Solutions for Problem Set Two

Problem 1. Necessity. Suppose that $\mathbb{P}_n$ converges weakly to some probability measure $\mathbb{P}$ on $(W^d, \mathcal{B}(\mathbb{R}^d))$. Then $\{\mathbb{P}_n\}$ is tight by Prokhorov's theorem. In addition, given $m \geqslant 1$ and $0 \leqslant t_1 < t_2 < \cdots < t_m$, let $\varphi \in C_b(\mathbb{R}^{d\times m})$ and define $\Phi \in C_b(W^d)$ by

$$\Phi(w) = \varphi(w_{t_1}, \cdots, w_{t_m}), \quad w \in W^d.$$

Then

$$\int_{\mathbb{R}^{d\times m}} \varphi dQ_n = \int_{W^d} \Phi d\mathbb{P}_n \to \int_{W^d} \Phi d\mathbb{P} = \int_{\mathbb{R}^{d\times m}} \varphi dQ,$$

where Q is the finite dimensional distribution of $\mathbb{P}$ at $(t_1, \cdots, t_m)$. Therefore, Q_n converges weakly to Q.

Sufficiency. We first show that the sequence $\mathbb{P}_n$ has exactly one weak limit point. Indeed, since $\{\mathbb{P}_n\}$ is tight, Prokhorov's theorem tells us that $\mathbb{P}_n$ has at least one weak limit point. Suppose that $\mathbb{P}'$ and $\mathbb{P}''$ are two weak limit points of $\mathbb{P}_n$. According to Assumption (i), we know that $\mathbb{P}'$ and $\mathbb{P}''$ have the same finite dimensional distributions. Therefore, by the monotone class theorem, $\mathbb{P}' = \mathbb{P}''$. In other words, $\mathbb{P}_n$ has exactly one weak limit point, which is denoted by $\mathbb{P}$. Now let $f \in C_b(W^d)$. Then as a bounded sequence in $\mathbb{R}^1$, $\int_{W^d} f d\mathbb{P}_n$ has exactly one limit point which is $\int_{W^d} f d\mathbb{P}$. Therefore, $\mathbb{P}_n$ converges weakly to $\mathbb{P}$.

Problem 2. (1) Let

$$p_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{2t}}, \quad t > 0, x \in \mathbb{R}^d.$$

We define a family of $\{Q_{\mathfrak{t}} : \mathfrak{t} \in \mathcal{T}\}$ of finite dimensional distributions on $\mathbb{R}^d$ in the following way. For $\mathfrak{t} = (t_1, \cdots, t_n)$ where $n \geqslant 1$ and $0 < t_1 < t_2 < \cdots < t_n$, define

$$Q_{\mathfrak{t}}(\Gamma) \triangleq \int_\Gamma p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \cdots p_{t_n - t_{n-1}}(x_n - x_{n-1}) dx_1 \cdots dx_n, \quad \Gamma \in \mathcal{B}(\mathbb{R}^{d\times n}). \tag{1}$$

The definition of $Q_{\mathfrak{t}}$ for general disordered $(t_1, \cdots, t_n) \in \mathcal{T}$ is easily obtained by permuting (1). The first consistency property is just definition, while the second consistency property follows from the fact that

$$\int_{\mathbb{R}^1} p_{t_i - t_{i-1}}(x_i - x_{i-1}) p_{t_{i+1}-t_i}(x_{i+1} - x_i) dx_i = p_{t_{i+1}-t_{i-1}}(x_{i+1} - x_{i-1})$$

if $t_{i-1} < t_i < t_{i+1}$, which can be shown by direct (but lengthy) computation. Therefore, according to Kolmogorov's extension theorem, there exists a unique probability measure $\mathbb{P}$ on the full path space $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}\left((\mathbb{R}^d)^{[0,\infty)}\right)\right)$ whose finite dimensional distributions coincide with $\{Q_{\mathfrak{t}} : \mathfrak{t} \in \mathcal{T}\}$. From the construction of $Q_{\mathfrak{t}}$, it is apparent that $\mathbb{P}$ satisfies the desired properties.

