Solutions for Problem Sheet 4

Problem 1. (1) The only thing which is not entirely trivial is that $O\cdot(B_t-B_s)\sim\mathcal{N}(0,(t-s)\mathrm{Id})$ and $\langle\mu,B_t-B_s\rangle\sim\mathcal{N}(0,1)$. But this can be seen by using characteristic functions. Of course the problem can also be solved simply by applying Lévy's characterization theorem once we notice that $O\cdot B_t=\int_0^t O\cdot dB_s$ and $\langle\mu,B_t\rangle=\int_0^t\langle\mu,dB_s\rangle$.

(2) We first show that $\mathbb{E}[B_s|B_t]=sB_t/t$ for $s<t$. Indeed, consider the time reversal $\widetilde{B}_r\triangleq rB_{1/r}$. From Problem 2, (1), we know that $\widetilde{B}_r$ is a Brownian motion. Let $u=1/s$, $v=1/t$ so that $u>v$. It follows that

$$\mathbb{E}[\widetilde{B}_u|\widetilde{B}_v]=\widetilde{B}_v+\mathbb{E}[\widetilde{B}_u-\widetilde{B}_v|\widetilde{B}_v]=\widetilde{B}_v=B_t/t.$$

But $\widetilde{B}_u=B_s/s$ and conditioning on $\widetilde{B}_v$ is the same as conditioning on B_t. Therefore,

$$\mathbb{E}[B_s|B_t]=\frac{s}{t}B_t.$$

Now for the general case, we have

$$\begin{aligned}\mathbb{E}[B_u|B_s,B_t]&=B_s+\mathbb{E}[B_u-B_s|B_s,B_t]\\&=B_s+\mathbb{E}[B_u-B_s|B_s,B_t-B_s]\\&=B_s+\mathbb{E}[B_u-B_s|B_t-B_s],\end{aligned}$$

where in the last equality, we used the fact that (B_u-B_s,B_t-B_s) and B_s are independent (c.f. Problem Sheet 1, Problem (1), (iii)). Therefore, from what we just proved, we have

$$\mathbb{E}[B_u|B_s,B_t]=B_s+\frac{u-s}{t-s}(B_t-B_s)=\frac{t-u}{t-s}B_s+\frac{u-s}{t-s}B_t.$$

Problem 2. (1) It is easy to see that $(X_t)_{t>0}$ has the right distribution as a Brownian motion, and $t\mapsto X_t$ is continuous for $t>0$. The only fact which is not so clear is the continuity at $t=0$. By the defintion of X_t, this is equivalent to showing that with probability one, $B_t/t\to 0$ as $t\to\infty$. Indeed, from the strong law of large numbers, we know that

$$\lim_{n\to\infty}\frac{B_n}{n}=\lim_{n\to\infty}\frac{\sum_{k=1}^n(B_k-B_{k-1})}{n}=0\ \text{ a.s.}$$

Moreover, since

$$\begin{aligned}\mathbb{P}\left(\sup_{n-1\leqslant t\leqslant n}\frac{|B_t-B_{n-1}|}{n}>\varepsilon\right)&\leqslant\frac{1}{(n\varepsilon)^2}\mathbb{E}\left[\sup_{n-1\leqslant t\leqslant n}|B_t-B_{n-1}|^2\right]\\&\leqslant\frac{4}{(n\varepsilon)^2}\mathbb{E}[|B_n-B_{n-1}|^2]\\&\leqslant\frac{4}{(n\varepsilon)^2},\end{aligned}$$

from the first Borel-Cantelli's lemma, we know that with probability one,

$$\lim_{n\to\infty}\sup_{n-1\leqslant t\leqslant n}\frac{|B_t-B_{n-1}|}{n}=0.$$

Therefore, $B_t/t\to 0$ almost surely as $t\to\infty$.

(2) Since $\{X_t:\ t\geqslant 0\}$ is a Brownian motion, this part follows from Proposition 4.2 in the lecture notes.

