# Final Problem Set

Due 12/15 Friday

*There are two main problems in this assignment. Each problem contains several sub-questions which are closely related and pointing to a main central motivating question. The difficulty of these questions is comparable to homework problems. You are encouraged to collaborate for discussion, to use all resources that you find helpful or to ask myself for hints at any time. But solutions must be written down by yourself and reflect your own understanding on the subject to some extent.*

*You must submit your solutions before 12/15 Friday. You can either submit a handwritten version or a PDF version. Please be aware that if you would like to submit your work after 12/08 Friday (last class of semester), you can only email me your solutions as I will be away from Pittsburgh.*

*Enjoy the problems!*

**Problem 1.** In Itô's formula, we know that the chain rule comes with a second order quadratic term which makes Itô's calculus quite different from ordinary calculus. In this problem, we explore another kind of stochastic calculus which is consistent with ordinary calculus, and try to understand its connection with Itô's calculus.

(1) Let $X_t, Y_t$ be two continuous semimartingales. Given $t \geqslant 0$, let $\mathcal{P}_n$ be a sequence of finite partitions over $[0,t]$ such that $\mathrm{mesh}(\mathcal{P}_n) \to 0$. Show that the limit

$$\int_0^t X_s \circ dY_s \triangleq \lim_{n\to\infty} \sum_{t_i \in \mathcal{P}_n} \frac{X_{t_{i-1}} + X_{t_i}}{2} \cdot \big(Y_{t_i} - Y_{t_{i-1}}\big)$$

exists in probability, and the process $t \mapsto \int_0^t X_s \circ dY_s$ is a modification of a continuous semimartingale. In addition, for $f \in C^3(\mathbb{R}^1)$, show that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.$$

(2) Recall that a vector field on $\mathbb{R}^n$

$$\mathbb{R}^n \ni x \mapsto W(x) = \begin{pmatrix} W^1(x) \\ \vdots \\ W^n(x) \end{pmatrix}$$

can be regarded as a differential operator on $C^1$-functions by taking directional derivatives:

$$(Wf)(x) \triangleq \sum_{i=1}^n W^i(x) \frac{\partial f}{\partial x^i}, \quad f \in C^1(\mathbb{R}^n).$$

Let $B_t = (B_t^1, \cdots, B_t^d)$ be a $d$-dimensional Brownian motion, and let $V_1, \cdots, V_d \in C_b^3(\mathbb{R}^n; \mathbb{R}^n)$ be $d$ vector fields on $\mathbb{R}^n$.

(i) Show that on any given set-up $((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}), B_t)$, there exists a unique continuous semimartingale $X_t$ satisfying the SDE

$$dX_t = \sum_{\alpha=1}^d V_\alpha(X_t) \circ dB_t^\alpha, \quad t \geqslant 0,$$

with given initial condition $X_0 = x \in \mathbb{R}^n$, in the sense that

$$X_t = x + \sum_{\alpha=1}^d \int_0^t V_\alpha(X_s) \circ dB_s^\alpha.$$

In addition, for each $f \in C_b^3(\mathbb{R}^n)$, show that

$$f(X_t) = f(x) + \sum_{\alpha=1}^{d} \int_0^t (V_\alpha f)(X_s) \circ dB_s^\alpha.$$

(ii) There are two natural ways to construct approximations for an SDE associated with the vector fields $V_1, \cdots, V_d$. Let $\mathcal{P}_n = \{k/2^n : \ 0 \leqslant k \leqslant 2^n\}$ be the standard sequence of dyadic partitions over $[0, 1]$.

(a) For each $n \geqslant 1$, define the process $\left\{X_t^{(n)} : 0 \leqslant t \leqslant 1\right\}$ by $X_0^{(n)} = x$ and

$$X_t^{(n)} = X_{\frac{k-1}{2^n}}^{(n)} + \sum_{\alpha=1}^{d} V_\alpha \left(X_{\frac{k-1}{2^n}}^{(n)}\right) \cdot \left(B_t - B_{\frac{k-1}{2^n}}\right), \quad \frac{k-1}{2^n} \leqslant t \leqslant \frac{k}{2^n},$$

for $1 \leqslant k \leqslant 2^n$.

