Solutions for Problem Set Five

Problem 1. (1) Necessity. Suppose that $M \in H_0^2$. Then $M_t \to M_\infty$ in L^2. Therefore, we may take limit on the identity

$$\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]$$

to conclude that

$$\mathbb{E}[\langle M \rangle_\infty] = \mathbb{E}[M_\infty^2] < \infty.$$

Sufficiency. Suppose that $\mathbb{E}[\langle M \rangle_\infty] < \infty$. According to the BDG inequalities, we know that

$$\mathbb{E}\left[\sup_{t \geqslant 0} |M_t|^2\right] < \infty. \tag{1}$$

Let τ_n be a sequence of stopping times converging to infinity such that M^{τ_n} is a martingale for each n. Then for $s < t$ and $A \in \mathcal{F}_s$, we have $\mathbb{E}[M_{\tau_n \wedge t}\mathbf{1}_A] = \mathbb{E}[M_{\tau_n \wedge s}\mathbf{1}_A]$. Moreover, (1) implies that $\{M_{\tau_n \wedge t} : \ n \geqslant 1\}$ and $\{M_{\tau_n \wedge s} : \ n \geqslant 1\}$ are both bounded in L^2 and hence uniformly integrable. Therefore, we conclude that M_t is a martingale. The L^2-boundedness follows again from (1).

(2) Necessity. Suppose that $\langle M \rangle_t = f(t)$ for some deterministic continuous increasing function f vanishing at $t = 0$. According to (1) (more precisely, a local version of (1), that $\{M_t, \mathcal{F}_t : \ 0 \leqslant t \leqslant T\}$ is a square integrable martingale if and only of $\mathbb{E}[\langle M \rangle_T] < \infty$). Exactly the same argument as in the proof of Lévy's characterization theorem shows that

$$\mathbb{E}\left[\mathrm{e}^{i\theta(M_t - M_s)} | \mathcal{F}_s\right] = \mathrm{e}^{-\frac{1}{2}\theta^2(f(t) - f(s))}.$$

Therefore, M_t is a Gaussian martingale with independent increments (indeed $M_t - M_s$ and $\mathcal{F}_s$ are independent).

Sufficiency. Suppose that M_t is a Gaussian martingale. Let $\{\mathcal{F}_t^M\}$ be the augmented natural filtration of M_t. It follows that $\mathcal{F}_t^M \subseteq \mathcal{F}_t$ and M_t is an $\{\mathcal{F}_t^M\}$-martingale. Moreover, since

$$\mathbb{E}[M_s(M_t - M_s)] = \mathbb{E}\left[M_s \mathbb{E}[M_t - M_s | \mathcal{F}_s]\right] = 0,$$

we conclude that M_t has independent increments. Define $f(t) \triangleq \mathbb{E}[M_t^2]$. It is not hard to see that $f(t)$ is continuous, increasing and vanishes at $t = 0$. Moreover,

$$\begin{aligned} \mathbb{E}\left[M_t^2 - f(t)|\mathcal{F}_s^M\right] &= \mathbb{E}\left[(M_t - M_s + M_s)^2|\mathcal{F}_s^M\right] - f(t) \\ &= f(t) - f(s) + M_s^2 - f(t) \\ &= M_s^2 - f(s). \end{aligned}$$

Therefore, the quadratic variation process of M_t with respect to the filtration $\{\mathcal{F}_t^M\}$ is $f(t)$. According to Proposition 5.7 in the lecture notes, we conclude that

$$\lim_{\|\mathcal{P}_n\| \to 0} \sum_{t_i \in \mathcal{P}_n} (M_{t_i} - M_{t_{i-1}})^2 = f(t)$$

in probability. But since $M \in \mathcal{M}_0^{\mathrm{loc}}$ with respect to the filtration $\{\mathcal{F}_t\}$, the quadratic variation process of M_t with respect to $\{\mathcal{F}_t\}$ also satisfies Proposition 5.7. Therefore, $\langle M \rangle_t = f(t)$. As in the necessity part, we can also conclude that $M_t - M_s$ and $\mathcal{F}_s$ are independent.

