Question 1. *(1) [Filtered Probability Space] A filtration over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a increasing sequence $\{\mathcal{F}_t : t \geqslant 0\}$ of sub-σ-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leqslant s < t$. A filtered probability space is a probability space equipped with a filtration.*

[Stopping Time] Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ be a filtered probability space. An $\{\mathcal{F}_t\}$-stopping time is a random time $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leqslant t\} \in \mathcal{F}_t$ for every $t \geqslant 0$.

[Continuous Time Submartingale] A real-valued stochastic process $\{X_t : t \geqslant 0\}$ is called a continuous time submartingale if for every $t \geqslant 0$, X_t is $\mathcal{F}_t$-measurable and integrable, and for every $0 \leqslant s < t$, we have $\mathbb{E}[X_t|\mathcal{F}_s] \geqslant X_s$.

[Optional Sampling Theorem] Let $\{X_t, \mathcal{F}_t\}$ be a right continuous submartingale, and let σ, τ be two bounded $\{\mathcal{F}_t\}$-stopping times such that $\sigma \leqslant \tau$. Then the optional sampling theorem asserts that $\{X_\sigma, \mathcal{F}_\sigma; X_\tau, \mathcal{F}_\tau\}$ is a two-step submartinagle.

(2) From the optional sampling theorem, we know that

$$X_{\tau\wedge s} \leqslant \mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_{\tau\wedge s}] = \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_{\tau\wedge s}] + \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau>s\}}|\mathcal{F}_{\tau\wedge s}]$$

for $s \leqslant t$. First term equals $\mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_s]$ since $X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}} = X_{\tau\wedge s}\mathbf{1}_{\{\tau\leqslant s\}}$ is $\mathcal{F}_{\tau\wedge s}$-measurable. The second term equals $\mathbb{E}\left[\mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s]|\mathcal{F}_{\tau\wedge s}\right]$, where the integrand

$$\mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s] \in \mathcal{F}_{\tau\wedge s}.$$

Therefore,

$$X_{\tau\wedge s} \leqslant \mathbb{E}[X_{\tau\wedge t}\mathbf{1}_{\{\tau\leqslant s\}}|\mathcal{F}_s] + \mathbf{1}_{\{\tau>s\}}\mathbb{E}[X_{\tau\wedge s}|\mathcal{F}_s] = \mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s].$$

(3) (i) Since

$$u(\lambda, x) = \mathrm{e}^{\frac{1}{2}x^2} \cdot \mathrm{e}^{-\frac{1}{2}(x-\lambda)^2},$$

we can see that

$$H_n(x) = \mathrm{e}^{\frac{1}{2}x^2} \cdot \frac{d}{d^n\lambda}\mathrm{e}^{-\frac{1}{2}(x-\lambda)^2}\bigg|_{\lambda=0} = (-1)^n \mathrm{e}^{\frac{1}{2}x^2}\frac{d}{d^n x}\mathrm{e}^{-\frac{1}{2}x^2}.$$

(ii) Given $0 \leqslant s < t$, we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t}|\mathcal{F}_s^B\right] &= \mathrm{e}^{\lambda B_s - \frac{1}{2}\lambda^2 s} \cdot \mathbb{E}\left[\mathrm{e}^{\lambda(B_t - B_s) - \frac{1}{2}\lambda^2(t-s)}|\mathcal{F}_s^B\right] \\ &= \mathrm{e}^{\lambda B_s - \frac{1}{2}\lambda^2 s}.\end{aligned}$$

Therefore, the process $\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is an $\{\mathcal{F}_t^B\}$-martingale.

Now if we write

$$\mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t} = \mathrm{e}^{(\lambda\sqrt{t})\cdot\left(\frac{B_t}{\sqrt{t}}\right) - \frac{1}{2}(\lambda\sqrt{t})^2} = \sum_{n=0}^{\infty}\frac{\lambda^n t^{\frac{n}{2}}}{n!}H_n\left(\frac{B_t}{\sqrt{t}}\right),$$

the martingale property says that for $0 \leqslant s < t$, we have

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}\left[t^{\frac{n}{2}}H_n\left(\frac{B_t}{\sqrt{t}}\right)|\mathcal{F}_s^B\right] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}s^{\frac{n}{2}}H_n\left(\frac{B_s}{\sqrt{s}}\right).$$

Since this is true for all λ, by comparing the coefficients of λ^n, we conclude that the process $t^{\frac{n}{2}}H_n(B_t/\sqrt{t})$ is an $\{\mathcal{F}_t^B\}$-martingale.