(2) Since $|X_t - X_s|^n \leqslant C_{n,d} \sum_{i=1}^d |X_t^i - X_s^i|^n$, it is sufficient to consider the case when $d = 1$. In the one dimensional case, for $s < t$, since $(X_t - X_s)/\sqrt{t-s}$ is a standard normal random variable, we have

$$\mathbb{E}[|X_t - X_s|^{2n}] = \mathbb{E}\left[\left|\frac{X_t - X_s}{\sqrt{t-s}}\right|^{2n} \cdot |t-s|^n\right] = K_n |t-s|^{1+(n-1)}$$

for every $n \geqslant 1$, where K_n is the $2n$-th moment of the standard normal distribution (i.e. $K_n \triangleq \mathbb{E}[|Z|^{2n}]$ where $Z \sim \mathcal{N}(0,1)$). As $(n-1)/2n \to 1/2$ as $n \to \infty$, the result follows from Kolmogorov's continuity theorem.

(3) For the first assertion, for simplicity assume that $T = 1$. Then

$$\sup_{\substack{s,t \in [0,1] \\ s \neq t}} \frac{\left|\widetilde{X}_t - \widetilde{X}_s\right|}{\sqrt{t-s}} \geqslant \sup_{n \geqslant 1} \sup_{1 \leqslant k \leqslant n} \frac{\left|\widetilde{X}_{k/n} - \widetilde{X}_{(k-1)/n}\right|}{\sqrt{1/n}}. \tag{2}$$

Therefore, it suffices to show that the right hand side of (2) is infinite almost surely. Indeed, given $\lambda > 0$, let

$$A_n^\lambda = \left\{\sup_{1 \leqslant k \leqslant n} \frac{\left|\widetilde{X}_{k/n} - \widetilde{X}_{(k-1)/n}\right|}{\sqrt{1/n}} \leqslant \lambda\right\}, \quad n \geqslant 1.$$

Then

$$\begin{aligned} \mathbb{P}(A_n^\lambda) &= \mathbb{P}\left(\bigcap_{k=1}^n \left\{\frac{\left|\widetilde{X}_{k/n} - \widetilde{X}_{(k-1)/n}\right|}{\sqrt{1/n}} \leqslant \lambda\right\}\right) \\ &= (\mathbb{P}(|Z| \leqslant \lambda))^n \end{aligned}$$

for every n, where $Z \sim \mathcal{N}(0,1)$. As $\mathbb{P}(|Z| \leqslant \lambda) < 1$, we know that

$$\mathbb{P}\left(\sup_{n\geqslant 1}\sup_{1\leqslant k\leqslant n}\frac{\left|\widetilde{X}_{k/n}-\widetilde{X}_{(k-1)/n}\right|}{\sqrt{1/n}}\leqslant\lambda\right)\leqslant\mathbb{P}\left(\bigcap_{n=1}^{\infty}A_n^{\lambda}\right)=\lim_{n\to\infty}\mathbb{P}(A_n^{\lambda})=0.$$

This is true for every λ, which concludes that

$$\sup_{n\geqslant 1}\sup_{1\leqslant k\leqslant n}\frac{\left|\widetilde{X}_{k/n}-\widetilde{X}_{(k-1)/n}\right|}{\sqrt{1/n}}=\infty,\quad \text{a.s.}$$

The second assertion is proved in a similar way. First note that

$$\sup_{\substack{s,t\in[0,\infty)\\ s\neq t}}\frac{\left|\widetilde{X}_t-\widetilde{X}_s\right|}{(t-s)^{\gamma}}\geqslant\sup_{n\geqslant 1}\left|\widetilde{X}_n-\widetilde{X}_{n-1}\right|.$$

In addition, for every $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{n\geqslant 1}\left|\widetilde{X}_n-\widetilde{X}_{n-1}\right|\leqslant\lambda\right) &= \lim_{n\to\infty}\mathbb{P}\left(\left|\widetilde{X}_k-\widetilde{X}_{k-1}\right|\leqslant\lambda,\ \forall k\leqslant n\right)\\ &= \lim_{n\to\infty}\mathbb{P}(|Z|\leqslant\lambda)^n\\ &= 0.\end{aligned}$$

Therefore,

$$\sup_{n\geqslant 1}\left|\widetilde{X}_n-\widetilde{X}_{n-1}\right|=\infty,\quad \text{a.s.},$$

which implies the desired claim.