(3) Given $n \geqslant 1$, let $\tau_n \triangleq \inf\{t \geqslant 0 : \ \langle M \rangle_t \geqslant n\}$. Then $\langle M^{\tau_n} \rangle_t = \langle M \rangle_t^{\tau_n} \leqslant n$ for all t, which implies from (1) that $M^{\tau_n} \in H_0^2$. In particular, $M_t^{\tau_n}$ converges almost surely to a finite random variable as $t \to \infty$. One could of course take a single null set outside which this statement is true for all $n \geqslant 1$. Since

$$\{\langle M \rangle_\infty < \infty\} \subseteq \bigcup_n^\infty \{\tau_n = \infty\}.$$

It follows that with probability one, for every $\omega \in \{\langle M \rangle_\infty < \infty\}$, $M_t(\omega) = M_t^{\tau_n(\omega)}(\omega)$ (take n to be such that $\tau_n(\omega) = \infty$) converges to a finite limit. Therefore, with probability one, we have

$$\{\langle M \rangle_\infty < \infty\} \subseteq \left\{\lim_{t \to \infty} M_t \text{ exists finitely}\right\}.$$

On the other hand, by the generalized Dambis-Dubins-Schwarz theorem (c.f. Theorem 5.9), we know that $M_t = B_{\langle M \rangle_t}$ for some Brownian motion possibly defined on an enlarged space. According to Proposition 4.2 in the lecture notes, we know that with probability one,

$$\limsup_{t \to \infty} B_t = \infty, \ \liminf_{t \to \infty} B_t = -\infty.$$

But if $\langle M \rangle_\infty = \infty$, we have $\lim_{t \to \infty} C_t = \infty$ where C_t is the time-change associated with $\langle M \rangle_t$. Therefore, with probability one,

$$\{\langle M \rangle_\infty = \infty\} \subseteq \left\{\limsup_{t \to \infty} M_t = \infty, \ \liminf_{t \to \infty} M_t = -\infty\right\}.$$

Problem 2. (1) Since $f(x) \triangleq |x|^{-1}$ is harmonic on $\mathbb{R}^3\backslash\{0\}$ and from Problem Sheet 4, Problem 7, (2), (iv) that B_t never hits $x = 0$ on $(0, \infty)$, we conclude from Itô's formula that $X_t = 1/|B_{1+t}|$ is a continuous $\{\mathcal{F}^B_{1+t}\}$-local martingale. Moreover, we have

$$\mathbb{E}[|X_t|^2] = \mathbb{E}[|B_{1+t}|^{-2}] = \frac{C_2}{1+t},$$

where $C_2 \triangleq \mathbb{E}[|Z|^{-2}]$ with $Z \sim \mathcal{N}(0, 1)$. Therefore, $\{X_t\}$ is uniformly bounded in L^2. However, it is not a martingale because

$$\mathbb{E}[X_t] = \mathbb{E}[|B_{1+t}|^{-1}] = \frac{C_1}{\sqrt{1+t}}$$

is not a constant in t, where $C_1 \triangleq \mathbb{E}[|Z|^{-1}]$ with $Z \sim \mathcal{N}(0, 1)$.

(2) Let Y_t be a uniformly integrable continuous submartingale with a Doob-Meyer decomposition $Y_t = M_t + A_t$. Since $Y_t \to Y_\infty$ in L^1, we see that $A_\infty \in L^1$ which shows that A_∞ is of class (D). Moreover, $\{M_t\}$ is easily seen to by uniformly integrable, which implies from the optional sampling theorem that

$$M_\tau = \mathbb{E}[M_\infty|\mathcal{F}_\tau], \quad \forall \tau \text{ finite stopping time},$$

where $M_\infty \triangleq \lim_{t\to\infty} M_t$. Therefore, Y_t is of class (D).