(iii) From part (3) (i), we can easily see that

$$H_2(x) = x^2 - 1,\ H_4(x) = x^4 - 6x^2 + 3.$$

Therefore, the processes

$$\begin{aligned} M_t^{(2)} &\triangleq B_t^2 - t, \\ M_t^{(4)} &\triangleq B_t^4 - 6tB_t^2 + 3t^2 \end{aligned}$$

are $\{\mathcal{F}_t^B\}$-martingales. According to the optional sampling theorem, we have

$$\mathbb{E}[B_{\tau_r \wedge t}^2] = \tau_r \wedge t, \quad \forall t \geqslant 0.$$

Therefore, we have

$$\mathbb{E}[B_{\tau_r}^2] = r^2 = \mathbb{E}[\tau_r].$$

Similarly,

$$\mathbb{E}[M_{\tau_r}^{(4)}] = r^4 - 6r^2\mathbb{E}[\tau_r] + 3\mathbb{E}[\tau_r^2] = 0,$$

which implies that

$$\mathbb{E}[\tau_r^2] = \frac{6r^2\mathbb{E}[\tau_r] - r^4}{3} = \frac{5r^4}{3}.$$

Question 2. *(1) [Itô's formula] Let $X_t = (X_t^1, \cdots, X_t^d)$ be a vector of d continuous semimartingales. Suppose that $F \in C^2(\mathbb{R}^d)$. Then $F(X_t)$ is a continuous semimartingale given by*

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x^i}(X_s)dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^j}(X_s)d\langle X^i, X^j\rangle_s.$$

[Girsanov's theorem] Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ be a filtered probability space which satisfies the usual conditions, and let $B_t = (B_t^1, \cdots, B_t^d)$ be a d-dimensional $\{\mathcal{F}_t\}$-Brownian motion. Suppose that $X_t = (X_t^1, \cdots, X_t^d)$ is a stochastic process in $\mathbb{R}^d$ with $X^i \in L^2_{\mathrm{loc}}(B^i)$ for each i. Define

$$\mathcal{E}_t^X \triangleq \exp\left(\sum_{i=1}^d \int_0^t X_s^i dB_s^i - \frac{1}{2}\int_0^t |X_s|^2 ds\right), \quad t \geqslant 0.$$

Assume that $\{\mathcal{E}_t^X, \mathcal{F}_t :\ t \geqslant 0\}$ is a martingale. For each given $T > 0$, we define a probability measure $\widetilde{\mathbb{P}}_T$ on $\mathcal{F}_T$ by

$$\widetilde{\mathbb{P}}_T(A) \triangleq \mathbb{E}[\mathbf{1}_A \mathcal{E}_T^X], \quad A \in \mathcal{F}_T.$$

Define the process $\widetilde{B}_t = (\widetilde{B}_t^1, \cdots, \widetilde{B}_t^d)$ by

$$\widetilde{B}_t^i \triangleq B_t^i - \int_0^t X_s^i ds, \quad t \geqslant 0,\ 1 \leqslant i \leqslant d.$$

Then for each $T > 0$, $\{\widetilde{B}_t, \mathcal{F}_t :\ 0 \leqslant t \leqslant T\}$ is a d-dimensional Brownian motion under $\widetilde{\mathbb{P}}_T$.

(2) (i) Define

$$X_t \triangleq \exp\left(\int_0^t \sigma_s dB_s + \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right), \quad t \geqslant 0.$$

From Itô's formula, we see immediately that X_t satisfies the desired integral equation.

Now suppose that Y_t is another continuous semimartingale that also satisfies the integral equation. Let

$$Z_t \triangleq Y_t X_t^{-1} = Y_t \exp\left(-\int_0^t \sigma_s dB_s - \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right)$$

Itô's formula again, or more precisely, the integration by parts formula, will imply that the martingale part and the bounded variation part of the continuous semimartingale Z_t are identically zero. Therefore,

$$Z_t = Z_0 = 1,$$

which shows that $Y_t = X_t$. In other words, X_t is the unique continuous semimiartingale which satisfies the integral equation.