(3) Since

$$\frac{X_t}{t}=B_{1/t},\quad t>0,$$

the non-differentiability of X_t at $t=0$ also follows directly from Proposition 4.2. Now we show the almost everywhere non-differentiability of B. For each $t\geqslant 0$, let A_t be the event that B is differentiable at t. Then $\mathbb{P}(A_t)=0$ by applying what we just proved to the Brownian motion $\{B_{u+t}-B_t:\ u\geqslant 0\}$. According to Fubini's theorem, we have

$$\mathbb{E}\left[\int_0^\infty \mathbf{1}_{A_t}dt\right]=\int_0^\infty \mathbb{P}(A_t)dt=0.$$

Therefore, with probability one,

$$\int_0^\infty \mathbf{1}_{A_t}(\omega)dt=0,$$

which implies that $\omega\notin A_t$ for almost every $t\geqslant 0$. This means that $t\mapsto B_t(\omega)$ is almost everywhere non-differentiable.

Problem 3. We only need to consider the case when f is bounded and continuous. The case when f is bounded Borel measurable follows from a monotone class argument. Let

$$\sigma_n\triangleq\sum_{k=1}^\infty\frac{k}{2^n}\mathbf{1}_{\{(k-1)/2^n\leqslant\sigma<k/2^n\}}.$$

Define τ_n similarly. Apparently, σ_n,τ_n are stopping times, and $\sigma_n\downarrow\sigma,\tau_n\downarrow\tau$. Moreover, $\tau_n\in\mathcal{F}_\sigma$ since $\tau\in\mathcal{F}_\sigma$. From the Strong Markov property of Brownian motion, we know that

$$\mathbb{E}[f(B_{\sigma_n+k/2^n})|\mathcal{F}_{\sigma_n}]=P_{k/2^n}f(B_{\sigma_n}).$$

Therefore,

$$\mathbb{E}[f(B_{\sigma_n+k/2^n})|\mathcal{F}_\sigma]=\mathbb{E}[P_{k/2^n}f(B_{\sigma_n})|\mathcal{F}_\sigma]. \tag{1}$$

But we know that $\mathbf{1}_{\{\tau_n=\sigma_n+k/2^n\}}\in\mathcal{F}_\sigma$. By multiplying this function on both sides of (1) and summing over k, we arrive at

$$\mathbb{E}[f(B_{\tau_n})|\mathcal{F}_\sigma]=\mathbb{E}[P_{\tau_n-\sigma_n}f(B_{\sigma_n})|\mathcal{F}_\sigma].$$

By continuity and the dominated convergence theorem, we conclude that

$$\mathbb{E}[f(B_\tau)|\mathcal{F}_\sigma]=\mathbb{E}[P_{\tau-\sigma}f(B_\sigma)|\mathcal{F}_\sigma]=P_{\tau-\sigma}f(B_\sigma).$$

It is not true that $B_\tau-B_\sigma$ and $\mathcal{F}_\sigma$ are independent. Consider the one dimensional case. Let $\sigma\triangleq\inf\{t\geqslant 0:\ B_t=a\}$ for given $a>0$, and let $\tau\triangleq 2\sigma$. Suppose that $B_{2\sigma}-B_\sigma$ and $\mathcal{F}_\sigma$ are independent. Then $B_{2\sigma}$ and $\mathcal{F}_\sigma$ must be independent since $B_\sigma = a$ is a deterministic constant. Therefore, the conditional expectation

$$\mathbb{E}[f(B_{2\sigma})|\mathcal{F}_\sigma] = \mathbb{E}[f(B_{2\sigma})]$$

is a deterministic constant. However, according to what we just proved,

$$\mathbb{E}[f(B_{2\sigma})|\mathcal{F}_\sigma] = P_\sigma f(B_\sigma) = P_\sigma f(a) = \int_{\mathbb{R}^1} \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{(a-y)^2}{2\sigma}} f(y)dy.$$

This cannot be a deterministic constant for a large class of f as σ is random. Therefore, we have a contradiction, which shows that $B_{2\sigma} - B_\sigma$ and $\mathcal{F}_\sigma$ are not independent.

Problem 4. From direct computation, we have

$$D_1 = \begin{cases} 1, & X = 0, 1, 2; \\ -1, & X = -2, -1, \end{cases}$$

and

$$\begin{aligned} X_1 &= \mathbb{E}[X_1|D_1 = 1] \cdot \mathbf{1}_{\{D_1=1\}} + \mathbb{E}[X_1|D_1 = -1] \cdot \mathbf{1}_{\{D_1=-1\}} \\ &= 1 \cdot \mathbf{1}_{\{D_1=1\}} - \frac{3}{2} \cdot \mathbf{1}_{\{D_1=-1\}}. \end{aligned}$$