Now we show that X_t is not of class (D). Note that X_t is a non-negative supermartingale with a last element $X_\infty = 0$. Define $\tau_n \triangleq \inf\{t \geqslant 0 : \ |X_t| \geqslant n\}$. It follows that

$$X_{\tau_n} = \left(\frac{1}{|B_1|} \vee n\right) \mathbf{1}_{\{\tau_n<\infty\}}.$$

In general, τ_n is not finite almost surely. Indeed, from Problem Sheet 4, Problem 7, (2), (ii), we know that

$$\mathbb{P}(\tau_n < \infty|B_1) = \frac{1}{n|B_1|} \wedge 1,$$

and hence

$$\mathbb{P}(\tau_n < \infty) = \mathbb{E}\left[\mathbb{P}(\tau_n < \infty|B_1)\right] = \mathbb{E}\left[\frac{1}{n|B_1|} \wedge 1\right],$$

which is easily seen to be strictly less than 1 by direct computation. Therefore, we are going to show that the family $\{X_{\tau_n \wedge m} : \ n, m \geqslant 1\}$ is not uniformly integrable.

We first show that there exists $c > 0$, such that for every $\lambda > 0$, there exists some $n \geqslant 1$ with

$$\mathbb{E}[X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}}] \geqslant c. \tag{2}$$

Indeed, observe that

$$X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}} = \left(\frac{1}{|B_1|}\vee n\right)\mathbf{1}_{\left\{\frac{1}{|B_1|}\vee n>\lambda,\ \tau_n<\infty\right\}}.$$

If $n \geqslant \lambda$, then

$$X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}} \geqslant n\mathbf{1}_{\{\tau_n<\infty\}},$$

and hence

$$\mathbb{E}\left[X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}}\right] \geqslant n\mathbb{P}(\tau_n<\infty) = \mathbb{E}\left[\frac{1}{|B_1|}\wedge n\right].$$

Apparently, there exists $n_0 \geqslant 1$, such that for any $n > n_0$, we have

$$\mathbb{E}\left[\frac{1}{|B_1|}\wedge n\right] \geqslant \frac{1}{2}\mathbb{E}\left[\frac{1}{|B_1|}\right] =: c > 0.$$

Taking $n = n_0 \vee \lambda$ will verify (2).

On the other hand, for every $n \geqslant 0$, we have

$$X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}} = \lim_{m\to\infty} X_{\tau_n\wedge m}\mathbf{1}_{\{X_{\tau_n\wedge m}>\lambda\}},$$

and Fatou's lemma shows that

$$\mathbb{E}\left[X_{\tau_n}\mathbf{1}_{\{X_{\tau_n}>\lambda\}}\right] \leqslant \liminf_{m\to\infty}\mathbb{E}\left[X_{\tau_n\wedge m}\mathbf{1}_{\{X_{\tau_n\wedge m}>\lambda\}}\right].$$

Therefore, for the previous particular choice of n, we can further find m, such that

$$\mathbb{E}\left[X_{\tau_n\wedge m}\mathbf{1}_{\{X_{\tau_n\wedge m}>\lambda\}}\right] \geqslant \frac{c}{2}.$$

This proves that $\{X_{\tau_n\wedge m}:\ n,m\geqslant 1\}$ cannot be uniformly integrable, and hence X_t is not of class (D). In particular, it does not have a Doob-Meyer decomposition.

Problem 3. (1) Since $\mathbf{1}_{\Gamma_1\cap\Gamma_2} = \mathbf{1}_{\Gamma_1}\cdot\mathbf{1}_{\Gamma_2}$, and $\mathbf{1}_{\Gamma_2\backslash\Gamma_1} = \mathbf{1}_{\Gamma_2} - \mathbf{1}_{\Gamma_1}$ if $\Gamma_1 \subseteq \Gamma_2$, it is seen that $\mathcal{P}$ is closed under complement and finite union. Moreover, if $\Gamma_n \uparrow \Gamma$, then $\mathbf{1}_{\Gamma_n} \uparrow \mathbf{1}_{\Gamma}$. From this we also see that $\mathcal{P}$ is closed under increasing limit. Therefore, $\mathcal{P}$ is a σ-algebra. To see that $\mathcal{P}$ is a sub-σ-algebra of $\mathcal{B}([0,\infty))\otimes\mathcal{F}$, we only need to observe that

$$\Gamma\bigcap([0,t]\times\Omega) = \{(s,\omega)\in[0,t]\times\Omega:\ \mathbf{1}_{\Gamma}(s,\omega)=1\} \in \mathcal{B}([0,t])\otimes\mathcal{F}_t.$$