(b) For each $n \geqslant 1$, define the process $\left\{B_t^{(n)} : 0 \leqslant t \leqslant 1\right\}$ to be the piecewise linear interpolation of $\{B_t : 0 \leqslant t \leqslant 1\}$ over $\mathcal{P}_n$, i.e. $B_{k/2^n}^{(n)} = B_{k/2^n}$ for each $k$ and $B_t^{(n)}$ is linear on each sub-interval $[(k-1)/2^n, k/2^n]$. For every given $\omega$, since $t \mapsto B_t^{(n)}(\omega)$ is a piecewise linear path, we know from classical ODE theory that there exists a unique solution $t \mapsto Y_t^{(n)}(\omega)$ to the ODE

$$\begin{cases} dy_t = \sum_{\alpha=1}^d V_\alpha(y_t) dB_t^{(n)}(\omega), & 0 \leqslant t \leqslant 1, \\ y_0 = x. \end{cases}$$

This defines a process $\left\{Y_t^{(n)} : \ 0 \leqslant t \leqslant 1\right\}$ pathwisely.

Let $X_t$ be the solution to the SDE

$$\begin{cases} dX_t = \sum_{\alpha=1}^d V_\alpha(X_t) dB_t^\alpha, & 0 \leqslant t \leqslant 1, \\ X_0 = x, \end{cases}$$

and let $Y_t$ be the solution to the SDE

$$\begin{cases} dY_t = \sum_{\alpha=1}^d V_\alpha(Y_t) \circ dB_t^\alpha, & 0 \leqslant t \leqslant 1, \\ Y_0 = x. \end{cases}$$

Show that

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant 1} \mathbb{E}\left[\left\|X_t^{(n)} - X_t\right\|^2\right] = 0$$

while

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant 1}\mathbb{E}\left[\left\|Y_t^{(n)}-Y_t\right\|^2\right]=0.$$

(4) Consider the unit sphere $S^2=\{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2+z^2=1\}$ in $\mathbb{R}^3$.

(i) Given a family $\{V_1,\cdots,V_d\}$ of $C_b^3$-vector fields on $\mathbb{R}^3$, let $X_t$ be the solution to the SDE

$$\begin{cases} dX_t=\sum_{\alpha=1}^d V_\alpha(X_t)\circ dB_t^\alpha, & t\geqslant 0,\\ X_0=\xi\in S^2. \end{cases}$$

Suppose that for each $1\leqslant\alpha\leqslant d$, $V_\alpha$ is tangent to the sphere $S^2$ at every point on $S^2$. Show that with probability one, $X_t\in S^2$ for all $t\geqslant 0$. Moreover, the solution depends only on the restriction of $V_1,\cdots,V_d$ on $S^2$.

(ii) Let $\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3\}$ be the standard vector basis of $\mathbb{R}^3$. For $\alpha=1,2,3$, let $\xi\mapsto W_\alpha(\xi)$ be the vector field on $S^2$ defined by setting $W_\alpha(\xi)$ to be the orthogonal projection of $\mathrm{e}_\alpha$ onto the tangent plane of $S^2$ at $\xi$. Given $\xi\in S^2$, let $X_t^\xi$ be the solution to the SDE

$$dX_t=\sum_{\alpha=1}^3 W_\alpha(X_t)\circ dB_t^\alpha \qquad (1.1)$$

with initial condition $X_0^\xi=\xi\in S^2$. From Part (i) we know that $X_t^\xi$ is well-defined and lives on $S^2$.

