s, X_s^x)$, we get

$$\begin{aligned} f(X_t^x) - v(t,x) &= -\int_0^t \partial_t v(t-s, X_s^x)ds + \int_0^t \nabla^* v(t-s, X_s^x) \cdot \sigma(t-s) \cdot dB_s \\ &\quad + \int_0^t \mathcal{L}_{t-s} v(t-s, X_s^x)ds \\ &= \int_0^t \nabla^* v(t-s, X_s^x) \cdot \sigma(t-s) \cdot dB_s. \end{aligned}$$

Therefore,

$$v(t,x) = \mathbb{E}\left[f(X_t^x)\right].$$

This in particular shows uniqueness.

Problem 7. (1) Let B_t^x be the one dimensional Brownian motion starting at $x \in (a,b)$. Define τ^x to be the first time when B_t^x exits the interval (a,b). Applying Itô's formula for $u(t-s, B^x_{\tau^x \wedge s})$ $(0 \leqslant s \leqslant t)$, we have

$$\begin{aligned} u(t-s, B^x_{\tau^x\wedge s}) - u(t,x) &= -\int_0^s u_t(t-r, B^x_{\tau^x\wedge r})dr + \int_0^{\tau^x\wedge s} u_x(t-r, B^x_{\tau^x\wedge r})dB_r^x \\ &\quad + \frac{1}{2}\int_0^{\tau^x\wedge s} u_{xx}(t-r, B^x_{\tau^x\wedge r})dr \\ &= \int_0^{\tau^x\wedge s} K(B_r^x)dr + \int_0^{\tau^x\wedge s} u_x(t-r, B^x_{\tau^x\wedge r})dB_r^x, \end{aligned}$$

where we have used the PDE for u and the fact that

$$\int_{\tau^x\wedge s}^s u_t(t-r, B^x_{\tau^x\wedge r})dr = 0.$$

Therefore, by applying the integration by parts formula for the process

$$M_s \triangleq u(t-s, B^x_{\tau^x\wedge s}) \exp\left(-\int_0^{\tau^x\wedge s} K(B_r^x)dr\right), \quad 0 \leqslant s \leqslant t,$$

we obtain that

$$M_s = u(t,x) + \int_0^{\tau^x\wedge s} u_x(t-r, B^x_{\tau^x\wedge r}) \exp\left(-\int_0^{\tau^x\wedge r} K(B_v^x)dv\right) dB_r^x.$$

In particular, M_s is a continuous local martingale. But we also know that M_s is uniformly bounded. Therefore, it is a martingale, and hence

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[M_t\right] \\ &= \mathbb{E}\left[f(B^x_{\tau^x\wedge t})\exp\left(-\int_0^{\tau^x\wedge t}K(B^x_s)ds\right)\right] \\ &= \mathbb{E}\left[f(B^x_t)\exp\left(-\int_0^t K(B^x_s)ds\right);t<\tau^x\right]. \end{aligned}$$

(2) For any continuous f which is compactly supported on (a,b), the corresponding initial-boundary value problem can be solved by the method of separation of variables. The solution is given by

$$u(t,x)=\sum_{n=0}^{\infty}\frac{2}{b-a}\left(\int_a^b f(y)\sin\frac{n\pi(y-a)}{b-a}dy\right)\mathrm{e}^{-\frac{n^2\pi^2 t}{2(b-a)^2}}\sin\frac{n\pi(x-a)}{b-a}.$$

From Part (1), we also know that

$$u(t,x)=\mathbb{E}\left[f(B^x_t);t<\tau^x\right].$$

Since f is arbitrary, the result follows immediately.