(2) First of all, it suffices to prove the claim on $t \in [0,T]$. Indeed, if for each $T > 0$, there exists a process

$$Y^T : \ \mathbb{R}^1 \times [0,T] \times \Omega \to \mathbb{R}^1$$

which verifies the claim for $t \in [0,T]$, then the process

$$Y \triangleq \left(\limsup_{T\to\infty} Y^T \right) \cdot \mathbf{1}_{\{\limsup_{T\to\infty} Y^T \text{ is finite}\}}$$

will have the desired properties on $[0,\infty)$.

Now consider a fixed time interval $[0,T]$. It is apparent that the claim is true for Φ of the form

$$\Phi_t^a(\omega) = f(a)H_t(\omega), \tag{3}$$

where f is a bounded $\mathcal{B}(\mathbb{R}^1)$-measurable function and H is a bounded progressively measurable process. Let $\mathcal{S}$ be the vector space spanned by such Φ. Then the claim is true for all $\Phi \in \mathcal{S}$.

If Φ is a general bounded $\mathcal{B}(\mathbb{R}^1) \otimes \mathcal{P}$-measurable process, a standard measure theoretic argument shows that there exists a sequence $\Phi_n \in \mathcal{S}$, such that $|(\Phi_n)_t^a(\omega)| \leqslant |\Phi_t^a(\omega)|$ and $(\Phi_n)_t^a(\omega) \to \Phi_t^a(\omega)$ for every $(a,t,\omega) \in \mathbb{R}^1 \times [0,T] \times \Omega$. For each n, let Y_n be the process with the desired properties associated with Φ_n. It follows from the stochastic dominated convergence theorem that for every $a \in \mathbb{R}^1$,

$$Y_n^a \to I^X(\Phi^a) \tag{4}$$

in probability uniformly on $[0,T]$, and

$$Y_n^\mu \triangleq \int_{\mathbb{R}^1} Y_n^a \mu(da) = I^X(\Phi_n^\mu) \to I^X(\Phi^\mu) \tag{5}$$

in probability uniformly on $[0,T]$, where $\Phi_n^\mu \triangleq \int_{\mathbb{R}^1} \Phi_n^a \mu(da)$ (similarly for Φ^μ).

Of course we want to define Y as the limit of Y_n. More precisely, we want to take a subsequence n_k (depending on a), such that along this subsequence we can define Y as the pointwise limit of Y_n. Here the main difficulty lies in choosing a subsequence $n_k(a)$ in a way which is measurable in a.

To do so, we first define

$$U_{n,m}^a(\omega) \triangleq \sup_{0\leqslant t\leqslant T} \left|(Y_n)_t^a(\omega) - (Y_m)_t^a(\omega)\right|.$$

It is apparent that $(a,\omega) \mapsto U^a_{n,m}(\omega)$ is $\mathcal{B}(\mathbb{R}^1)\otimes\mathcal{F}$-measurable. Moreover, we know that for each $a\in\mathbb{R}^1$, $U^a_{n,m}$ converges to zero in probability as $n,m\to\infty$. We define $n_0(a)\triangleq 1$, and for each $k\geqslant 1$, define

$$n_k(a) \triangleq \inf\left\{n \geqslant k \vee n_{k-1}(a) : \sup_{m,m'\geqslant n} \mathbb{P}(U^a_{m,m'} > 2^{-k}) \leqslant 2^{-k}\right\}.$$

Then it is easy to see that n_k is $\mathcal{B}(\mathbb{R}^1)$-measurable, and for every $a\in\mathbb{R}^1$, $n_k(a)\uparrow\infty$.

