Solutions for Problem Sheet 1

Problem 1. (1) (i) We have

$$\mathbb{E}[X\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[\mathbb{E}[X\mathbb{E}[Y|\mathcal{G}]|\mathcal{G}]] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\cdot\mathbb{E}[Y|\mathcal{G}]].$$

Similarly for $\mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]]$.

(ii) We call a bounded measurable function satisfying property $\mathbf{P}$ if

$$\mathbb{E}[f(X,Y)|\mathcal{G}] = \mathbb{E}[f(x,Y)]|_{x=X}.$$

Let $\mathcal{E} = \{E \in \mathcal{B}(\mathbb{R}^2) : \ \mathbf{1}_E \text{ satisfies property } \mathbf{P}\}$. Then $\mathcal{E}$ is a monotone class containing the π-system $\mathcal{C} \triangleq \{A \times B : \ A, B \in \mathcal{B}(\mathbb{R}^1)\}$. By the monotone class theorem in measure theory, we conclude that $\mathcal{E} = \mathcal{B}(\mathbb{R}^2)$. In other words, $\mathbf{1}_E$ satisfies property $\mathbf{P}$ for every $E \in \mathcal{B}(\mathbb{R}^2)$.

Note that the property $\mathbf{P}$ is linear in f. By writing $f = f^+ - f^-$, we only need to consider the case when f is bounded and non-negative. But then there exists a sequence f_n of simple functions on $\mathbb{R}^2$ such that $0 \leqslant f_n \uparrow f$. We know that each f_n satisfies property $\mathbf{P}$. By the monotone convergence theorem for both conditional and unconditional expectations, we conclude that f satisfies property $\mathbf{P}$.

(iii) Since both sides are $\sigma(\mathcal{G},\mathcal{H})$-measurable, it suffices to show that

$$\int_E X d\mathbb{P} = \int_E \mathbb{E}[X|\mathcal{G}]d\mathbb{P}, \quad \forall E \in \sigma(\mathcal{G},\mathcal{H}). \tag{1}$$

Let $\mathcal{E} = \{E \in \sigma(\mathcal{G},\mathcal{H}) : \ \text{equation (1) holds}\}$, and let $\mathcal{C} = \{A \cap B : \ A \in \mathcal{G}, B \in \mathcal{H}\}$. Apparently, $\mathcal{C}$ is a π-system. For any $A \in \mathcal{G}, B \in \mathcal{H}$, we have

$$\mathbb{E}[X\mathbf{1}_A\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_A]\mathbb{P}(B) = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_A]\mathbb{P}(B) = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_A\mathbf{1}_B].$$

Therefore, $\mathcal{C} \subseteq \mathcal{E}$. Moreover, it is easy to see that $\mathcal{E}$ is a monotone class. By the monotone class theorem, we conclude that $\sigma(\mathcal{G},\mathcal{H}) = \mathcal{E}$.

(2) By assumption, we know that for every $r \in \mathbb{R}^1$,

$$\mathbb{E}\left[(X-Y)\mathbf{1}_{\{X\leqslant r\}}\right] = \mathbb{E}\left[(X-Y)\mathbf{1}_{\{Y\leqslant r\}}\right] = 0.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[(X-Y)\mathbf{1}_{\{X\leqslant r, Y>r\}}\right] + \mathbb{E}\left[(X-Y)\mathbf{1}_{\{X\leqslant r, Y\leqslant r\}}\right] &= 0,\\
\mathbb{E}\left[(X-Y)\mathbf{1}_{\{X> r, Y\leqslant r\}}\right] + \mathbb{E}\left[(X-Y)\mathbf{1}_{\{X\leqslant r, Y\leqslant r\}}\right] &= 0.
\end{aligned}$$

It follows that

$$\mathbb{E}\left[(X-Y)\mathbf{1}_{\{X>r, Y\leqslant r\}}\right] + \mathbb{E}\left[(Y-X)\mathbf{1}_{\{X\leqslant r, Y>r\}}\right] = 0.$$

But the integrand inside each of the above expectations is non-negative. Therefore,

$$(X-Y)\mathbf{1}_{\{X>r,Y\leqslant r\}}=(Y-X)\mathbf{1}_{\{X\leqslant r,Y>r\}}=0 \ \text{ a.s.}$$