Problem 3. (1) Let τ be a finite random time defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ which has a bounded density $f(t)$ with respect the Lebesgue measure (i.e. $\mathbb{P}(\tau\in A)=\int_A f(t)dt$ for $A\in\mathcal{B}([0,\infty))$). Define a stochastic process X_t by

$$X(t,\omega)=\begin{cases}1, & \text{if } t\geqslant\tau(\omega);\\ 0, & \text{otherwise.}\end{cases}$$

Then for every $\alpha>0$ and $s<t$,

$$\begin{aligned}\mathbb{E}[|X_t-X_s|^{\alpha}] &= \mathbb{E}[1\cdot\mathbf{1}_{\{s<\tau\leqslant t\}}]=\mathbb{P}(s<\tau\leqslant t)\\ &= \int_s^t f(u)du\leqslant\|f\|_{\infty}(t-s).\end{aligned}$$

However, there is no modification of X whose sample paths are continuous.

(2) Let τ be as in (1) and define a stochastic process X_t by

$$X(t,\omega) = \begin{cases} 1, & \text{if } \tau(\omega) = t; \\ 0, & \text{otherwise.} \end{cases}$$

Then for each fixed t, $X_t = 0$ almost surely because $\mathbb{P}(\tau = t) = 0$. Therefore, the conditions in Kolmogorov's continuity theorem are verified. But every sample path of X is discontinuous because $\tau(\omega) < \infty$ for every ω.

If we further assume that every sample path of X is right continuous with left limits, then the assertion is true. Indeed, following the notation in the proof of the theorem, for every $\omega \in \Omega^*$, we have

$$d(X_t(\omega), X_s(\omega)) \leqslant 2^\gamma \left(1 + \frac{2}{2^\gamma - 1}\right) |t - s|^\gamma \tag{3}$$

for each $s, t \in D$ with $0 < |t - s| < 2^{-n^*(\omega)}$. Since every sample path of X is right continuous with left limits, we know that (3) is true for all $s, t \in [0, 1]$ with $0 < |t - s| < 2^{-n^*(\omega)}$. Therefore, $t \mapsto X_t(\omega)$ is continuous for every $\omega \in \Omega^*$.

(3) From Theorem 1.10 in Section 1, we need to show that

$$\lim_{a\to\infty} \sup_n \mathbb{P}(|X_0^{(n)}| > a) = 0,$$

and

$$\lim_{\delta\downarrow 0} \sup_n \mathbb{P}(\Delta(\delta, k; X^{(n)}) > \varepsilon) = 0$$

for each $\varepsilon > 0$ and $k \geqslant 1$.

The first assertion follows immediately from Chebyshev's inequality and the first assumption in the problem. For the second claim, as in the proof of Kolmogorov's continuity theorem, let $0 < \gamma < \beta/\alpha$. For notation simplicity, we write $Y_t = X_t^{(n)}$ (it is important that the estimates below are uniform in n). Then for fixed $k \geqslant 1$, we have

$$\mathbb{P}\left(\left|Y_{\frac{l}{2^m}} - Y_{\frac{l-1}{2^m}}\right| > \frac{1}{2^{\gamma m}}\right) \leqslant M_k 2^{\alpha\gamma m} 2^{-m(1+\beta)}$$

for each $m \geqslant 1$ and $1 \leqslant l \leqslant 2^m k$. Therefore,

$$\mathbb{P}\left(\max_{1\leqslant l\leqslant 2^m k}\left|Y_{\frac{l}{2^m}} - Y_{\frac{l-1}{2^m}}\right| > \frac{1}{2^{\gamma m}}\right) \leqslant k M_k 2^{-m(\beta-\alpha\gamma)}.$$

Given $\varepsilon, \eta > 0$, let $p \geqslant 1$ be such that

$$kM_k \sum_{m=p}^{\infty} 2^{-m(\beta-\alpha\gamma)} = \frac{kM_k 2^{-p(\beta-\alpha\gamma)}}{1-2^{-(\beta-\alpha\gamma)}} < \eta$$

and

$$2^{\gamma}\left(1+\frac{2}{2^{\gamma}-1}\right)2^{-\gamma p} < \varepsilon.$$

Define

$$\Omega_p = \bigcup_{m=p}^{\infty}\left\{\max_{1\leqslant l\leqslant 2^m k}\left|Y_{\frac{l}{2^m}} - Y_{\frac{l-1}{2^m}}\right| > \frac{1}{2^{\gamma m}}\right\}.$$