Now

$$\{D_1 = 1, D_2 = 1\} = \{X = 1, 2\}, \qquad \{D_1 = 1, D_2 = -1\} = \{X = 0\},$$
$$\{D_1 = -1, D_2 = 1\} = \{X = -1\}, \quad \{D_1 = -1, D_2 = -1\} = \{X = -2\}.$$

It follows that

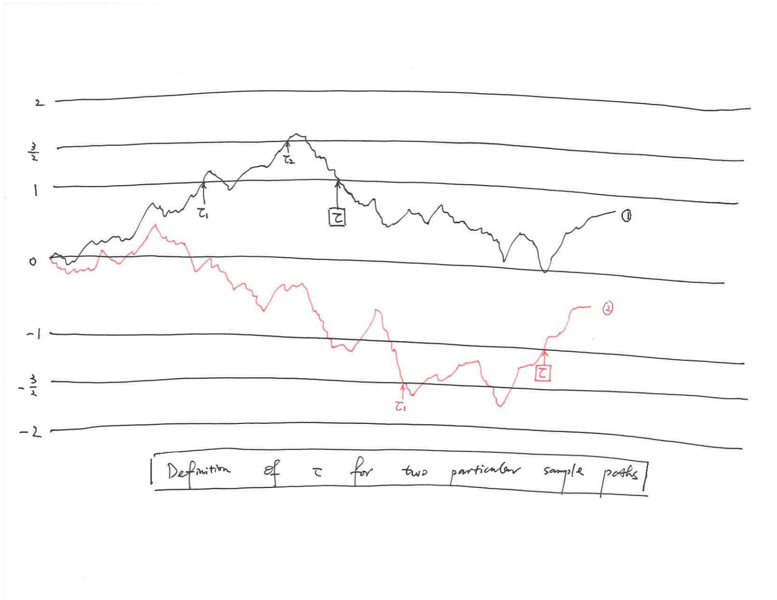
$$\begin{aligned} X_2 &= \frac{3}{2}\mathbf{1}_{\{D_1=1,D_2=1\}} + 0 \cdot \mathbf{1}_{\{D_1=1,D_2=-1\}} + (-1) \cdot \mathbf{1}_{\{D_1=-1,D_2=1\}} + (-2) \cdot \mathbf{1}_{\{D_1=-1,D_2=-1\}} \\ &= \frac{3}{2}\mathbf{1}_{\{X_1=1,D_2=1\}} + 0 \cdot \mathbf{1}_{\{X_1=1,D_2=-1\}} + (-1) \cdot \mathbf{1}_{\{X_1=-3/2,D_2=1\}} \\ &\quad +(-2) \cdot \mathbf{1}_{\{X_1=-3/2,D_2=-1\}}. \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} X_3 &= 2 \cdot \mathbf{1}_{\{D_1=1,D_2=1,D_3=1\}} + 1 \cdot \mathbf{1}_{\{D_1=1,D_2=1,D_3=-1\}} + 0 \cdot \mathbf{1}_{\{D_1=1,D_2=-1\}} \\ &\quad +(-1) \cdot \mathbf{1}_{\{D_1=-1,D_2=1\}} + (-2) \cdot \mathbf{1}_{\{D_1=-1,D_2=-1\}} \\ &\quad 2 \cdot \mathbf{1}_{\{X_1=1,X_2=3/2,D_3=1\}} + 1 \cdot \mathbf{1}_{\{X_1=1,X_2=3/2,D_3=-1\}} + 0 \cdot \mathbf{1}_{\{X_1=1,X_2=0\}} \\ &\quad +(-1) \cdot \mathbf{1}_{\{X_1=-3/2,X_2=-1\}} + (-2) \cdot \mathbf{1}_{\{X_1=-3/2,X_2=-2\}}, \end{aligned}$$

and $X_n = X_3$ for $n \geqslant 3$.

The stopping time τ is defined in the following way. Let τ_1 be the first exit time of the interval $(-3/2, 1)$. Define τ_2 as follows: if $B_{\tau_1} = 1$, then τ_2 is the exit time of the interval $(0, 3/2)$ after τ_1, and if $B_{\tau_1} = -3/2$, then τ_2 is the exist time of the interval $(-2, -1)$. Define τ_3 as follows: if $(B_{\tau_1}, B_{\tau_2}) = (1, 3/2)$, then τ_3 is the exist time of the interval $(1, 2)$ after τ_2, and in all other cases, $\tau_3 \triangleq \tau_2$. The desired stopping time τ will be $\tau \triangleq \tau_3$ (in the proof of the Skorokhod embedding theorem, in this case we have $X_n = X_3$ and $\tau_n = \tau_3$ for $n \geqslant 3$, so $\tau = \tau_3$). See the Figure below for an illustration of the construction of τ.