This implies that

$$\mathbb{P}(X>r,Y\leqslant r)=\mathbb{P}(X\leqslant r,Y>r)=0.$$

And this is true for all $r\in\mathbb{R}^1$. The result then follows from the fact that

$$\{X\neq Y\}\subseteq\{X>Y\}\bigcup\{X<Y\}\subseteq\bigcup_{n\in\mathbb{Z}}\Big(\{X>n\geqslant Y\}\bigcup\{Y>n\geqslant X\}\Big).$$

Problem 2. (1) For $\lambda>0$, we have

$$|\mathbb{E}[X|\mathcal{G}_i]|\mathbf{1}_{\{|\mathbb{E}[X|\mathcal{G}_i]|>\lambda\}}\leqslant\mathbb{E}[|X||\mathcal{G}_i]\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{G}_i]>\lambda\}}.$$

Therefore, by taking expectations on both sides, we obtain that

$$\mathbb{E}\left[|\mathbb{E}[X|\mathcal{G}_i]|\mathbf{1}_{\{|\mathbb{E}[X|\mathcal{G}_i]|>\lambda\}}\right]\leqslant\mathbb{E}[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{G}_i]>\lambda\}}].$$

But

$$\begin{aligned}\mathbb{E}[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{G}_i]>\lambda\}}] &= \mathbb{E}[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{G}_i]>\lambda\}};|X|>\sqrt{\lambda}]+\mathbb{E}[|X|\mathbf{1}_{\{\mathbb{E}[|X||\mathcal{G}_i]>\lambda\}};|X|\leqslant\sqrt{\lambda}]\\ &\leqslant \mathbb{E}[|X|;|X|>\sqrt{\lambda}]+\sqrt{\lambda}\cdot\frac{1}{\lambda}\mathbb{E}[\mathbb{E}[|X||\mathcal{G}_i]]\\ &= \mathbb{E}[|X|;|X|>\sqrt{\lambda}]+\frac{1}{\sqrt{\lambda}}\mathbb{E}[|X|],\end{aligned}$$

which goes to zero uniformly in $i\in\mathcal{I}$ as $\lambda\to\infty$ since X is integrable. Therefore, $\{\mathbb{E}[X|\mathcal{G}_i]:\ i\in\mathcal{I}\}$ is uniformly integrable.

(2) Let $M=\sup_{t\in T}\mathbb{E}[\varphi(|X_t|)]$. For $\varepsilon>0$, let $R=M/\varepsilon$. Then there exists some $\Lambda>0$, such that for any $x>\Lambda$, we have $\varphi(x)/x>R$. Therefore, for $\lambda>\Lambda$, we have

$$\mathbb{E}[|X_t|\mathbf{1}_{\{|X_t|>\lambda\}}]\leqslant\frac{1}{R}\mathbb{E}[\varphi(|X_t|)]\leqslant\frac{M}{R}=\varepsilon,\quad\forall t\in T.$$

Consequently, $\{X_t:\ t\in T\}$ is uniformly integrable.

Problem 3. (1) $\mathbb{P}(X_n>\alpha\log n)=\mathrm{e}^{-\alpha\log n}=1/n^{\alpha}$. Therefore, by the Borel-Cantelli lemma, we have

$$\mathbb{P}(X_n>\alpha\log n\text{ for infinitely many }n)=\begin{cases}0, & \alpha>1;\\ 1, & 0<\alpha\leqslant 1.\end{cases}$$

(2) Let $A_\alpha=\{X_n>\alpha\log n\text{ for infinitely many }n\}$. Since $\mathbb{P}(A_1)=1$, we know that $L\geqslant 1$ almost surely. Moreover,

$$\{L>1\}\subseteq\bigcup_{k=1}^{\infty}\left\{L>1+\frac{1}{k}\right\}\subseteq\bigcup_{k=1}^{\infty}A_{1+\frac{1}{2k}}.$$

It follows that $\mathbb{P}(L>1)=0$. Therefore, $L=1$ almost surely.