On the other hand, recall that the spherical Laplacian $\Delta_{S^2}$ is given by

$$(\Delta_{S^2}f)(\xi)\triangleq\left(\Delta\widehat{f}\right)(\xi),\quad f\in C^2(S^2),$$

where $\widehat{f}:\ \mathbb{R}^3\backslash\{0\}\to\mathbb{R}^1$ is defined by

$$\widehat{f}(x)\triangleq f\left(\frac{\xi}{\|\xi\|}\right),\quad \xi=(x,y,z)\in\mathbb{R}^3\backslash\{0\},$$

and $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the standard Laplacian on $\mathbb{R}^3$.

Show that the generator of $X_t^\xi$ is $\frac{1}{2}\Delta_{S^2}$ in the sense that

$$\lim_{t\to 0}\frac{\mathbb{E}\left[f(X_t^\xi)\right]-f(\xi)}{t}=\frac{1}{2}\left(\Delta_{S^2}f\right)(\xi),\quad \forall f\in C^\infty(S^2).$$

(iii) Denote S (respectively, N) as the south pole (respectively, the north pole) of the unit sphere $S^2$. Let $X_t^{\mathrm{S}}$ be the unique solution to the SDE (1.1) starting at

$X_0^{\mathrm{S}} = \mathrm{S}$. What is the probability that the process $X_t^{\mathrm{S}}$ hits N in finite time?

*The prerequisites for solving this problem are Section 5 (stochastic integration) and Section 6.1 (Itô's theory of stochastic differential equations).*

**Problem 2.** Let $b: \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field on $\mathbb{R}^d$ with bounded derivatives of all orders. Consider the diffusion

$$dX_t = dB_t + b(X_t)dt \qquad (2.1)$$

with initial condition $X_0 = 0$, where $B_t$ is a standard $d$-dimensional Brownian motion. Let $\phi: [0,1] \to \mathbb{R}^d$ be a twice continuously differentiable path such that $\phi_0 = 0$. In this problem, we would like to study the asymptotic behavior of the probability

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |X_t - \phi_t| < \varepsilon\right)$$

as $\varepsilon \to 0$.

(1) We first consider the case when $b = 0$ and $\phi = 0$.

Let $D \triangleq \{x \in \mathbb{R}^d : |x| < 1\}$. It is a classical result in PDE theory that there exist an increasing sequence

$$0 < \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \uparrow \infty$$

of real numbers and an orthonormal basis $\{\phi_n : n \geqslant 0\} \subseteq C_b^\infty(D)$ of $L^2(D)$, such that

$$\begin{cases} -\Delta\phi_n = \lambda_n \phi_n, \\ \lim_{|x|\uparrow 1} \phi_n(x) = 0, \end{cases}$$

for each $n$. In addition, $\phi_0 > 0$ everywhere in $D$, and there exists $N = N(d)$ such that the series

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^N} \phi_n(x)\phi_n(y)$$

converges absolutely and uniformly on $\overline{D} \times \overline{D}$.

(i) For each $f \in C(\overline{D})$ vanishing at the boundary $\partial D$, verify that the function

$$u(t,x) \triangleq \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}\lambda_n t} \langle f, \phi_n \rangle_{L^2(D)} \phi_n(x)$$

is the solution to the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0, & (t,x) \in (0,\infty) \times D, \\ u(0,x) = f(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,\infty) \times \partial D. \end{cases}$$

(ii) Let

$$\tau_x \triangleq \inf\{t \geqslant 0 : \ |x + B_t| \geqslant 1\}.$$

Under the same assumptions as in Part (i), show that

$$u(t,x) = \mathbb{E}\left[f(x+B_t); \tau_x > t\right].$$

(iii) Show that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |B_t| < \varepsilon\right) \sim C\mathrm{e}^{-\lambda_0/\varepsilon^2}$$

as $\varepsilon \to 0$, where

$$C \triangleq \phi_0(0) \cdot \int_D \phi_0(x)dx.$$

As notation we write $\varphi(\varepsilon) \sim \psi(\varepsilon)$ if $\lim_{\varepsilon\to 0} \frac{\varphi(\varepsilon)}{\psi(\varepsilon)} = 1$.

(2) Now we move to the case with general $b$ and $\phi$ satisfying the conditions stated at the beginning.