Now we define $\Psi_k \triangleq \Phi_{n_k}$ and $Z_k \triangleq Y_{n_k}$, and let

$$V^a_{n,m}(\omega) \triangleq \sup_{0\leqslant t\leqslant T} \left|(Z_n)^a_t(\omega) - (Z_m)^a_t(\omega)\right|.$$

By the definition of n_k, we know that

$$\mathbb{P}(V^a_{k,k+p} > 2^{-k}) \leqslant 2^{-k}$$

for all $a\in\mathbb{R}^1$ and $k,p\geqslant 1$. According to the Borel-Cantelli lemma, for every $a\in\mathbb{R}^1$, with probability one, $(Z_n)^a$ is a Cauchy sequence in $C([0,T];\mathbb{R}^1)$. More precisely, let

$$A \triangleq \left\{(a,\omega) : \lim_{n,m\to\infty} V^a_{n,m}(\omega) = 0\right\} \in \mathcal{B}(\mathbb{R}^1)\otimes\mathcal{F}.$$

Then for every $a\in\mathbb{R}^1$,

$$\int_\Omega \mathbf{1}_{A^c}(a,\omega)\mathbb{P}(d\omega) = 0.$$

According to Fubini's theorem, we conclude that $\mu\otimes\mathbb{P}(A)=0$ and with probability one,

$$\int_{\mathbb{R}^1} \mathbf{1}_{A^c}(a,\omega)\mu(da) = 0.$$

Finally, we define

$$Y \triangleq \left(\limsup_{k\to\infty} Z_k\right)\cdot \mathbf{1}_{\{\limsup_{k\to\infty} Z_k \text{ is finite}\}}.$$

Apparently Y is $\mathcal{B}(\mathbb{R}^1)\otimes\mathcal{P}$-measurable. According to (4) and (5), and the fact that Y is the uniform limit of Z_k on A where $\mu\otimes\mathbb{P}(A)=0$, we conclude that with probability one, for each $a\in\mathbb{R}^1$, $Y^a = I^X(\Phi^a)$, and

$$Y^\mu \triangleq \int_{\mathbb{R}^1} Y^a\mu(da) = I^X(\Phi^\mu).$$

Here a technical point is to see that with probability one, $\int_{\mathbb{R}^1} Z_k^a \mu(da) \to \int_{\mathbb{R}^1} Y^a \mu(da)$ in probability uniformly on $[0,T]$. One could see this by first considering the case where X is bounded (in which case one has convergence in L^2) and then using the standard localization argument to remove the localization (c.f. the proof of Proposition 5.14).

Problem 4. (1) Define

$$X_t \triangleq \exp\left(\int_0^t \sigma_s dB_s + \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right), \quad t \geqslant 0.$$

From Itô's formula, we see immediately that X_t satisfies the desired integral equation.

Now suppose that Y_t is another process that also satisfies the integral equation. Let

$$Z_t \triangleq Y_t X_t^{-1} = Y_t \exp\left(-\int_0^t \sigma_s dB_s - \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right)$$

Itô's formula again, or more precisely, the integration by parts formula, will imply that the martingale part and the bounded variation part of the continuous semimartingale Z_t are identically zero. Therefore,

$$Z_t = Z_0 = 1,$$

which shows that $Y_t = X_t$. In other words, there exists a unique continuous, $\{\mathcal{F}_t\}$-adapted process which satisfies the integral equation.

(2) First of all, we know that

$$X_t - 1 - \int_0^t X_s \mu_s ds = \int_0^t X_s \sigma_s dB_s, \quad 0 \leqslant t \leqslant T,$$

is a continuous local martingale under $\mathbb{P}$. Suppose we want to find a process q_t which is used to define the change of measure in the way that

$$\widetilde{\mathbb{P}}_T(A) \triangleq \mathbb{E}\left[\exp\left(\int_0^T q_s dB_s - \frac{1}{2}\int_0^T q_s^2 ds\right)\mathbf{1}_A\right], \quad A \in \mathcal{F}_T.$$

Then we know from Theorem 5.16 in the lecture notes that the process

$$\int_0^t X_s \sigma_s dB_s - \int_0^t q_s X_s \sigma_s ds = X_t - 1 - \int_0^t X_s \mu_s ds - \int_0^t q_s X_s \sigma_s ds$$

is a continuous local martingale under $\widetilde{\mathbb{P}}_T$ (provided that the exponential martingale is a true martingale so that $\widetilde{\mathbb{P}}_T$ is a probability measure). Now we want this process to be $X_t - 1$, therefore we just need to choose

$$q_t \triangleq -\mu_t \sigma_t^{-1}.$$

Since μ_t is uniformly bounded and $\sigma \geqslant C$, in this way we can see easily that Novikov's condition holds for the continuous local martingale $\int_0^t q_s dB_s$, which verifies that the exponential martingale is a true martingale.