(3) For each $x \in \mathbb{R}^1$, we have

$$\mathbb{P}(M_n \leqslant x) = \mathbb{P}\left(\max_{1 \leqslant i \leqslant n} X_i \leqslant x + \log n\right) = (1 - \mathrm{e}^{-x-\log n})^n,$$

provided that $x + \log n > 0$. Therefore,

$$\lim_{n \to \infty} \mathbb{P}(M_n \leqslant x) = \mathrm{e}^{-\mathrm{e}^{-x}}, \quad \forall x \in \mathbb{R}^1.$$

Apparently, the function $F(x) \triangleq \mathrm{e}^{-\mathrm{e}^{-x}}$ defines a continuous distribution function on $\mathbb{R}^1$. Therefore, M_n converges weakly to F.

Problem 4. (1) $\Longrightarrow$ (2). Suppose that $\mathbb{P}_n$ converges weakly to $\mathbb{P}$. According to Theorem 1.7, we know that $\mathbb{P}_n(A) \to \mathbb{P}(A)$ for every $A \in \mathcal{B}(\mathbb{R}^1)$ satisfying $\mathbb{P}(\partial A) = 0$. In particular, let x be a continuity point of F and let $A = (-\infty, x]$. Then $\mathbb{P}(\partial A) = dF(\{x\}) = 0$. Therefore,

$$F_n(x) = \mathbb{P}_n(A) \to \mathbb{P}(A) = F(x).$$

(2) $\Longrightarrow$ (1). Suppose that F_n converges in distribution to F. Let C_F be the set of continuity points of F. Since C_F^c is at most countable, we conclude that C_F is dense in $\mathbb{R}^1$.

Let $\varphi \in C_b(\mathbb{R}^1)$. Given $\varepsilon > 0$, let $a, b \in C_F$ be such that $a < 0 < b$ and

$$F(a) < \varepsilon, \ 1 - F(b) < \varepsilon.$$

Then there exists $N \geqslant 1$, such that for any $n > N$,

$$|F_n(a) - F(a)| < \varepsilon, |F_n(b) - F(b)| < \varepsilon.$$

It follows that

$$F_n(a) < 2\varepsilon, \ 1 - F_n(b) < 2\varepsilon, \quad \forall n > N.$$

Therefore,

$$\begin{aligned} \left|\int_{\mathbb{R}^1} \varphi dF_n - \int_{\mathbb{R}^1} \varphi dF\right| &\leqslant \left|\int_{(a,b]} \varphi(dF_n - dF)\right| + \|\varphi\|_\infty (dF_n((a,b]^c) + dF((a,b]^c)) \\ &\leqslant \left|\int_{(a,b]} \varphi(dF_n - dF)\right| + 6\|\varphi\|_\infty \varepsilon \end{aligned} \tag{2}$$

for every $n > N$.

Since φ is uniformly continuous on $[a, b]$, there exists $\delta > 0$, such that whenever $x, y \in [a, b]$ with $|x - y| < \delta$, we have $|\varphi(x) - \varphi(y)| < \varepsilon$. Choose a finite partition $\mathcal{P}: \ a = x_0 < x_1 < \cdots < x_k = b$ of $[a, b]$, such that $x_0, x_1, \cdots, x_k \in C_F$ and $|x_i - x_{i-1}| < \delta$ for each i. Define a step function ψ by taking $\psi(x) = \varphi(x_{i-1})$ for $x \in [x_{i-1}, x_i]$. It follows that

$$\sup_{x \in [a,b]} |\varphi(x) - \psi(x)| \leqslant \varepsilon.$$

Therefore,

$$\begin{aligned}\left|\int_{(a,b]}\varphi(dF_n-dF)\right| &\leqslant 2\sup_{x\in[a,b]}|\varphi(x)-\psi(x)|+\left|\int_{(a,b]}\psi(dF_n-dF)\right| \\ &\leqslant 2\varepsilon+\sum_i|\varphi(x_{i-1})|\cdot((F_n(x_i)-F(x_i))-(F_n(x_{i-1})-F(x_{i-1}))).\end{aligned}\tag{3}$$

Note that the partition $\mathcal{P}$ we chose before does not depend on n.

By substituting (3) into (2) and letting $n\to\infty$, we arrive at

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^1}\varphi dF_n-\int_{\mathbb{R}^1}\varphi dF\right|\leqslant(2+6\|\varphi\|_\infty)\varepsilon.$$

Since ε is arbitrary, we conclude that $\int_{\mathbb{R}^1}\varphi dF_n\to\int_{\mathbb{R}^1}\varphi dF$ as $n\to\infty$. Therefore, $\mathbb{P}_n$ converges weakly to $\mathbb{P}$.