(ii) First of all, we know that

$$X_t - 1 - \int_0^t X_s \mu_s ds = \int_0^t X_s \sigma_s dB_s, \quad 0 \leqslant t \leqslant T,$$

is a continuous local martingale under $\mathbb{P}$. Suppose we want to find a process q_t which is used to define the change of measure in the way that

$$\widetilde{\mathbb{P}}_T(A) \triangleq \mathbb{E}\left[\exp\left(\int_0^T q_s dB_s - \frac{1}{2}\int_0^T q_s^2 ds\right)\mathbf{1}_A\right], \quad A \in \mathcal{F}_T.$$

Then we know from Theorem 5.16 in the lecture notes that the process

$$\int_0^t X_s \sigma_s dB_s - \int_0^t q_s X_s \sigma_s ds = X_t - 1 - \int_0^t X_s \mu_s ds - \int_0^t q_s X_s \sigma_s ds$$

is a continuous local martingale under $\widetilde{\mathbb{P}}_T$ (provided that the exponential martingale is a true martingale so that $\widetilde{\mathbb{P}}_T$ is a probability measure). Now we want this process to be $X_t - 1$, therefore we just need to choose

$$q_t \triangleq -\mu_t \sigma_t^{-1}.$$

Since μ_t is uniformly bounded and $\sigma \geqslant C$, in this way we can see easily that Novikov's condition holds for the continuous local martingale $\int_0^t q_s dB_s$, which verifies that the exponential martingale is a true martingale.

(3) In matrix notation, we need to solve

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix},$$

which gives

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Therefore, define

$$\mathbb{Q}(A) \triangleq \mathbb{E}\left[\mathbf{1}_A \exp(-3B_T^1 + B_T^2 - 5T)\right], \ A \in \mathcal{F}_T,$$

and

$$\widetilde{B}_t \triangleq B_t - \begin{pmatrix} -3 \\ 1 \end{pmatrix} t,\ 0 \leqslant t \leqslant T.$$

It follows that under $\mathbb{Q}$, $\widetilde{B}_t$ *is a Brownian motion, and* X_t *satisfies*

$$\begin{cases} dX_t = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} d\widetilde{B}_t, & 0 \leqslant t \leqslant T; \\ X_0 = y, \end{cases}$$

which is apparently a martingale.

Question 3. *(1) [Exactness] We say that the SDE is exact if on any given set-up* $((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}), \xi, B_t)$, *there exists exactly one (up to indistinguishability) continuous,* $\{\mathcal{F}_t\}$*-adapted* n*-dimensional process* X_t, *such that with probability one,*

$$\int_0^t \left(\|\alpha(s, X)\|^2 + \|\beta(s, X)\|\right) ds < \infty, \quad \forall t \geqslant 0, \quad \text{(a)}$$

and

$$X_t = \xi + \int_0^t \alpha(s, X) dB_s + \int_0^t \beta(s, X) ds, \quad t \geqslant 0. \quad \text{(b)}$$

[Weak Solution] Let μ *be a probability measure on* $\mathbb{R}^n$. *We say that the SDE has a weak solution with initial distribution* μ *if there exists a set-up* $((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}), \xi, B_t)$ *together with a continuous,* $\{\mathcal{F}_t\}$*-adapted* n*-dimensional process* X_t, *such that*

(i) ξ *has distribution* μ;

(ii) X_t *satisfies (a) and (b).*

If for every probability measure μ *on* $\mathbb{R}^n$, *the SDE has a weak solution with initial distribution* μ, *we say that it has a weak solution.*

[Pathwise Uniqueness] We say that pathwise uniqueness holds for the SDE if the following statement is true. Given any set-up $((\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}), \xi, B_t)$, *if* X_t *and* X'_t *are two continuous,* $\{\mathcal{F}_t\}$*-adapted* n*-dimensional process satisfying (a) and (b), then* $\mathbb{P}(X_t = X'_t\ \forall t \geqslant 0) = 1$.