Problem 5. (1) From Itô's formula, we have

$$B_T^2 = T + 2\int_0^T B_t dB_t,$$

so $\Phi_t = 2B_t$.

Similarly,

$$\begin{aligned} B_T^3 &= 3\int_0^T B_t^2 dB_t + 3\int_0^T B_t dt \\ &= 3\int_0^T B_t^2 dB_t + 3TB_T - 3\int_0^T t dB_t \\ &= \int_0^T \left(3B_t^2 + 3T - 3t\right) dB_t, \end{aligned}$$

so $\Phi_t = 3B_t^2 + 3T - 3t$.

(2) Fix $T > 0$, define $\sigma_T \triangleq \inf\{t \geqslant T : \ B_t = 0\}$. Consider $\Phi_t(\omega) \triangleq \mathbf{1}_{[0,\sigma_T(\omega)]}(t)$. Apparently $0 < \sigma_T < \infty$ almost surely (note that $B_T \neq 0$, and B is unbounded from above and from below almost surely), so we know that

$$0 < \int_0^\infty \Phi_t^2 dt = \sigma_T < \infty$$

almost surely. However,

$$\int_0^\infty \Phi_t dB_t = B_{\sigma_T} - B_0 = 0.$$

Therefore, uniqueness for Theorem 5.11 does not hold in the space $L^2_{\text{loc}}(B)$.

(3) For $0 \leqslant t \leqslant 1$, let $M_t \triangleq \mathbb{E}[S_1 | \mathcal{F}_t^B]$. Since

$$S_1 = \max\left\{S_t, \sup_{t \leqslant u \leqslant 1} B_u\right\} = \max\left\{S_t, B_t + \sup_{t \leqslant u \leqslant 1}(B_u - B_t)\right\},$$

and the Brownian motion has independent increments, we know that

$$M_t = F(S_t, B_t, t), \tag{6}$$

where

$$\begin{aligned} F(x,y,t) &\triangleq \mathbb{E}\left[\max\left\{x, y+\sup_{t\leqslant u\leqslant 1}\left(B_u - B_t)\right\}\right] \\ &= \mathbb{E}\left[\max\left\{x, y+S_{1-t}\right\}\right]. \end{aligned}$$

By using the distribution formula for S_{1-t}, we see that

$$F(x,y,t) = \int_{-\infty}^{\infty} \max\{x, y+\sqrt{1-t}|u|\}\varphi(u)du,$$

where φ is the density for a standard Gaussian distribution.

Since F is continuous, we see that M_t is a continuous martingale (more generally, the reader should think about why every càdlàg $\{\mathcal{F}_t^B\}$-martingale is continuous). Moreover, $F \in C^2$ on $t < 1$. Therefore, according to Itô's formula, we have

$$M_t = M_0 + \int_0^t \frac{\partial F}{\partial y}(S_u, B_u, u)dB_u, \quad t < 1.$$

Now

$$\frac{\partial F}{\partial y}(x,y,t) = \int_{-\infty}^{\infty} \mathbf{1}_{\{y+\sqrt{1-t}|u|\geqslant x\}}\varphi(u)du =: f(x,y,t), \quad t<1,$$

Therefore,

$$\begin{aligned} M_t &= M_0 + \int_0^t f(S_u, B_u, u)dB_u \\ &= \mathbb{E}[S_1] + \int_0^t 2\left(1-\Phi\left(\frac{S_u - B_u}{\sqrt{1-u}}\right)\right)dB_u, \ t<1, \end{aligned}$$

where $\Phi(x) \triangleq \int_{-\infty}^x \varphi(u)du$ is the standard Gaussian distribution function. Note that $f(x,y,t)$ is well defined even for $t=1$. Letting $t \uparrow 1$, we conclude that

$$\begin{aligned} S_1 &= \mathbb{E}[S_1] + \int_0^1 2\left(1-\Phi\left(\frac{S_u - B_u}{\sqrt{1-u}}\right)\right)dB_u \\ &= \sqrt{\frac{2}{\pi}} + \int_0^1 2\left(1-\Phi\left(\frac{S_u - B_u}{\sqrt{1-u}}\right)\right)dB_u. \end{aligned}$$

Problem 6. (1) Since the Brownian motion is rotationally invariant, we know that the distribution of B_τ is rotationally invariant on the unit sphere S^{d-1}. Let μ be the unique rotationally invariant probability measure (the normalized volume measure) on S^{d-1}. Then the distribution of B_τ is ν.

