Solutions for Problem Set Three

Problem 1. (1) The supermartingale property with respect to the filtration $\{\mathcal{F}_{\tau\wedge t}\}$ is a direct consequence of the optional sampling theorem for bounded stopping times. As for the original filtration, first observe that

$$X_{\tau\wedge s} \geqslant \mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_{\tau\wedge s}] = \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_{\tau\wedge s}] + \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau> s\}}|\mathcal{F}_{\tau\wedge s}]$$

for $s \leqslant t$. The first term equals $\mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_s]$ since $X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}} = X_{\tau\wedge s}\mathbf{1}_{\{\tau\leqslant s\}}$ is $\mathcal{F}_{\tau\wedge s}$-measurable. The second term equals $\mathbb{E}\left[\mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s]|\mathcal{F}_{\tau\wedge s}\right]$, where the integrand

$$\mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s] \in \mathcal{F}_{\tau\wedge s}.$$

Therefore,

$$X_{\tau\wedge s} \geqslant \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_s] + \mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge s}|\mathcal{F}_s] = \mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s].$$

(2) Let $s < t$ and $A \in \mathcal{F}_s$. Define $\sigma = s\mathbf{1}_A + t\mathbf{1}_{A^c}$ and $\tau = t$. It is obvious that σ, τ are bounded $\{\mathcal{F}_t\}$-stopping times. Therefore,

$$\mathbb{E}[X_\sigma] = \mathbb{E}[X_s\mathbf{1}_A] + \mathbb{E}[X_t\mathbf{1}_{A^c}] \leqslant \mathbb{E}[X_\tau] = \mathbb{E}[X_t],$$

which implies the desired submartingale property.

Problem 2. (1) Let $s < t$ and $A \in \mathcal{F}_s$. Since $\mathcal{F}_s \subseteq \mathcal{F}_t$, we have

$$\int_A M_t d\mathbb{P} = \mathbb{Q}(A) = \int_A M_s d\mathbb{P}.$$

Therefore, $\{M_t, \mathcal{F}_t\}$ is a martingale.

(2) Necessity. Suppose that $\{M_t\}$ is uniformly integrable. Then $M_t \to M_\infty$ almost surely and in L^1 for some $M_\infty \in \mathcal{F}_\infty$. Let $A \in \mathcal{F}_t$ for some $t \geqslant 0$. Then for any $u > t$, we have $A \in \mathcal{F}_u$ and thus

$$\mathbb{Q}(A) = \int_A M_u d\mathbb{P}.$$

By letting $u \to \infty$, we obtain that

$$\mathbb{Q}(A) = \int_A M_\infty d\mathbb{P}.$$

This is indeed true for all $A \in \mathcal{F}_\infty$ by the monotone class theorem, since $\mathcal{F}_\infty$ is generated by the π-system $\cup_{t\geqslant 0}\mathcal{F}_t$. Therefore, $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_\infty$ with the Radon-Nikodym derivative given by M_∞.

Sufficiency. Suppose that $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_\infty$ with $d\mathbb{Q}/d\mathbb{P} = Z$ for some $Z \in \mathcal{F}_\infty$. Then for each $t \geqslant 0$ and $A \in \mathcal{F}_t$, we have

$$\mathbb{Q}(A) = \int_A M_t d\mathbb{P} = \int_A Z d\mathbb{P}.$$

Therefore, $M_t = \mathbb{E}[Z|\mathcal{F}_t]$ which implies that $\{M_t\}$ is uniformly integrable.

Apparently, from the above argument we have already proved that $M_\infty \triangleq \lim_{t\to\infty} M_t$ is the Radon-Nikodym derivative of $\mathbb{Q}$ against $\mathbb{P}$ on $\mathcal{F}_\infty$. To see the final part, since in this case M_t is an $\{\mathcal{F}_t\}$-martingale with a last element M_∞, from the optional sampling theorem, we know that

$$\mathbb{Q}(A) = \int_A M_\infty d\mathbb{P} = \int_A M_\tau d\mathbb{P}, \quad \forall A \in \mathcal{F}_\tau.$$

Therefore, $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_\tau$ and $M_\tau = d\mathbb{Q}/d\mathbb{P}$ on $\mathcal{F}_\tau$.