Problem 5. (1) Write $B_t = B_t^x + iB_t^y$ where B_t^x is a standard Brownian motion and B_t^y is a Brownian motion starting at position 1. Note that B^x and B^y are independent. Therefore,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\lambda i\cdot B_t}|\mathcal{F}_s^B\right] &= \mathbb{E}\left[\mathrm{e}^{\lambda i\cdot(B_t-B_s)}\right]\cdot\mathrm{e}^{\lambda i\cdot B_s}\\ &= \mathbb{E}\left[\mathrm{e}^{i\lambda(B_t^x-B_s^x)-\lambda(B_t^y-B_s^y)}\right]\cdot\mathrm{e}^{\lambda i\cdot B_s}\\ &= \mathrm{e}^{\lambda i\cdot B_s},\end{aligned}$$

which shows that $X_t \triangleq \mathrm{e}^{\lambda i\cdot B_t}$ is an $\{\mathcal{F}_t^B\}$-martingale.

(2) A crucial observation is that $\tau = \inf\{t \geqslant 0 : B_t^y = 0\}$, which is independent of B^x and has density

$$f_\tau(t) = \frac{1}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{1}{2t}},\ t > 0.$$

Now let $\varphi \in B_b(\mathbb{R}^1)$. Then we have

$$\begin{aligned}\mathbb{E}[\varphi(B_\tau)] &= \int_0^\infty \mathbb{E}[\varphi(B_\tau)|\tau = t]f_\tau(t)dt\\ &= \int_0^\infty \mathbb{E}[\varphi(B_t^x)|\tau = t]f_\tau(t)dt\\ &= \int_0^\infty \mathbb{E}[\varphi(B_t^x)]f_\tau(t)dt\\ &= \int_0^\infty \left(\int_{\mathbb{R}^1}\varphi(u)\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{u^2}{2t}}du\right)\frac{1}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{1}{2t}}dt.\end{aligned}$$

By using Fubini's theorem and integrating out t by a change of variables $s = 1/t$, we arrive at

$$\mathbb{E}[\varphi(B_\tau)] = \int_{\mathbb{R}^1} \varphi(u) \frac{1}{\pi(u^2+1)} du.$$

Therefore, B_τ is Cauchy distributed.

Problem 6. (1) Note that under $\mathbb{P}^{x,c}$, the coordinate process is a Brownian motion starting at x with drift c. Therefore, for any $n \geqslant 1$, $t_1 < \cdots < t_n = t$, and $f \in C_b(\mathbb{R}^n)$, we have

$$\begin{aligned}
\int_{W^1} f(w_{t_1}, \cdots, w_{t_n}) d\mathbb{P}^{x,c} &= \int_{W^1} f(w_{t_1} + ct_1, \cdots, w_{t_n} + ct_n) d\mathbb{P}^{x,0} \\
&= \int_{\mathbb{R}^n} f(u_1 + ct_1, \cdots, u_n + ct_n) p_{t_1}(u_1 - x) \\
&\quad \cdot p_{t_2 - t_1}(u_2 - u_1) \cdots p_{t_n - t_{n-1}}(u_n - u_{n-1}) du \\
&= \int_{\mathbb{R}^n} f(v_1, \cdots v_n) p_{t_1}(v_1 - x - ct_1) \cdot p_{t_2 - t_1}(v_2 - v_1 - c(t_2 - t_1)) \\
&\quad \cdots p_{t_n - t_{n-1}}(v_n - v_{n-1} - c(t_n - t_{n-1})) dv \\
&= \int_{\mathbb{R}^n} f(v_1, \cdots, v_n) \mathrm{e}^{c(v_n - x) - \frac{1}{2}c^2 t} \gamma(dv) \\
&= \int_{W^1} f(w_{t_1}, \cdots, w_{t_n}) \mathrm{e}^{c(w_t - x) - \frac{1}{2}c^2 t} d\mathbb{P}^{x,0},
\end{aligned}$$

where

$$p_t(u) \triangleq \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{u^2}{2t}}$$

and $\gamma(dv)$ is the distribution of $(w_{t_1}, \cdots, w_{t_n})$ under $\mathbb{P}^{x,0}$. Therefore, the result follows.