(i) The following martingale inequality is in general very useful. Let $M \in \mathcal{M}_0^{\mathrm{loc}}$ and let $\tau$ be a stopping time. Show that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant \tau} |M_t| \geqslant x, \ \langle M\rangle_\tau \leqslant y\right) \leqslant \mathrm{e}^{-\frac{x^2}{2y}}, \quad \forall x, y > 0.$$

(ii) Define $\psi_t \triangleq \phi_t - \int_0^t b(\phi_s)ds$. Let $\mathbb{Q}_\psi$ be the probability measure defined by

$$\frac{d\mathbb{Q}_\psi}{d\mathbb{P}} \triangleq \exp\left(\int_0^1 \langle \dot{\psi}_t, dB_t\rangle - \frac{1}{2}\int_0^1 |\dot{\psi}_t|^2 dt\right),$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$. By using Girsanov's theorem, show that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |X_t - \phi_t| < \varepsilon\right) \sim \exp\left(-\frac{1}{2}\int_0^1 |\dot{\psi}_t|^2 dt\right) \cdot \mathbb{Q}_\psi\left(\sup_{0\leqslant t\leqslant 1} |X_t - \phi_t| < \varepsilon\right)$$

as $\varepsilon \to 0$.

(iii) Define $\widetilde{b}(t,x) \triangleq b(x+\phi_t) - b(\phi_t)$. Let $Y_t \triangleq X_t - \phi_t$. Show that under $\mathbb{Q}_\psi$, $Y_t$ is the solution to the SDE

$$\begin{cases} dY_t = d\widetilde{B}_t + \widetilde{b}(t, Y_t)dt, & 0 \leqslant t \leqslant 1, \\ Y_0 = 0, \end{cases}$$

where $\widetilde{B}_t$ is some Brownian motion under $\mathbb{Q}_\psi$.

(iv) By using Girsanov's theorem again, show that

$$\begin{aligned}&\mathbb{Q}_\psi\left(\sup_{0\leqslant t\leqslant 1}|X_t-\phi_t|<\varepsilon\right)\\&=\mathbb{E}\left[\exp\left(\int_0^1\langle\widetilde{b}(t,\beta_t),d\beta_t\rangle-\frac{1}{2}\int_0^1|\widetilde{b}(t,\beta_t)|^2dt\right);\ \sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right],\end{aligned}$$

where $\beta_t$ is a standard one dimensional Brownian motion.

(v) By using integration by parts, show that

$$\begin{aligned}&\mathbb{Q}_\psi\left(\sup_{0\leqslant t\leqslant 1}|X_t-\phi_t|<\varepsilon\right)\\&\sim\exp\left(\int_0^1\mathrm{div}b(\phi_t)dt\right)\cdot\mathbb{E}\left[\exp\left(\sum_{i,j=1}^d\int_0^1\beta_t^i\partial_jb^i(\beta_t+\phi_t)d\beta_t^j\right);\ \sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right]\end{aligned}$$

as $\varepsilon\to 0$, where $\mathrm{div}b\triangleq\sum_{i=1}^d\partial_ib^i$ is the divergence of the vector field $b$.

(vi) This part is devoted to studying the quantity

$$\mathbb{E}\left[\exp\left(\sum_{i,j=1}^d\int_0^1\beta_t^i\partial_jb^i(\beta_t+\phi_t)d\beta_t^j\right)\middle|\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right].$$

(vi–i) Let $W_t=(W_t^1,W_t^2)$ be a standard two dimensional Brownian motion. Define

$$L_t\triangleq\frac{1}{2}\int_0^tW_s^1dW_s^2-W_s^2dW_s^1.$$

Show that there exists a one dimensional Brownian motion $\gamma_t$ which is independent of the process $\{|W_t|:t\geqslant 0\}$, such that $L_t=\gamma_{A_t}$, where

$$A_t\triangleq\frac{1}{4}\int_0^t|W_s|^2ds.$$

(vi–ii) For $1\leqslant i\neq j\leqslant d$, set

$$\xi_t^{ij}\triangleq\int_0^t\beta_s^i\circ d\beta_s^j,$$

where the integral is defined in the sense of Problem 1 (1). By using the result of (vi–i), show that

$$\lim_{\lambda\to\infty}\sup_{0<\varepsilon<1}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}|\xi_t^{ij}|>\lambda\varepsilon\middle|\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right)=0.$$