Problem 5. (1) Necessity. Suppose that $\{\mathbb{P}_n\}$ is tight. Then there exists $M>0$, such that

$$\mathbb{P}_n([-M,M])\geqslant\frac{3}{4},\quad\forall n\geqslant 1.$$

It follows that $|\mu_n|\leqslant M$ for all n. Indeed, if this is not the case, suppose for instance that $\mu_n>M$ for some n. Then

$$\frac{1}{2}\leqslant\mathbb{P}_n([\mu_n,\infty))\leqslant\mathbb{P}_n((M,\infty))<\frac{1}{4},$$

which is a contradiction. In addition, we have

$$\begin{aligned}\frac{3}{4} &\leqslant \mathbb{P}_n([-M,M])=\frac{1}{\sqrt{2\pi}\sigma_n}\int_{-M}^{M}\mathrm{e}^{-\frac{(x-\mu_n)^2}{2\sigma_n^2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{-M-\mu_n}{\sigma_n}}^{\frac{M-\mu_n}{\sigma_n}}\mathrm{e}^{-\frac{x^2}{2}}dx\leqslant\frac{1}{\sqrt{2\pi}}\int_{\frac{-2M}{\sigma_n}}^{\frac{2M}{\sigma_n}}\mathrm{e}^{-\frac{x^2}{2}}dx.\end{aligned}\tag{4}$$

This implies that σ_n is bounded. Indeed, if $\sigma_n\uparrow\infty$ along a subsequence, then the right hand side of (4) goes to zero along this subsequence, which is a contradiction.

Sufficiency. Suppose that $|\mu_n|\leqslant M_1,\sigma_n\leqslant M_1$ for some $M_1>0$. Then for any $M>M_1$, we have

$$\mathbb{P}_n([-M,M])=\frac{1}{\sqrt{2\pi}}\int_{\frac{-M-\mu_n}{\sigma_n}}^{\frac{M-\mu_n}{\sigma_n}}\mathrm{e}^{-\frac{x^2}{2}}dx\geqslant\frac{1}{\sqrt{2\pi}}\int_{-\frac{M-M_1}{\sigma_n}}^{\frac{M-M_1}{\sigma_n}}\mathrm{e}^{-\frac{x^2}{2}}dx\geqslant\frac{1}{\sqrt{2\pi}}\int_{-\frac{M-M_1}{M_1}}^{\frac{M-M_1}{M_1}}\mathrm{e}^{-\frac{x^2}{2}}dx.\tag{5}$$

Since the right hand side of (5) converges to 1 as $M\to\infty$, we conclude that

$$\lim_{M\to\infty}\inf_{n\geqslant 1}\mathbb{P}_n([-M,M])=1.$$

In other words, $\{\mathbb{P}_n\}$ is tight.

(2) Sufficiency. Suppose that $\mu_n\to\mu$ and $\sigma_n^2\to\sigma^2$. Then

$$\mathrm{e}^{i\mu_n t-\frac{1}{2}\sigma_n^2 t}\to\mathrm{e}^{i\mu t-\frac{1}{2}\sigma^2 t}$$

for every $t \in \mathbb{R}^1$ as $n \to \infty$. Therefore, $\mathbb{P}_n$ converges weakly to $\mathcal{N}(\mu, \sigma^2)$.

Necessity. Suppose that $\{\mathbb{P}_n\}$ is weakly convergent. From the first part we already know that $\{\mu_n\}$ and $\{\sigma_n^2\}$ are both bounded. Assume that μ and μ' are two limit points of μ_n. We may further assume without loss of generality that $\mu_{n_k} \to \mu, \sigma_{n_k}^2 \to \sigma^2$, and $\mu_{n_l'} \to \mu', \sigma_{n_l'}^2 \to \sigma'^2$ along two subsequences n_k and n_l'. By the sufficiency part and the uniqueness of weak limits, we know that $\mathcal{N}(\mu, \sigma^2) = \mathcal{N}(\mu', \sigma'^2)$, and hence $\mu = \mu'$ and $\sigma^2 = \sigma'^2$. Therefore, μ_n converges to some $\mu \in \mathbb{R}^1$. Similarly, we conclude that σ_n^2 has exactly one limit point, which means that it converges to some $\sigma^2 \geqslant 0$.