Course Plan for 21880 Stochastic Calculus

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1 Course description

Roughly speaking, in a classical sense, stochastic calculus means the calculus for Brownian motion, or more generally, the calculus for semimartingales. A central motivation of stochastic calculus is the following. In classical analysis, the ordinary differential equation

$$dx_t = b(x_t)dt \tag{1}$$

describes the time evolution of an n-dimensional deterministic nonlinear system, where b is a vector field on $\mathbb{R}^n$. However, it is generally interesting and important to consider the situation where an additional random perturbation is presented in the description of system (1). Formally the resulting system can be written in matrix form as

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \tag{2}$$

where $\sigma = (\sigma_1, \cdots, \sigma_d)$ is a family of d vector fields on $\mathbb{R}^n$, and dB_t represents some kind of d-dimensional random perturbation along every direction σ_i.

From an analytic point of view, the probability distribution of the random system (2) gives the fundamental solution (in the generalized sense) to the parabolic PDE

$$\frac{\partial u}{\partial t} - \mathcal{A}^* u = 0,$$

where $\mathcal{A}^*$ is the formal adjoint of the second order differential operator

$$\mathcal{A} \triangleq \frac{1}{2}\sum_{i,j=1}^{n}(\sigma \cdot \sigma^*)^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i}.$$

Equivalently, solutions to Cauchy problems for the parabolic PDE

$$\frac{\partial u}{\partial t} - \mathcal{A}u = 0$$

admit stochastic representations in terms of the solution to the random system (2). Such way of studying PDEs from a probabilistic point of view proves to be rather useful both on the theoretical and computational sides.

The ultimate goal of this course is to develop a mathematical theory for the study of the random system (2) in the original spirit of K. Itô which was further developed by H. Kunita and S. Watanabe, and to explore many nice consequences of this elegant theory.

We start our journey by a short review on discrete probability theory in a measure-theoretic flavor, followed by the basic notions of continuous time stochastic processes. Here a core concept which is new to us is a mathematical way of describing information up to certain amount of time which can possibly be random. Keeping track of information plays a fundamental role in the theory. After that, we introduce the powerful tool of martingale methods, which leads to an elegant intrinsic construction of stochastic integrals. As a fundamental example in the theory, we then spend some time studying the Brownian motion in a very much martingale flavor. The rest of the course will be devoted to the study of stochastic integration and differential equations. If it has to be one, a core result of the theory will be Itô's formula, and a core technique of our study will be martingale methods.

2 Prerequisites

The main prerequisite for the course is measure-theoretic discrete probability theory. In particular, it is rather important to be comfortable with the conditional expectation and its basic properties. No knowledge on discrete martingales is assumed. The first section of the lecture notes contains essentially all preliminary notions needed for the course. Therefore, it will be very helpful if you could go through the materials carefully and understand the statements without proofs in that section.

3 Course syllabus

3.1 Review of probability theory

- Conditional expectations
- Uniform integrability

- The Borel-Cantelli lemma
- The law of large numbers and the central limit theorem
- Weak convergence of probability measures

3.2 Generalities on continuous time stochastic processes

- Construction of stochastic processes: Kolmogorov's extension theorem
- Kolmogorov's continuity theorem
- Filtrations and stopping times

3.3 Continuous time martingales

- The martingale transform–discrete stochastic integration
- The martingale convergence theorems
- Doob's optional sampling theorems
- Doob's martingale inequalities
- Doob's regularization theorems
- The Doob-Meyer decomposition theorem

3.4 Brownian motion

- Invariance properties of Brownian motion
- The strong Markov property
- The reflection principle
- The Skorokhod embedding theorem
- The Donsker invariance principle
- Passage time distributions: Laplace transforms and densities
- Sample path properties: oscillations, irregularity and the p-variation of Brownian motion

3.5 Stochastic Integration

- Square-integrable martingales
- Construction of stochastic integrals
- Itô's formula
- The Burkholder-Davis-Gundy inequality
- Lévy's characterization of Brownian motion
- Continuous local martingales as time-changed Brownian motions
- Martingale representation theorems
- The Cameron-Martin-Girsanov transformation
- Local times for continuous semimartingales
- Lévy's theorem for Brownian local time

3.6 Stochastic differential equations and diffusion processes

- Itô's theory of stochastic differential equations
- Different notions of solutions and the Yamada-Watanabe theorem
- Existence of weak solutions
- Pathwise uniqueness results
- A comparison theorem for one dimensional SDEs
- Two useful techniques: transformation of drift and time-change
- Examples: linear SDEs and Bessel processes
- Itô's diffusion processes and partial differential equations

4 Homework and grading

4.1 Homework

The lecture notes contain 6 sections, and there is a problem set at the end of each section. These problem sets form an essential complementary component of the subject. Many of the problems, thought challenging, are very useful, enlightening and enjoyable. In particular, the problems that are marked with a "$\star$" will be used in the lectures. Therefore, it is important to have at least a good understanding on these problems if not all.

You are encouraged to work in small groups for solving homework problems. You can always ask myself for hints at any time. Submission of solutions are not required but very welcomed. Standard solutions will be uploaded on the course website in consistence with the pace of the course. In lectures, I will always assume that you have understood the problems marked with a "$\star$".

4.2 Grading

At the beginning of October, a take home exam which contains 2 main problems will be distributed. Solutions are due *December 15th 2017*. You are encouraged to work in small groups for finding reasonable approaches to the problems. However, solutions must be written down by yourself and should reflect your own understanding to some extent. If you are not able to find a complete solution, credits will still be granted for partial attempts and reasonable thoughts.

Final Grade = Attendance (30%) + Take home exam (70%).

5 References

For a review of discrete probability theory:

[1] K.L. Chung, *A course in probability theory*, 3rd edition, Elsevier, 2010.

[2] D. Williams, *Probability with martingales*, Cambridge University Press, 1991.

For stochastic calculus:

[3] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes,* 2nd edition, North-Holland, 1989.

[4] I.A. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus,* 2nd edition, Springer-Verlag,1991.

[5] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd edition, Springer, 2004.

[6] L.C.G. Rogers and D. Williams, *Diffusions, Markov processes and martingales*, Volume 1 and 2, 2nd edition, John Wiley & Sons, 1994.