It follows that $\mathbb{P}(\Omega_p) < \eta$. Now we show that for every $\delta < 2^{-p}$, we have

$$\{\Delta(\delta, k; Y) > \varepsilon\} \subseteq \Omega_p, \tag{4}$$

which completes the proof. Indeed, let $\omega \notin \Omega_p$, then

$$\left|Y_{\frac{l}{2^m}}(\omega) - Y_{\frac{l-1}{2^m}}(\omega)\right| \leqslant \frac{1}{2^{\gamma m}}$$

for each $m \geqslant p$ and $1 \leqslant l \leqslant 2^m k$. Let $D \triangleq \cup_{m=1}^{\infty} D_m$, where $D_m \triangleq \{l/2^m : 0 \leqslant l \leqslant 2^m k\}$. The same argument as in the proof of Kolmogorov's continuity theorem allows us to conclude that for each $s, t \in D$ with $0 < |s-t| < 2^{-p}$, we have

$$|Y_t(\omega) - Y_s(\omega)| \leqslant 2^{\gamma}\left(1+\frac{2}{2^{\gamma}-1}\right)\cdot|t-s|^{\gamma} < 2^{\gamma}\left(1+\frac{2}{2^{\gamma}-1}\right)2^{-\gamma p} < \varepsilon.$$

Since Y has continuous sample paths, the above inequality is true for all $s, t \in [0, k]$. This implies that

$$\Delta(\delta, k; Y(\omega)) \leqslant \varepsilon$$

provided $\delta < 2^{-p}$. Therefore, (4) holds for $\delta < 2^{-p}$.

Problem 4. (1) The intuition behind this property is the following. If we have the information up to time t, we know whether $\{\tau \leqslant t\}$ occurs since τ is an $\{\mathcal{F}_t\}$-stopping time. If it occurs, then we have the information up to τ. But σ is $\mathcal{F}_\tau$-measurable, so we are able to determine the value of σ, and of course the occurrence of $\{\sigma \leqslant t\}$ or not. If $\{\tau \leqslant t\}$ does not occur, then $\tau > t$. But $\sigma \geqslant \tau$, so we conclude that $\sigma > t$.

The mathematical proof is the following. For $t \geqslant 0$, we have

$$\{\sigma > t\} = \{\tau > t\} \bigcup \{\sigma > t,\ \tau \leqslant t\}.$$

By assumption, we know that $\{\tau > t\} \in \mathcal{F}_t$ and $\{\sigma > t\} \cap \{\tau \leqslant t\} \in \mathcal{F}_t$. Therefore, $\{\sigma > t\} \in \mathcal{F}_t$, which implies that σ is an $\{\mathcal{F}_t\}$-stopping time.

(2) The following observation is generally useful.

Proposition. Suppose that $\{\mathcal{F}_t\}$ is a right continuous filtration. Then τ is an $\{\mathcal{F}_t\}$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for every $t \geqslant 0$. In this case, $A \in \mathcal{F}_\tau$ if and only if $A \cap \{\tau < t\} \in \mathcal{F}_t$ for every $t \geqslant 0$.

Proof. We only proof the sufficiency of the first part. All other parts are either easy or similar. Suppose that τ satisfies $\{\tau < t\} \in \mathcal{F}_t$ for every $t \geqslant 0$. Since $\{\mathcal{F}_t\}$ is right continuous, it suffices to show that $\{\tau \leqslant t\} \in \mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$ for each given t. Indeed, for every $u > t$, we have $\{\tau \leqslant t\} = \cap_{n>(u-t)^{-1}}\{\tau < t + 1/n\} \in \mathcal{F}_u$. Therefore, the desired property holds. **Q.E.D.**

(i) For the first part, since $\{\tau < t\} = \cup_{n=1}^{\infty}\{\tau_n < t\} \in \mathcal{F}_t$, from the above proposition we know that τ is an $\{\mathcal{F}_t\}$-stopping time. For the second part, suppose that $A \in \cap_{n=1}^{\infty}\mathcal{F}_{\tau_n}$. Then $A \cap \{\tau < t\} = \cup_{n=1}^{\infty}(A \cap \{\tau_n < t\}) \in \mathcal{F}_t$. Therefore, again from the above proposition we know that $A \in \mathcal{F}_\tau$. The other direction is obvious.