Now we show that B_τ and τ are independent. For a given orthogonal matrix O, define $B_t^O \triangleq O \cdot B_t$, and $\tau^O \triangleq \inf\{t \geqslant 0 : \ |B_t^O| = 1\}$. A crucial observation is that $\tau^O = \tau$. Therefore, for any bounded measurable function f on S^{d-1} and g on $[0, \infty)$, we have

$$\mathbb{E}[f(B_\tau^O)g(\tau)] = \mathbb{E}[f(B_\tau)g(\tau)].$$

In particular, this shows that the conditional distribution of B_τ given τ is again rotationally invariant, which implies that it has to be ν. Therefore,

$$\begin{aligned}
\mathbb{E}[f(B_\tau)g(\tau)] &= \mathbb{E}\left[g(\tau)\mathbb{E}[f(B_\tau)|\tau]\right] \\
&= \mathbb{E}\left[g(\tau)\cdot \int_{S^{d-1}} f(x)\nu(dx)\right] \\
&= \left(\int_{S^{d-1}} f(x)\nu(dx)\right)\cdot \mathbb{E}[g(\tau)] \\
&= \mathbb{E}[f(B_\tau)]\cdot \mathbb{E}[g(\tau)].
\end{aligned}$$

This shows that B_τ and τ are independent.

(2) Consider the continuous path space $(W^d, \mathcal{B}(W^d))$. Let μ be the Wiener measure on W^d. Let $B_t(w) \triangleq w_t$ be the coordinate process, which is a Brownian motion under μ, and let $\{\mathcal{B}_t(W^d)\}$ be the natural filtration of B_t. Define $\widetilde{\mathbb{P}}$ to be the unique extension of the family

$$\widetilde{\mathbb{P}}_T(A) \triangleq \int_A \mathrm{e}^{\langle c, B_T(w)\rangle - \frac{1}{2}|c|^2 T}\mu(dw), \quad A \in \mathcal{B}_T(W^d), \ T > 0,$$

of probability measures to $\mathcal{B}(W^d)$. It follows that under $\widetilde{\mathbb{P}}$, B_t is a Brownian motion with drift vector c. The reader might refer to the discussion after the proof of Theorem 5.17 for this part.

Now let f, g be two bounded measurable functions. Since $\mathrm{e}^{\langle c, B_t\rangle - \frac{1}{2}|c|^2 t}$ is a martingale, it follows from the optional sampling theorem that

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[f(B_{\tau\wedge t})g(\tau\wedge t)\right] &= \mathbb{E}\left[f(B_{\tau\wedge t})g(\tau\wedge t)\mathrm{e}^{\langle c, B_t\rangle - \frac{1}{2}|c|^2 t}\right] \\
&= \mathbb{E}\left[f(B_{\tau\wedge t})g(\tau\wedge t)\mathrm{e}^{\langle c, B_{\tau\wedge t}\rangle - \frac{1}{2}|c|^2 \tau\wedge t}\right].
\end{aligned}$$

Since $|B_{\tau\wedge t}| \leqslant 1$, by the dominated convergence theorem, we have

$$\widetilde{\mathbb{E}}\left[f(B_\tau)g(\tau)\right] = \mathbb{E}\left[f(B_\tau)g(\tau)\mathrm{e}^{\langle c,B_\tau\rangle - \frac{1}{2}|c|^2\tau}\right].$$

The same reason shows that

$$\mathbb{E}\left[\mathrm{e}^{\langle c,B_\tau\rangle - \frac{1}{2}|c|^2\tau}\right] = 1.$$