(2) (i) Let $\widetilde{B}_t$ *be the reflection of* B_t *at* $x = 1$ *defined by*

$$\widetilde{B}_t \triangleq \begin{cases} B_t, & t < \tau_1; \\ 2 - B_t, & t \geqslant \tau_1. \end{cases}$$

Let $\widetilde{S}_t \triangleq \sup_{s \leqslant t} \widetilde{B}_s$. *From the reflection principle, we know that* $\widetilde{B}_t$ *is also a Brownian motion. Now observe that* $\{S_t \geqslant 1\} = \left\{\widetilde{S}_t \geqslant 1\right\}$. *Therefore,*

$$\begin{aligned} \mathbb{P}(\tau_1 \leqslant t) &= \mathbb{P}(S_t \geqslant 1) \\ &= \mathbb{P}(S_t \geqslant 1, B_t \leqslant 1) + \mathbb{P}(S_t \geqslant 1, B_t > 1) \\ &= \mathbb{P}(\widetilde{S}_t \geqslant 1, \widetilde{B}_t \leqslant 1) + \mathbb{P}(S_t \geqslant 1, B_t > 1) \\ &= \mathbb{P}(S_t \geqslant 1, B_t \geqslant 1) + \mathbb{P}(S_t \geqslant 1, B_t > 1) \\ &= 2\mathbb{P}(B_t \geqslant 1) \\ &= \frac{2}{\sqrt{2\pi t}} \int_1^\infty \mathrm{e}^{-\frac{u^2}{2t}} du. \end{aligned}$$

By differentiation, we arrive at

$$\mathbb{P}(\tau_1 \in dt) = \frac{x}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{x^2}{2t}} dt, \quad t > 0. \quad \textit{(c)}$$

(ii) Let $\tau_0 \triangleq \inf\{t \geqslant 0 : \ X_t = 0\}$, *and define*

$$Y_t \triangleq X_t^{-1/2}, \quad t < \tau_0 \wedge e.$$

According to Itô's formula, we conclude that

$$dY_t = dB_t, \quad t < \tau_0 \wedge e.$$

This in particular implies that $\tau_0 = \infty$ *almost surely and therefore we have*

$$X_t = \frac{1}{(1+B_t)^2}, \quad t < e.$$

It follows that $e = \inf\{t \geqslant 0 : \ B_t = -1\}$. *From the density formula (c) for* e, *it is easy to see that*

$$\mathbb{P}(e < \infty) = 1, \quad \mathbb{E}[e] = \infty.$$

(3) Let $\tau_0 \triangleq \{t \geqslant 0 : \ X_t = 0\}$, *and define*

$$Y_t \triangleq X_t^{-\theta}, \quad t < \tau_0 \wedge e.$$

It follows from Itô's formula that

$$dY_t = \left(-\theta\alpha + \left(\frac{\theta(\theta+1)}{2}\gamma^2 - \theta\beta \right) Y_t \right) dt - \theta\gamma Y_t dB_t, \quad t < \tau_0 \wedge e.$$

Therefore, if we define

$$Z_t \triangleq \exp\left(\gamma B_t + \left(\beta - \frac{\gamma^2}{2} \right) t \right),$$

then

$$Y_t = Z_t^{-\theta} \left(Y_0 - \alpha\theta \int_0^t Z_s^{\theta} ds \right), \quad t < \tau_0 \wedge e.$$

This implies that $\tau_0 = \infty$ *almost surely, and we have*

$$X_t = Z_t \left(x^{-\theta} - \alpha\theta \int_0^t Z_s^{\theta} ds \right)^{-\frac{1}{\theta}}, \quad t < e.$$

In particular, we conclude that

$$e = \inf\left\{ t \geqslant 0 : \ x^{-\theta} - \alpha\theta \int_0^t Z_s^{\theta} ds = 0 \right\}.$$

If $\alpha \leqslant 0$, *it is apparent that*

$$x^{-\theta} - \alpha\theta \int_0^t Z_s^{\theta} ds > 0, \quad \forall t \geqslant 0.$$

Therefore, $\mathbb{P}(e = \infty) = 1$.

Question 4. *(1) According to Green's theorem in calculus, the value of the integral*

$$\frac{1}{2}\int_0^t (x_s dy_s - y_s dx_s)$$

is the geometric (signed) area enclosed by the path $\{\gamma_s:\ 0\leqslant s\leqslant t\}$ *and the segment connecting* γ_0,γ_t.