(ii) The intuition is the following. Suppose that we have the information up to time t. If we observe that $\{\sigma > t\}$, then of course we can conclude that $\{\sigma + \tau > t\}$ happens. If we observe that $\{\sigma \leqslant t\}$, then we know the information of "$\mathcal{G}_{t-\sigma}$" (this thing is actually not well defined because $t - \sigma$ is not a stopping time, but we can still think in this way naively). Therefore, we can determine the occurrence of $\{\tau \leqslant t - \sigma\} = \{\sigma + \tau \leqslant t\}$ because τ is a $\{\mathcal{G}_t\}$-stopping time.

The rigorous proof is the following. For any given $t \geqslant 0$, we have $\{\sigma + \tau < t\} = \cup_{r \in (0,t) \cap \mathbb{Q}}\{\sigma < r, \tau < t - r\}$. Since $\{\tau < t - r\} \in \mathcal{F}_{\sigma + (t-r)}$, we know that

$$\{\tau < t - r\} \cap \{\sigma + (t - r) < t\} = \{\tau < t - r, \sigma < r\} \in \mathcal{F}_t.$$

Therefore, $\{\sigma + \tau < t\} \in \mathcal{F}_t$. From the above proposition, this implies that $\sigma + \tau$ is an $\{\mathcal{F}_t\}$-stopping time.

Problem 5. (1) It will be sufficient if we can prove that

$$\mathbb{E}[F \cdot \varphi(X_{t+u_1} - X_t, \cdots, X_{t+u_n} - X_t)] = \mathbb{E}[F]\mathbb{E}[\varphi(X_{t+u_1} - X_t, \cdots, X_{t+u_n} - X_t)], \quad (5)$$

for any bounded $\mathcal{G}_{t+}^X$-measurable F and $\varphi \in C_b\left((\mathbb{R}^d)^n\right)$ where $n \geqslant 1$, $0 \leqslant u_1 < \cdots < u_n < \infty$.

Indeed, for any $\varepsilon > 0$, by assumption we know that $\mathcal{G}^X_{t+\varepsilon}$ and $\mathcal{U}_{t+\varepsilon}$ are independent. Since F is also $\mathcal{G}^X_{t+\varepsilon}$-measurable, we have

$$\begin{aligned}&\mathbb{E}[F \cdot \varphi(X_{t+u_1+\varepsilon} - X_{t+\varepsilon}, \cdots, X_{t+u_n+\varepsilon} - X_{t+\varepsilon})]\\ &= \mathbb{E}[F]\mathbb{E}[\varphi(X_{t+u_1+\varepsilon} - X_{t+\varepsilon}, \cdots, X_{t+u_n+\varepsilon} - X_{t+\varepsilon})].\end{aligned}$$

Since X_t has right continuous sample paths, the desired identity (5) follows from letting $\varepsilon \to 0$.

(2) For fixed $t \geqslant 0$, we first show that $\mathcal{G}^X_{t+} \subseteq \mathcal{F}^X_t$. To this end, let ξ be an arbitrary bounded $\mathcal{G}^X_{t+}$-measurable random variable. Define $\eta = \xi - \mathbb{E}[\xi|\mathcal{G}^X_t]$. If we can show that $\eta = 0$, then we know that ξ is equivalent to a $\mathcal{G}^X_t$-measurable random variable, which implies that ξ is $\mathcal{F}^X_t$-measurable. Our claim then follows.

Now we show that $\eta = 0$. Let $\mathcal{C} \triangleq \{A \cap B : \ A \in \mathcal{G}_t, B \in \mathcal{U}_t\}$. Then $\mathcal{C}$ is a π-system which generates $\mathcal{G}^X_\infty = \sigma(X_t : \ t \geqslant 0)$. Since η is $\mathcal{G}^X_\infty$-measurable, it suffices to show that: for any $A \in \mathcal{G}^X_t$ and $B \in \mathcal{U}_t$, we have $\mathbb{E}[\eta \mathbf{1}_{A\cap B}] = 0$. Indeed, since $\eta \mathbf{1}_A$ is $\mathcal{G}^X_{t+}$-measurable, from (1) we know that

$$\mathbb{E}[\eta \mathbf{1}_{A\cap B}] = \mathbb{E}[\eta \mathbf{1}_A]\mathbb{P}(B).$$

But $\mathbb{E}[\eta \mathbf{1}_A] = 0$ for $A \in \mathcal{G}^X_t$ by the definition of conditional expectation. Therefore, $\mathbb{E}[\eta \mathbf{1}_{A\cap B}] = 0$. This implies that $\eta = 0$.