(vi–iii) By using the result of (vi-ii) and integration by parts, show that

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\left|\sum_{i,j=1}^{d}\int_0^1 \partial_j b^i(\beta_t+\phi_t)d\xi_t^{ij}\right|>\alpha\,\middle|\,\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right)=0$$

for all $\alpha>0$. In addition, for every $\lambda>0$, show that

$$\sup_{0<\varepsilon<1}\mathbb{E}\left[\exp\left(\lambda\sum_{i,j=1}^{d}\int_0^1 \partial_j b^i(\beta_t+\phi_t)d\xi_t^{ij}\right)\middle|\,\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right]<\infty.$$

Combining these two results, conclude that

$$\mathbb{E}\left[\exp\left(\sum_{i,j=1}^{d}\int_0^1 \partial_j b^i(\beta_t+\phi_t)d\xi_t^{ij}\right)\middle|\,\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right]=1.$$

(vi–iv) By using the result of (vi–iii), show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\exp\left(\sum_{i,j=1}^{d}\int_0^1 \beta_t^i\partial_j b^i(\beta_t+\phi_t)d\beta_t^j\right)\middle|\,\sup_{0\leqslant t\leqslant 1}|\beta_t|<\varepsilon\right]$$
$$=\exp\left(-\frac{1}{2}\int_0^1 \mathrm{div}b(\phi_t)dt\right).$$

(vii) Combining all the results obtained so far, conclude that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}|X_t-\phi_t|<\varepsilon\right)\sim C\exp\left(-\frac{1}{2}\int_0^1\left(|\dot{\psi}_t|^2-\mathrm{div}b(\phi_t)\right)dt\right)\mathrm{e}^{-\lambda_0/\varepsilon^2}$$

as $\varepsilon\to 0$, where $C$ and $\lambda_0$ are the constants appearing in Part (1) (iii) for the Brownian motion case.

(viii) Now assume that dimension $d=1$. Fix $x\neq 0\in\mathbb{R}^1$. Let $\mathcal{I}_{0,x}$ be the set of twice continuously differentiable paths $\phi:\ [0,1]\to\mathbb{R}^1$ such that $\phi_0=0$ and $\phi_1=1$. Observe that in the result of (vii), the constants $C,\lambda_0$ are independent of $\phi$. Therefore, the path $\phi\in\mathcal{I}_{0,x}$ which maximizes the probability

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}|X_t-\phi_t|<\varepsilon\right)$$

in the asymptotics $\varepsilon\to 0$ corresponds to the path which minimizes the functional

$$J(\phi)\triangleq\int_0^1\left(|\dot{\psi}_t|^2-\mathrm{div}b(\phi_t)\right)dt,\quad \phi\in\mathcal{I}_{0,x}.$$

If $\phi$ is such a minimizer, we can imagine intuitively that the trajectory of $X_t$ has better chance of "concentrating" around $\phi$ than around any other paths in $\mathcal{I}_{0,x}$.

(viii–i) Suppose that $\phi \in \mathcal{I}_{0,x}$ is a minimizer of the functional $J$. Establish a second order ODE for $\phi$.

(viii-ii) For both of the cases when $b = 0$ and $b(y) = \alpha y$ ($\alpha \neq 0$), identify the minimizer $\phi \in \mathcal{I}_{0,x}$ of the functional $J$ explicitly.

*Part (1) (ii) requires some ideas from Section 6.8 (Itô's diffusion processes and partial differential equations). If you do not want to jump ahead too much, you can leave this question to the end. Assuming the correctness of this result, all other questions can be done based on knowledge up to Section 5.8 (the Cameron-Martin-Girsanov transformation).*