Solutions for Problem Sheet 6

Problem 1. Exactness all follow from continuity and local Lipschitz property for the coefficients.

(1) We have

$$A(t) = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix},\ a(t) = 0,\ \sigma(t) = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \mathrm{e}^{tA} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dB_s, \quad t \geqslant 0.$$

(2) We have

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix},\ a(t) = \begin{pmatrix} 0 \\ \frac{G(t)}{L} \end{pmatrix},\ \sigma(t) = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \mathrm{e}^{tA} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \frac{G(s)}{L} \end{pmatrix} ds + \int_0^t \mathrm{e}^{(t-s)A} \cdot \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix} dB_s.$$

(3) Let $\tau_0 \triangleq \inf\{t \geqslant 0 :\ X_t = 0\}$ and let e be the explosion time. Define $Y_t \triangleq \ln X_t$. According to Itô's formula, we have

$$dY_t = r(K - X_t)dt + \beta dB_t - \frac{\beta^2}{2}dt, \quad t < \tau_0 \wedge e.$$

Define $Z_t \triangleq \ln X_t + r\int_0^t X_s ds$. It follows that

$$dZ_t = \left(rK - \frac{\beta^2}{2}\right) dt + \beta dB_t.$$

Therefore,

$$\ln\left(\frac{X_t}{x}\right) + r\int_0^t X_s ds = \beta B_t + \left(rK - \frac{\beta^2}{2}\right) t,$$

and hence

$$X_t \cdot \exp\left(r\int_0^t X_s ds\right) = x\exp\left(\beta B_t + \left(rK - \frac{\beta^2}{2}\right) t\right).$$

Integrating with respect to dt, we have

$$\exp\left(r\int_0^t X_s ds\right) - 1 = rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right) s\right) ds.$$

Therefore,

$$\int_0^t X_s ds = \frac{1}{r} \cdot \ln\left(1 + rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right)s\right)ds\right).$$

Differentiating with respect to t, we arrive at

$$X_t = \frac{x\exp\left(\beta B_t + \left(rK - \frac{\beta^2}{2}\right)t\right)}{1 + rx\int_0^t \exp\left(\beta B_s + \left(rK - \frac{\beta^2}{2}\right)s\right)ds}, \quad t < \tau_0 \wedge e. \tag{1}$$

This in particular implies that $\tau_0 = e = \infty$ almost surely, and (1) defines the global solution to the SDE.

Problem 2. (1) Since B_t is a Gaussian process, we know that X_t is also a Gaussian process. The mean function is $m(t) \triangleq \mathbb{E}[X_t] = 0$, and the covariance function is

$$\begin{aligned}
\rho(s,t) &\triangleq \mathbb{E}[X_s X_t] \\
&= \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] \\
&= s \wedge t - st - st + st \\
&= s \wedge t - st \\
&= \begin{cases} s(1-t), & s \leqslant t; \\ t(1-s), & s > t. \end{cases}
\end{aligned}$$

(2) The SDE is a linear SDE with $A(t) = -(1-t)^{-1}$, $a(t) = 0$ and $\sigma(t) = 1$. By the general formula for the solution, we have $\Phi(t) = 1 - t$, and

$$Y_t = (1-t)\int_0^t \frac{dB_s}{1-s}, \quad 0 \leqslant t < 1.$$

Since the integrand is deterministic, it is immediate that Y_t is a Gaussian process. The mean function is zero, and for $s < t < 1$, we have

$$\mathbb{E}[Y_s Y_t] = (1-s)(1-t)\mathbb{E}\left[\int_0^s \frac{du}{(1-u)^2}\right] = s(1-t).$$

In particular, Y_t has the same mean and covariance functions as X_t. Since they are both Gaussian processes, we conclude that

$$(X_t)_{0 \leqslant t < 1} \stackrel{\text{law}}{=} (Y_t)_{0 \leqslant t < 1}.$$

Moreover, since $X_1 = 0$, and by continuity, the probability $\mathbb{P}(\lim_{t\uparrow 1} X_t = 0)$ is determined by the distribution of $(X_t)_{0 \leqslant t < 1}$, therefore we conclude that

$$\mathbb{P}\left(\lim_{t \uparrow 1} Y_t = 0\right) = 1.$$

(3) Let B_t be a one dimensional Brownian motion, and let $S_t \triangleq \sup_{0 \leqslant s \leqslant 1} B_s$. The joint density $f_{(S_1,B_1)}(x,y)$ of (S_1, B_1) is given by

$$\mathbb{P}(S_1 \in dx,\ B_1 \in dy) = \frac{2(2x-y)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(2x-y)^2}{2}} dxdy, \quad x \geqslant 0, x \geqslant y.$$

Now a crucial observation is, the process X_t has the same distribution as the Brownian motion B_t conditioned on $B_1 = 0$. More precisely, for $0 < t_1 < \cdots < t_n < 1$, the joint density of $(X_{t_1}, \cdots, X_{t_n})$ is the same as the conditional density of $(B_{t_1}, \cdots, B_{t_n})$ conditioned on $B_1 = 0$. Therefore, the desired probability is

$$\mathbb{P}\left(S_1 \geqslant x | B_1 = 0\right) = \int_x^\infty \frac{f_{(S_1,B_1)}(z,0)dz}{f_{B_1}(0)} = \int_x^\infty 4z \cdot \mathrm{e}^{-2z^2} dz = \mathrm{e}^{-2x^2}.$$

Problem 3. (1) The coefficients are given by $\sigma(x) = -2|x|^{3/2}$ and $b(x) = 3x^2$. They are continuous and locally Lipschitz. Therefore, the SDE is exact.