Finally, we show that $\mathcal{F}^X_t$ is right continuous. Let $u_n \downarrow t$. Then $\mathcal{F}^X_{t+} = \cap_{n=1}^\infty \sigma(\mathcal{G}_{u_n}, \mathcal{N})$. Since we have shown that $\sigma(\mathcal{G}^X_{t+}, \mathcal{N}) = \sigma(\mathcal{G}^X_t, \mathcal{N})$, it suffices to show that $\cap_{n=1}^\infty \sigma(\mathcal{G}^X_{u_n}, \mathcal{N}) = \sigma(\mathcal{G}^X_{t+}, \mathcal{N})$. The argument here is a standard argument in measure theory when we construct the completion of a measure space.

The key point is the following general fact: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra, and let $\mathcal{N}$ be the set of $\mathbb{P}$-null sets, then $F \in \sigma(\mathcal{G}, \mathcal{N})$ if and only if there exists some $G \in \mathcal{G}$, such that $F \Delta G \triangleq (F \backslash G) \cup (G \backslash F) \in \mathcal{N}$. This fact can be easily shown by proving that the set of F satisfying the latter property is a σ-algebra.

Coming back to our assertion, let $F \in \cap_{n=1}^\infty \sigma(\mathcal{G}^X_{u_n}, \mathcal{N})$. Then for every $n \geqslant 1$, there exists $G_n \in \mathcal{G}^X_{u_n}$ such that $F \Delta G_n \in \mathcal{N}$. Define $G = \cap_{n=1}^\infty \cup_{m=n}^\infty G_m$. Then it is not hard to see that $G \in \mathcal{G}^X_{t+}$. Moreover,

$$F \backslash G \subseteq \bigcup_{n=1}^\infty F \backslash G_n \in \mathcal{N}, \quad G \backslash F \subseteq \bigcup_{n=1}^\infty G_n \backslash F \in \mathcal{N}.$$

Therefore, $F \Delta G \in \mathcal{N}$, which implies that $F \in \sigma(\mathcal{G}^X_{t+}, \mathcal{N})$. Hence $\cap_{n=1}^\infty \sigma(\mathcal{G}^X_{u_n}, \mathcal{N}) \subseteq \sigma(\mathcal{G}^X_{t+}, \mathcal{N})$. The other direction is trivial.

Problem 6. This is a hard problem although the assertion is so natural to expect.

One direction is easy. Since X is $\{\mathcal{F}_t^X\}$-adapted and continuous, from Proposition 2.2 we know that it is progressively measurable. It follows from Proposition 2.6 that for every $t \geqslant 0$, $X_{\tau\wedge t}$ is $\mathcal{F}^X_{\tau\wedge t}$-measurable, and is thus $\mathcal{F}_\tau$-measurable. Therefore, $\sigma(X_{\tau\wedge t}: \ t \geqslant 0) \subseteq \mathcal{F}_\tau^X$.

The other direction is hard. It requires a good microscopic intuition on filtrations and stopping times. We do it step by step.

We always interpret a particular sample point $w \in \Omega$ as doing a particular experiment.

We first take a more careful look at natural filtrations.

Let $t \geqslant 0$. An event $A \in \mathcal{F}_t^X$ means that the occurrence of A can be determined by an observation of the trajectory of X up to time t. Therefore, if we consider two experiments $w, w' \in \Omega$ in which w triggers A (i.e. $w \in A$), and if we assume that both experiments lead to the same observation of trajectory up to time t (i.e. the trajectory up to time t corresponding to the experiment w is exactly the same as the one corresponding to w'), then we should conclude that w' triggers A as well ($w' \in A$). The starting point of this problem is to understand this philosophy in a mathematical way. Here is the way to write it down precisely. Note that we are considering the coordinate process $X_t(w) = w_t$.