(2) Since (S_t, X_t) is $\mathcal{F}_t$-measurable, for any $f \in C_b(\mathbb{R}^2)$, from (1) we have

$$\mathbb{E}^{0,c}[f(S_t, X_t)] = \mathbb{E}^{0,0}\left[f(S_t, X_t) \mathrm{e}^{cX_t - \frac{1}{2}c^2 t}\right].$$

According to Proposition 4.9 in the lecture notes, this equals

$$\int_{\{x \geqslant 0, x \geqslant y\}} f(x,y) \mathrm{e}^{cy - \frac{1}{2}c^2 t} \frac{2(2x-y)}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(2x-y)^2}{2t}} dx dy.$$

Therefore,

$$\mathbb{P}^{0,c}(S_t \in dx, X_t \in dy) = \frac{2(2x-y)}{\sqrt{2\pi t^3}} \mathrm{e}^{cy - \frac{1}{2}c^2 t - \frac{(2x-y)^2}{2t}}, \quad x \geqslant 0, x \geqslant y.$$

Problem 7. (1) The first part follows from Itô's formula and the boundedness of e_θ. The second part follows from integrating the martingale property of $\mathrm{e}_\theta(B_t)$ against $\phi(\theta)d\theta$. Note that we can integrate because $\|\mathrm{e}_\theta\| \leqslant 1$ and $\phi(\theta)$ is rapidly decreasing.

(2) (i) Trivial.

(ii) Choose $f \in C_c^\infty(\mathbb{R}^d)$ such that on the annulus $A_{a,b} \triangleq \{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$, $f(x) = \log|x|$ for $d = 2$ and $f(x) = |x|^{2-d}$ for $d \geqslant 3$. Since

$$f(B_t) - f(0) - \frac{1}{2}\int_0^t \Delta f(B_s) ds$$

is a bounded martingale and $\Delta f(B_s)=0$ on $[0,\tau_a\wedge\tau_b]$, according to the optional sampling theorem, we have

$$\begin{aligned} f(0) &= \mathbb{E}[f(B_{\tau_a\wedge\tau_b})] \\ &= f(B_{\tau_a})\mathbb{P}_d^x(\tau_a<\tau_b)+f(B_{\tau_b})\left(1-\mathbb{P}_d^x(\tau_a<\tau_b)\right). \end{aligned}$$

By the definition of f on the annulus $A_{a,b}$, we obtain that

$$\mathbb{P}_d^x(\tau_a<\tau_b)=\begin{cases} \frac{\log b-\log|x|}{\log b-\log a}, & d=2; \\ \frac{|x|^{2-d}-b^{2-d}}{a^{2-d}-b^{2-d}}, & d\geqslant 3. \end{cases} \tag{2}$$

Since

$$\{\tau_a<\infty\}=\bigcup_{b>|x|}\{\tau_b>\tau_a\},$$

we also obtain that

$$\mathbb{P}_d^x(\tau_a<\infty)=\lim_{b\to\infty}\mathbb{P}_d^x(\tau_a<\tau_b)=\begin{cases} 1, & d=2; \\ \left(\frac{a}{|x|}\right)^{d-2}, & d\geqslant 3. \end{cases}$$

(iii) We first consider the case when $d=2$. Let $B(x_0,\varepsilon)$ be an open ball contained in U, and take $N\geqslant 1$ such that $U\bigcup\{0\}\subseteq B(x_0,N)$. Define

$$\begin{aligned} \theta_1 &\triangleq \inf\{t\geqslant 0:\ |X_t|=N\},\quad \tau_1\triangleq\inf\{t\geqslant\theta_1:\ |X_t|=\varepsilon\}, \\ \theta_2 &\triangleq \inf\{t\geqslant\tau_1,\ |X_t|=N+1\},\quad \tau_2\triangleq\inf\{t\geqslant\theta_2:\ |X_t|=\varepsilon\}, \\ &\cdots \\ \theta_n &\triangleq \inf\{t\geqslant\tau_{n-1},\ |X_t|=N+n-1\},\quad \tau_n\inf\{t\geqslant\theta_n:\ |X_t|=\varepsilon\}. \end{aligned}$$