(2) The result follows from the comparison theorem, since the unique solution to the SDE

$$\begin{cases} dY_t = -2|Y_t|^{\frac{3}{2}} dB_t, & t \geqslant 0, \\ Y_0 = 0, \end{cases}$$

is the zero solution.

(3) Let $\tau_0 \triangleq \inf\{t \geqslant 0 : \ Y_t = 0\}$, and define $Z_t \triangleq Y_t^{-1/2}$. According to Itô's formula, we conclude that

$$dZ_t = dB_t, \ \ t < \tau_0 \wedge e.$$

Therefore,

$$Y_t = \frac{1}{(1+B_t)^2}, \ \ t < \tau_0 \wedge e.$$

This in particular implies that $\tau_0 = \infty$ almost surely (otherwise we have $1/(1+B_{\tau_0})^2 = 0$ which is absurd). In other words, Y_t never reaches zero and we have

$$Y_t = \frac{1}{(1+B_t)^2}, \ \ t < e.$$

It follows that $e = \inf\{t \geqslant 0 : \ B_t = -1\}$. Therefore,

$$\mathbb{P}(e > t) = 1 - 2\mathbb{P}(B_t \geqslant 1) = 1 - 2\int_1^\infty \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{u^2}{2t}} du.$$

From the density formula for e, it is easy to see that

$$\mathbb{P}(e < \infty) = 1, \ \ \mathbb{E}[e] = \infty.$$

Problem 4. (1) According to Itô's formula, we have

$$d\mathcal{E}_t^G = \mathcal{E}_t^G dG_t.$$

Therefore,

$$\begin{aligned} dZ_t &= \left(\int_0^t (\mathcal{E}_s^G)^{-1} dH_s\right) d\mathcal{E}_t^G + dH_t + d\mathcal{E}_t^G \cdot dH_t \\ &= Z_t dG_t + dH_t + \mathcal{E}_t^G d\langle G, H\rangle_t \\ &= Z_t dG_t + dH_t. \end{aligned}$$

(2)By the comparison theorem (c.f. Theorem 6.11 in the lecture notes), we may assume without loss of generality that $X_0^1 = X_0^2$. Now suppose that b^1 is Lipschitz continuous. Then

$$\begin{aligned}
X_t^2 - X_t^1 &= \int_0^t \left(b^2(X_s^2) - b^1(X_s^1)\right) ds + \int_0^t \left(\sigma(s, X_s^2) - \sigma(s, X_s^1)\right) dB_s \\
&= \int_0^t \left(b^2(X_s^2) - b^1(X_s^2)\right) ds + \int_0^t \left(\sigma(s, X_s^2) - \sigma(s, X_s^1)\right) dB_s \\
&\quad + \int_0^t \left(b^1(X_s^2) - b^1(X_s^1)\right) ds \\
&= H_t + \int_0^t (X_s^2 - X_s^1) dG_s,
\end{aligned}$$

where

$$\begin{aligned}
H_t &\triangleq \int_0^t \left(b^2(X_s^2) - b^1(X_s^2)\right) ds, \\
G_t &\triangleq \int_0^t \mathbf{1}_{\{X_s^1 \neq X_s^2\}} (X_s^2 - X_s^1)^{-1} \left((\sigma(s, X_s^2) - \sigma(s, X_s^1)) dB_s + (b^1(X_s^2) - b^1(X_s^1)) ds\right).
\end{aligned}$$

Define $\mathcal{E}_t^G$ to be the stochastic exponential of G as before, and define

$$Z_t \triangleq \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} dH_s.$$

It follows from the first part that

$$Z_t = H_t + \int_0^t Z_s dG_s.$$

But we have already seen that $X_t^2 - X_t^1$ also satisfies the same (linear) equation. Apparently we have uniqueness in this context. Therefore,

$$Z_t = X_t^2 - X_t^1 = \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} \left(b^2(X_s^2) - b^1(X_s^2)\right) ds.$$

Since $\mathcal{E}_t^G > 0$ for all t, by the assumption $b^2 > b^1$, we conclude that with probability one, $X_t^2 > X_t^1$ for all $t > 0$.

The result does not hold if σ is not Lipschitz continuous. For example, consider the two SDEs

$$dX_t^i = 2\sqrt{|X_t^i|} dB_t + \alpha^i dt,$$

where $\alpha^1 = 0$ and $\alpha^2 = 1$. Suppose that $X_0^1 = X_0^2 = 0$, then $X_t^1 = 0$ and X_t^2 is the square of a one dimensional Brownian motion. Since the Brownian motion visits the origin infinitely often, we see that the result fails in this case.