Proposition 1: Let $\mathcal{G}$ be the set of $A \in \mathcal{F}$ which satisfies the following property: for any $w, w' \in \Omega$, if $w \in A$ and $w_s = w'_s$ for all $s \in [0, t]$, then $w' \in A$. Then $\mathcal{G} = \mathcal{F}_t^X$.

Proof. From definition it is apparent that $\mathcal{G}$ is a σ-algebra and X_s is $\mathcal{G}$-measurable for every $s \in [0, t]$. Therefore, $\mathcal{F}_t^X \subseteq \mathcal{G}$.

Conversely, let $A \in \mathcal{G}$. Since $A \in \mathcal{F} = \mathcal{B}(W^d)$, from general properties of product σ-algebras over an arbitrary index set, we know that A has the form

$$A = \{w \in W^d : \ (w_{t_1}, w_{t_2}, \cdots) \in \Gamma\}$$

for some countable sequence $t_n \in [0, \infty)$ and $\Gamma \in \Pi_1^\infty \mathcal{B}(\mathbb{R}^d)$. Moreover, for every $w \in \Omega$ we know that the path $w_s^t \triangleq w_{t\wedge s}$ ($s \geqslant 0$) coincides with w on $[0, t]$. Therefore, from the definition of $\mathcal{G}$, we conclude that for every $w \in \Omega$, $w \in A$ if and only if $w^t \in A$. In other words,

$$A = \{w \in W^d : \ (w_{t\wedge t_1}, w_{t\wedge t_2}, \cdots) \in \Gamma\}.$$

But $\{w \in W^d : \ (w_{t\wedge t_1}, w_{t\wedge t_2}, \cdots) \in \Gamma\} \in \mathcal{F}_t^X$ since $\mathcal{F}_t^X = \sigma(X_{t\wedge s} : \ s \geqslant 0)$. Therefore, $A \in \mathcal{F}_t^X$. **Q.E.D.**

To extend Proposition 1 to the case where $t = \tau$, we need a more careful look at stopping times.

If τ is a stopping time, then we know that for every $t \geqslant 0$, the occurrent of the event $\{\tau = t\}$ can be determined by an observation of the trajectory of X up to time t. Let $w \in \Omega$ be an experiment and think of $\tau(\omega)$ is a deterministic number. It follows that the occurrence of the event $\{w' \in \Omega : \ \tau(w') = \tau(w)\}$ is determined by an observation of trajectory up to time $\tau(w)$. Now suppose that $w' \in \Omega$ is another experiment such that $w' = w$ on $[0, \tau(w)]$. This implies that w and w' give the same observation of trajectory up to time $\tau(w)$. Therefore, they should both trigger $\{\tau = \tau(w)\}$ or both not trigger it. But w triggers this event since $\tau(w) = \tau(w)$ trivially, therefore w' should also trigger this event (this is essentially the philosophy of the previous Proposition 1). In other words, we should have $\tau(w') = \tau(w)$. The way of making this philosophy precise is the following.

Proposition 2. Let $\tau : \ \Omega \to [0, \infty]$ be an $\mathcal{F}$-measurable map. Then τ is an $\{\mathcal{F}_t^X\}$-stopping time if and only if the following property holds: for any $w, w' \in \Omega$ with $w = w'$ on $[0, \tau(w)] \cap [0, \infty)$, we have $\tau(w') = \tau(w)$.

Proof. Necessity. Suppose that τ is an $\{\mathcal{F}_t^X\}$-stopping time. Let w, w' be such that $w = w'$ on $[0, \tau(w)] \cap [0, \infty)$. If $\tau(w) = \infty$, then $w = w'$ and thus $\tau(w') = \tau(w) = \infty$. Therefore, we may assume that $\tau(w) < \infty$. In this case, we know that $A \triangleq \{\tau = \tau(w)\} \in \mathcal{F}_{\tau(w)}^X$. Since $w \in A$, according to Proposition 1, we know that $w' \in A$. Therefore, $\tau(w') = \tau(w)$.