Problem 3. Since $|X_t|$ is a right continuous submartingale, Doob's L^p-inequality implies that

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^p\right] \leqslant q^p \mathbb{E}[|X_t|^p] \leqslant q^p M,$$

where $M \triangleq \sup_{t\geqslant 0} \mathbb{E}[|X_t|^p]$ and $q = p/(p-1)$. In particular, Fatou's lemma implies that $\sup_{t\geqslant 0} |X_t|^p \in L^p$. On the other hand, since $\{X_t\}$ is uniformly integrable (because it is bounded in L^p), X_t converges to some X_∞ almost surely and in L^1. Now

$$|X_t - X_\infty|^p \leqslant 2^p(|X_t|^p + |X_\infty|^p) \leqslant 2^{p+1} \sup_{t\geqslant 0} |X_t|^p \in L^1.$$

The dominated convergence theorem then implies that

$$\lim_{t\to\infty} \mathbb{E}[|X_t - X_\infty|^p] = 0.$$

Problem 4. (1) Let $f(t) = \log t - t/e$ ($t > 0$), then $f'(t) = 1/t - 1/e$. Therefore, $f(t) \leqslant f(e) = 0$. Now we prove that $a \log^+ b \leqslant a \log^+ a + b/e$ for $a, b > 0$. If $b \leqslant 1$, this is trivial. If $b > 1, a \leqslant 1$, then

$$a \log^+ b = a \log b \leqslant \log b \leqslant \frac{b}{e} = a \log^+ a + \frac{b}{e}.$$

If $a, b > 1$, then the desired inequality follows from the fact that $\log(b/a) \leqslant (b/a)/e$.

(2) Similar to the proof of Doob's L^p-inequality, we have

$$\begin{aligned} \mathbb{E}[\rho(X_T^*)] &\leqslant \mathbb{E}\left[\int_0^{X_T^*} \rho(d\lambda)\right] \\ &= \mathbb{E}\left[\int_0^{\infty} \mathbf{1}_{\{X_T^* \geqslant \lambda\}} \rho(d\lambda)\right] \\ &= \int_0^{\infty} \mathbb{P}(X_T^* \geqslant \lambda)\rho(d\lambda) \\ &\overset{\text{Doob}}{\leqslant} \int_0^{\infty} \frac{1}{\lambda}\mathbb{E}[X_T \mathbf{1}_{\{X_T^* \geqslant \lambda\}}]\rho(d\lambda) \\ &= \mathbb{E}\left[X_T \int_0^{X_T^*} \lambda^{-1}\rho(d\lambda)\right]. \end{aligned}$$

(3) Let $\rho(t) = (t-1)^+$ ($t \geqslant 0$). Then from the second part, we have

$$\begin{aligned} \mathbb{E}[X_T^*] - 1 &\leqslant \mathbb{E}[(X_T^* - 1); X_T^* \geqslant 1] \\ &\leqslant \mathbb{E}\left[X_T \int_0^{X_T^*} \lambda^{-1}\rho(d\lambda)\right] \\ &= \mathbb{E}\left[X_T \int_1^{X_T^*} \lambda^{-1}\rho(d\lambda); X_T^* \geqslant 1\right] \\ &= \mathbb{E}[X_T \log X_T^*; X_T^* \geqslant 1] \\ &= \mathbb{E}[X_T \log^+ X_T^*] \\ &\leqslant \mathbb{E}[X_T \log^+ X_T^*] + \frac{1}{\mathrm{e}}\mathbb{E}[X_T^*]. \end{aligned}$$

Rearranging the terms yields the desired inequality.

Problem 5. From the assumption that

$$\sup_{0 \leqslant t < \infty} X_t(\omega) = \infty, \quad \inf_{0 \leqslant t < \infty} X_t(\omega) = -\infty,$$

it is apparent that every τ_n is well-defined finitely. Since X_t is $\{\mathcal{F}_t\}$-adapted and has continuous sample paths, according to Proposition 2.7 in the lecture notes, we know that τ_1 is an $\{\mathcal{F}_t\}$-stopping time. To see why τ_2 is also an $\{\mathcal{F}_t\}$-stopping time, define $\widetilde{X}_t \triangleq X_{\tau_1+t} - X_{\tau_1}$ and $\mathcal{G}_t \triangleq \mathcal{F}_{t+\tau_1}$. It follows that $\widetilde{X}_t$ is $\{\mathcal{G}_t\}$-adapted and has continuous sample paths. Therefore, the same reason implies that $\tau_2 - \tau_1$ is a $\{\mathcal{G}_t\}$-stopping time. According to Problem Sheet 2, Problem 4, (2), (ii), we conclude that τ_2 is an $\{\mathcal{F}_t\}$-stopping time. Inductively, we know that every τ_n is an $\{\mathcal{F}_t\}$-stopping time.