Moreover, from the first part, B_τ and τ are independent under μ. It follows that

$$\begin{aligned} & \mathbb{E}\left[f(B_\tau)g(\tau)\mathrm{e}^{\langle c,B_\tau\rangle - \frac{1}{2}|c|^2\tau}\right] \\ = \ & \mathbb{E}\left[f(B_\tau)\mathrm{e}^{\langle c,B_\tau\rangle}\right] \cdot \mathbb{E}\left[g(\tau)\mathrm{e}^{-\frac{1}{2}|c|^2\tau}\right] \\ = \ & \mathbb{E}\left[f(B_\tau)\mathrm{e}^{\langle c,B_\tau\rangle}\right] \cdot \mathbb{E}\left[\mathrm{e}^{-\frac{1}{2}|c|^2\tau + \langle c,B_\tau\rangle}\right] \cdot \mathbb{E}\left[g(\tau)\mathrm{e}^{-\frac{1}{2}|c|^2\tau}\right] \\ = \ & \mathbb{E}\left[f(B_\tau)\mathrm{e}^{\langle c,B_\tau\rangle - \frac{1}{2}|c|^2\tau}\right] \cdot \mathbb{E}\left[g(\tau)\mathrm{e}^{\langle c,B_\tau\rangle - \frac{1}{2}|c|^2\tau}\right] \\ = \ & \widetilde{\mathbb{E}}[f(B_\tau)] \cdot \widetilde{\mathbb{E}}[g(\tau)]. \end{aligned}$$

Therefore,

$$\widetilde{\mathbb{E}}\left[f(B_\tau)g(\tau)\right] = \widetilde{\mathbb{E}}[f(B_\tau)] \cdot \widetilde{\mathbb{E}}[g(\tau)],$$

which shows that B_τ and τ are independent under $\widetilde{\mathbb{P}}$.

Problem 7. From Tanaka's formula, we know that $|X_t|$ is a continuous semimartingale given by

$$|X_t| = |X_0| + \int_0^t \mathrm{sgn}(X_s)dX_s + L_t^0(X).$$

It follows that $\langle |X| \rangle_t = \langle M \rangle_t = \langle X \rangle_t$, where $X_t = X_0 + M_t + A_t$ is the semimartingale decomposition of X_t. Therefore, according to Corollary 5.5, we have

$$\begin{aligned} L_t^a(|X|) &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[a,a+\varepsilon)}(|X_s|)d\langle |X| \rangle_s \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[a,a+\varepsilon)}(|X_s|)d\langle X \rangle_s. \end{aligned}$$

From the above identity, we can already see that $L_t^a(|X|) = 0$ if $a < 0$. If $a \geqslant 0$, again from Corollary 5.5, we further have

$$\begin{aligned} L_t^a(|X|) &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[a,a+\varepsilon)}(X_s)d\langle X \rangle_s + \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{(-a-\varepsilon,-a]}(X_s)d\langle X \rangle_s \\ &= L_t^a(X) + L_t^{(-a)-}(X). \end{aligned}$$

Problem 8. Let $X_t \triangleq \lambda B_t^+ - \mu B_t^-$, where $\lambda \neq \mu > 0$. Let L_t^a be the local time process of X_t which is continuous in t and càdlàg in a. Then

$$L_t^0 - L_t^{0-} = 2\int_0^t \mathbf{1}_{\{X_s=0\}} dA_s.$$

On the one hand, according to the Tanaka's formula for Brownian motion, we have

$$A_t = \frac{\lambda - \mu}{2} l_t,$$

where l_t is the local time at 0 of Brownian motion. On the other hand,

$$\{s: \ X_s = 0\} = \{s: \ \lambda B_s^+ = \mu B_s^-\} = \{s: \ B_s = 0\}.$$

But we know that the random measure dl_t is supported on $\{t \geqslant 0: \ B_t = 0\}$. Therefore,

$$L_t^0 - L_t^{0-} = (\lambda - \mu)\int_0^t \mathbf{1}_{\{s: \ B_s=0\}} dl_s = (\lambda - \mu) l_t,$$

which is strictly non-zero almost surely.