Sufficiency. Suppose that τ satisfies the assumed property. We are going to use Proposition 1 to show that $\{\tau \leqslant t\} \in \mathcal{F}_t^X$ for every given $t \geqslant 0$. Indeed, let $w \in \{\tau \leqslant t\}$ so that $\tau(w) \leqslant t$ and let $w' \in \Omega$ be such that $w = w'$ on $[0, t]$. This particularly implies that $w = w'$ on $[0, \tau(w)] \cap [0, \infty)$. Therefore, by assumption we have $\tau(w') = \tau(w) \leqslant t$. From Proposition 1, we know that $\{\tau \leqslant t\} \in \mathcal{F}_t^X$. **Q.E.D.**

Now we are able to generalize Proposition 1 to the stopping time case. The underlying philosophy is of course the same.

Proposition 3. Let τ be an $\{\mathcal{F}_t^X\}$-stopping time. Let $\mathcal{H}$ be the set of $A \in \mathcal{F}$ which satisfies the following property: for any $w, w' \in \Omega$, if $w \in A$ and $w = w'$ for all $[0, \tau(w)] \cap [0, \infty)$, then $w' \in A$. Then $\mathcal{H} = \sigma(X_t^\tau : \ t \geqslant 0)$.

Proof. Keeping Proposition 2 in mind, the proof is exactly the same as the proof of Proposition 1. **Q.E.D.**

The next thing is to characterize $\mathcal{F}_\tau^X$ in a similar way. For $w \in \Omega$, define $w_t^\tau = w_{\tau \wedge t}$ $(t \geqslant 0)$. Then $w = w^\tau$ on $[0, \tau(w)] \cap [0, \infty)$ and $\tau(w) = \tau(w^\tau)$. Therefore, if w triggers A, then w^τ should also trigger A.

Proposition 4. Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_\tau^X$ if and only if for every $w \in \Omega$, $w \in A \Longleftrightarrow w^\tau \in A$.

Proof. Necessity. Suppose that $A \in \mathcal{F}_\tau^X$. For $w \in \Omega$, if $\tau(w) = \infty$, then

$w = w^\tau$, in which case the claim is trivial. Therefore, we may assume that $\tau(w) < \infty$. In this case we have $A \cap \{\tau \leqslant \tau(w)\} \in \mathcal{F}^X_{\tau(w)}$. If $w \in A$, then $w \in A \cap \{\tau \leqslant \tau(w)\}$. But $w = w^\tau$ on $[0, \tau(w)]$. By Proposition 1, we conclude that $w^\tau \in A \cap \{\tau \leqslant \tau(w)\} \subseteq A$. Conversely, if $w^\tau \in A$, since $\tau(w) = \tau(w^\tau)$ by Proposition 2, we know that $w^\tau \in A \cap \{\tau \leqslant \tau(w)\}$. It follows from Proposition 1 that $w \in A \cap \{\tau \leqslant \tau(w)\} \subseteq A$.

Sufficiency. Suppose that $A \in \mathcal{F}$ satisfies the assumed property. For given $t \geqslant 0$, we want to show that $A \cap \{\tau \leqslant t\} \in \mathcal{F}^X_t$. Let $w \in A \cap \{\tau \leqslant t\}$ and $w' = w$ on $[0, t]$. This implies that $\tau(w) \leqslant t$ and $w = w'$ on $[0, \tau(w)]$. Since $w \in A$, by assumption, we conclude that $w^\tau = (w')^\tau \in A$, which implies that $w' \in A$. Of course we also have $\tau(w) = \tau(w')$ by Proposition 2. Therefore, $w' \in A \cap \{\tau \leqslant t\}$. It follows from Proposition 1 that $A \cap \{\tau \leqslant t\} \in \mathcal{F}^X_t$. **Q.E.D.**

Now we are able to complete the proof of our main claim.

Proof of "$\mathcal{F}^X_\tau \subseteq \sigma(X^\tau_t : t \geqslant 0)$". Let $A \in \mathcal{F}^X_\tau$. Since $A \in \mathcal{F}$, by Proposition 3, it suffices to show that for given w, w', if $w \in A$ and $w = w'$ on $[0, \tau(w)] \cap [0, \infty)$, then $w' \in A$. Indeed, we only need to consider the case when $\tau(w) < \infty$. In this case we have $w^\tau = (w')^\tau$. Since $A \in \mathcal{F}^X_\tau$, by Proposition 4 we know that $w^\tau = (w')^\tau \in A$, which further implies that $w' \in A$. **Q.E.D.**