Now we study the distribution of the random sequence $\{X_{\tau_n} : n \geqslant 1\}$. Define $\sigma_n \triangleq \inf\{t \geqslant 0 : |X_t| > 2n\}$. Then $\tau_n < \sigma_n$ (in fact, $|X_t| \leqslant n$ for all $t \in [0, \tau_n]$) and $X_t^{\sigma_n} \triangleq X_{\sigma_n \wedge t}$ is a bounded $\{\mathcal{F}_t\}$-martingale. In particular, X_t has a last element $X_\infty = \lim_{t\to\infty} X_t$. By the optional sampling theorem, we conclude that

$$\mathbb{E}[X_{\tau_n} - X_{\tau_{n-1}} | \mathcal{F}_{\tau_{n-1}}] = \mathbb{E}[X_{\tau_n}^{\sigma_n} - X_{\tau_{n-1}}^{\sigma_n} | \mathcal{F}_{\tau_{n-1}}] = 0.$$

Now let $A_n^+ \triangleq \{X_{\tau_n} - X_{\tau_{n-1}} = 1\}$ and $A_n^- \triangleq \{X_{\tau_n} - X_{\tau_{n-1}} = -1\}$ respectively. It follows that

$$\mathbb{P}(A_n^+ | \mathcal{F}_{\tau_{n-1}}) = \mathbb{P}(A_n^- | \mathcal{F}_{\tau_{n-1}}) = \frac{1}{2} \quad \text{a.s.}$$

Therefore, for any $i_1, \cdots, i_n = \pm 1$, we have

$$\begin{aligned} & \mathbb{P}(X_{\tau_1} = i_1, X_{\tau_2} - X_{\tau_1} = i_2, \cdots, X_{\tau_n} - X_{\tau_{n-1}} = i_n) \\ = & \int_{\{X_{\tau_1} = i_1, \cdots, X_{\tau_{n-1}} - X_{\tau_{n-2}} = i_{n-1}\}} \mathbb{P}(\{X_{\tau_n} - X_{\tau_{n-1}} = i_n\} | \mathcal{F}_{\tau_{n-1}}) d\mathbb{P} \\ = & \frac{1}{2} \mathbb{P}(X_{\tau_1} = i_1, \cdots, X_{\tau_{n-1}} - X_{\tau_{n-2}} = i_{n-1}). \end{aligned}$$

Recursively, in the end this will imply that $X_{\tau_1}, X_{\tau_2} - X_{\tau_1}, \cdots, X_{\tau_n} - X_{\tau_{n-1}}$ are independent and identically distributed with distribution $\mathbb{P}(X_{\tau_1} = \pm 1) = 1/2$. Therefore, $\{X_{\tau_n} : n \geqslant 1\}$ is distributed as the standard simple random walk.

Problem 6. (1) We first prove a claim:

$$\mathbb{E}[|X_\tau - X_\sigma| | \mathcal{F}_\sigma] \leqslant M_X, \tag{1}$$

for any $\{\mathcal{F}_t\}$-stopping times $\sigma \leqslant \tau$. Indeed, since $\{X_t : 0 \leqslant t \leqslant \infty\}$ is a continuous martingale with a last element, the optional sampling theorem and the assumption imply that

$$\begin{aligned} \mathbb{E}\left[|X_\tau - X_\sigma| | \mathcal{F}_\sigma\right] &= \mathbb{E}\left[|\mathbb{E}[X_\infty | \mathcal{F}_\tau] - X_\sigma| \, | \mathcal{F}_\sigma\right] \\ &\leqslant \mathbb{E}\left[\mathbb{E}[|X_\infty - X_\sigma| | \mathcal{F}_\tau] | \mathcal{F}_\sigma\right] \\ &= \mathbb{E}[|X_\infty - X_\sigma| | \mathcal{F}_\sigma] \\ &\leqslant M_X. \end{aligned}$$

Now for $\lambda, \mu > 0$, let

$$\begin{aligned} \sigma &\triangleq \inf\{t \geqslant 0 : \ |X_t| \geqslant \lambda\}, \\ \tau &\triangleq \inf\{t \geqslant 0 : \ |X_t| \geqslant \lambda + \mu\}. \end{aligned}$$

According to (1), we have

$$\int_{\{\sigma<\infty\}} |X_\tau - X_\sigma| d\mathbb{P} \leqslant M_X \mathbb{P}(\sigma < \infty) \leqslant M_X \mathbb{P}(X^* \geqslant \lambda).$$

But since $\{X^* \geqslant \lambda + \mu\} \subseteq \{\sigma < \infty\}$ and $|X_\tau - X_\sigma| = \mu$ on $\{X^* \geqslant \lambda + \mu\}$, it follows that

$$\int_{\{\sigma<\infty\}} |X_\tau - X_\sigma| d\mathbb{P} \geqslant \mu \mathbb{P}(X^* \geqslant \lambda + \mu).$$

Therefore,

$$\mathbb{P}(X^* \geqslant \lambda + \mu) \leqslant \frac{M_X}{\mu} \mathbb{P}(X^* \geqslant \lambda).$$