Apparently, $\theta_n\uparrow\infty$ and hence $\tau_n\uparrow\infty$. Therefore, it is clear that

$$\bigcap_{n=1}^{\infty}\{\tau_n<\infty\}\subseteq\{\sigma=\infty\}.$$

Moreover, for each n,

$$\mathbb{P}_2^0(\tau_n<\infty)=\mathbb{E}_2^0[\mathbb{P}_2^0(\tau_n<\infty)|\mathcal{F}_{\sigma_n}],$$

and conditioned on $\mathcal{F}_{\sigma_n}$, B_{σ_n+t} is a Brownian motion starting at B_{σ_n}. According to the strong Markov property and (2), (ii), we have

$$\mathbb{P}_2^0(\tau_n<\infty)=1.$$

Therefore,

$$\mathbb{P}_2^0(\sigma=\infty)=1.$$

Now we consider the case when $d\geqslant 3$. Let $B(x_0,r)$ be an open ball such that $U\subseteq B(x_0,r)$. For each $R>r$ with $0\in B(x_0,R)$, define inductively

$$\theta_n\triangleq\inf\{t\geqslant\tau_{n-1},\ |X_t|=R\},\quad \tau_n\triangleq\inf\{t\geqslant\theta_n:\ |X_t|=r\},$$

where $\tau_0 \triangleq 0$. It follows that

$$\{\sigma=\infty\} \subseteq \bigcap_{n=1}^{\infty}\{\tau_n<\infty\}.$$

But in dimension greater than 2, we have

$$\mathbb{P}_d^0(\tau_n<\infty)=\left(\frac{r}{R}\right)^{d-2}.$$

Therefore,

$$\mathbb{P}_d^0(\sigma=\infty)\leqslant\left(\frac{r}{R}\right)^{d-2}.$$

As this is true for all R, we conclude that $\mathbb{P}_d^0(\sigma=\infty)=0$.

(iv) We first consider the case when $y\neq 0$. For each $r<|y|<R$, define

$$\tau_r \triangleq \inf\{t\geqslant 0:\ |X_t|=r\},\quad \tau_R \triangleq \inf\{t\geqslant 0:\ |X_t|=R\}.$$

It follows that

$$\{\sigma_y<\infty\}=\bigcup_{R>|y|}\{\sigma_y<\tau_R\}\subseteq\bigcup_{R>|y|}\left(\bigcap_{r<|y|}\{\tau_r<\tau_R\}\right).$$

Moreover, for each fixed R, in view of the formula (2), we have

$$\mathbb{P}_d^0\left(\bigcap_{r<|y|}\{\tau_r<\tau_R\}\right)=\lim_{r\downarrow 0}\mathbb{P}_d^0(\tau_r<\tau_R)=0,$$

for all $d\geqslant 2$. Therefore,

$$\mathbb{P}_d^0(\sigma_y<\infty)=0$$

for all $d\geqslant 2$.

Now we consider the case when $y=0$. For each $r>0$, define

$$\tau_r \triangleq \inf\{t\geqslant 0:\ |X_t|=r\},\quad \theta_r \triangleq \inf\{t\geqslant \tau_r:\ X_t=0\}.$$

Then we have

$$\{0<\sigma_y<\infty\}\subseteq\bigcup_{r>0}\{\theta_r<\infty\}.$$

But according to the result in the case when $y\neq 0$, we have

$$\mathbb{P}_d^0(\theta_r<\infty)=\mathbb{E}_d^0[\mathbb{P}_d^0(\theta_r<\infty|\mathcal{F}_{\tau_r})]=0.$$

Therefore,

$$\mathbb{P}_d^0(0<\sigma_y<\infty)=0.$$

It remains to show that $\mathbb{P}_d^0(\sigma_y=0)=0$. To this end, first observe that the probability $\mathbb{P}_d^0(\sigma_y=0)$ is determined by the distribution of Brownian motion. Therefore, we may use the Brownian motion $tX_{1/t}$ to compute this probability (so define $\widetilde{\sigma}_y=\inf\{t\geqslant 0:\ tX_{1/t}=y\}$). In this case, we have

$$\{\widetilde{\sigma}_y=0\}=\{\exists t_n\uparrow\infty,\ X_{t_n}=0\}\subseteq\{\theta_r<\infty\},$$

for any fixed $r>0$, where τ_r,θ_r are defined in the same way as before for the process X_t. Therefore,

$$\mathbb{P}_d^0(\sigma_y=0)=\mathbb{P}_d^0(\widetilde{\sigma}_y=0)\leqslant\mathbb{P}_d^0(\theta_r<\infty)=0.$$