(2) Let $\lambda > 0$. Note that for any $k \geqslant 1$, from (1) we have

$$\mathbb{P}(X^* \geqslant k\mathrm{e}M_X) \leqslant \frac{1}{\mathrm{e}} \mathbb{P}(X^* \geqslant (k-1)\mathrm{e}M_X) \leqslant \cdots \leqslant \mathrm{e}^{-k}.$$

Now if $\lambda \geqslant \mathrm{e}M_X$, let k be the unique positive integer such that $k\mathrm{e}M_X \leqslant \lambda < (k+1)\mathrm{e}M_X$. Then

$$\mathbb{P}(X^* \geqslant \lambda) \leqslant \mathbb{P}(X^* \geqslant k\mathrm{e}M_X) \leqslant \mathrm{e}^{-k} \leqslant \mathrm{e}^{1-\frac{\lambda}{\mathrm{e}M_X}} \leqslant \mathrm{e}^{2-\frac{\lambda}{\mathrm{e}M_X}}.$$

The inequality is trivial for $0 < \lambda < \mathrm{e}M_X$ since in this case $\mathrm{e}^{2-\frac{\lambda}{\mathrm{e}M_X}} > 1$.

To see the exponential integrability, first note that the first part implies that $X^* < \infty$ almost surely, and

$$\mathbb{P}(\mathrm{e}^{\alpha X^*} \geqslant \mathrm{e}^{\alpha\lambda}) \leqslant \mathrm{e}^{2-\frac{\lambda}{\mathrm{e}M_X}}, \quad \forall \lambda > 0.$$

Therefore,

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{\alpha X^*}] &= \int_0^\infty \mathbb{P}(\mathrm{e}^{\alpha X^*} \geqslant \mu) d\mu \\ &\leqslant 1 + \int_1^\infty \mathbb{P}(\mathrm{e}^{\alpha X^*} \geqslant \mu) d\mu \\ &= 1 + \alpha \int_0^\infty \mathbb{P}(\mathrm{e}^{\alpha X^*} \geqslant \mathrm{e}^{\alpha\lambda}) \mathrm{e}^{\alpha\lambda} d\lambda \\ &\leqslant 1 + \alpha \int_0^\infty \mathrm{e}^{2-\left(\frac{1}{\mathrm{e}M_X}-\alpha\right)\lambda} d\lambda, \end{aligned}$$

which is finite if $0 < \alpha < (\mathrm{e}M_X)^{-1}$. The L^p-integrability follows from then the exponential integrability.

Problem 7. (1) Let $\tau \in \mathcal{S}_T$. By the optional sampling theorem,

$$\mathbb{E}[X_\tau \mathbf{1}_{\{X_\tau > \lambda\}}] \leqslant \mathbb{E}[X_T \mathbf{1}_{\{X_\tau > \lambda\}}].$$

But

$$\mathbb{P}(X_\tau > \lambda) \leqslant \frac{\mathbb{E}[X_\tau]}{\lambda} \leqslant \frac{\mathbb{E}[X_T]}{\lambda} \to 0$$

uniformly in $\tau \in \mathcal{S}_T$ as $\lambda \to \infty$. Therefore, $\mathbb{E}[X_\tau \mathbf{1}_{\{X_\tau > \lambda\}}] \to 0$ uniformly in $\tau \in \mathcal{S}_T$ as $\lambda \to \infty$, which proves the claim that X_t is of class (DL). Suppose further that X_t is continuous. Let $\tau_n \uparrow \tau \in \mathcal{S}_T$. Then $X_{\tau_n} \to X_\tau$ almost surely as $n \to \infty$. But X_t is of class (DL), so $\{X_{\tau_n}\}$ is uniformly integrable. Therefore, $X_{\tau_n} \to X_\tau$ in L^1, which implies that X_t is regular.

(2) If X_t is non-negative and uniformly integrable, then X_t converges to some X_∞ almost surely and in L^1. Moreover, we have

$$X_t \leqslant \mathbb{E}[X_\infty | \mathcal{F}_t]$$

for every $t \geqslant 0$. The optional sampling theorem then implies that

$$X_\tau \leqslant \mathbb{E}[X_\infty | \mathcal{F}_\tau]$$

for every finite $\{\mathcal{F}_t\}$-stopping time τ. The uniform integrability of $\{X_\tau\}$ follows from the same argument as in the first part of the problem.

Since $X_t = M_t + A_t$ by the Doob-Meyer decomposition, we know that $\mathbb{E}[X_t] = \mathbb{E}[M_0] + \mathbb{E}[A_t]$. By letting $t \to \infty$, we conclude that $\mathbb{E}[X_\infty] = \mathbb{E}[M_0] + \mathbb{E}[A_\infty]$. In particular, $\mathbb{E}[A_\infty] < \infty$.