(2) (i) The solution is given by

$$g(s)=\sqrt{2\alpha}\cdot\frac{\mathrm{e}^{2\sqrt{2\alpha}(s-t)}-1}{\mathrm{e}^{2\sqrt{2\alpha}(s-t)}+1},\quad 0\leqslant s\leqslant t.$$

(ii) According to part (i), we have

$$\mathbb{E}\left[\exp\left(-\alpha\int_0^t b_s^2 ds\right)\right]=\mathbb{E}\left[-\left(\frac{1}{2}\int_0^t g'(s)b_s^2 ds+\frac{1}{2}\int_0^t g^2(s)b_s^2 ds\right)\right].$$

Now let $F(s,x)\triangleq g(s)x^2$*. By applying Itô's formula to* $F(s,b_s)$*, we obtain*

$$\int_0^t g'(s)b_s^2 ds+2\int_0^t g(s)b_s db_s+\int_0^t g(s)ds=0.$$

Therefore,

$$\mathbb{E}\left[\exp\left(-\alpha\int_0^t b_s^2 ds\right)\right]=\exp\left(\frac{1}{2}\int_0^t g(s)ds\right)\cdot\mathbb{E}\left[\exp\left(\int_0^t g(s)b_s db_s-\frac{1}{2}\int_0^t g^2(s)b_s^2 ds\right)\right].$$

On the other hand, it is apparent that

$$\exp\left(\frac{1}{2}\int_0^t g^2(s)b_s^2 ds\right)\leqslant\exp\left(\alpha\int_0^t b_s^2 ds\right).$$

By assumption (α is small), the martingale $\{\int_0^s g(u)b_u db_u:\ 0\leqslant s\leqslant t\}$ *satisfies Novikov's condition. Therefore, the exponential martingale*

$$\exp\left(\int_0^s g(u)b_u db_u-\frac{1}{2}\int_0^t g^2(u)b_u^2 du\right),\quad 0\leqslant s\leqslant t,$$

is a true martingale. In particular, we conclude that

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\alpha\int_0^t b_s^2 ds\right)\right] &= \exp\left(\frac{1}{2}\int_0^t g(s)ds\right)\\ &= \left(\frac{1+\mathrm{e}^{2\sqrt{2\alpha}t}}{2}\right)^{-\frac{1}{2}}\cdot\mathrm{e}^{\frac{\sqrt{2\alpha}}{2}t}.\end{aligned}$$

(iii) First of all, we know that ρ_t *satisfies the SDE*

$$\begin{aligned}d\rho_t &= 2(B_t^1 dB_t^1+B_t^2 dB_t^2)+2dt\\ &= 2\sqrt{\rho_t}db_t+2dt, \qquad (d)\end{aligned}$$

where

$$b_t \triangleq \int_0^t \frac{B_s^1 dB_s^1 + B_s^2 dB_s^2}{\sqrt{\rho_s}}, \quad t \geqslant 0,$$

is a Brownian motion according to Lévy's characterization theorem. On the other hand, we have

$$\langle L, b \rangle_t = -\frac{1}{2}\int_0^t \frac{B_s^1 B_s^2}{\sqrt{\rho_s}} ds + \frac{1}{2}\int_0^t \frac{B_s^1 B_s^2}{\sqrt{\rho_s}} ds = 0.$$

According to Knight's theorem, we know that the processes $W_t \triangleq L_{C_t}$ and b_t are independent Brownian motions. But we know from the Yamada-Watanabe theorem that the SDE (d) is exact. Therefore, the process ρ is a functional of b, and in particular it is measurable with respect to the σ-algebra generated by b. It follows that the processes W and ρ are independent.

(iv) First of all, we have

$$\mathrm{ch}_t(\lambda) = \mathbb{E}\left[\mathrm{e}^{i\lambda W_{\langle L \rangle_t}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{i\lambda W_{\langle L \rangle_t}} | \mathcal{F}^\rho\right]\right],$$

where $\mathcal{F}^\rho$ is the σ-algebra generated by ρ. Since

$$\langle L \rangle_t = \frac{1}{4}\int_0^t ((B_s^1)^2 + (B_s^2)^2) ds \in \mathcal{F}^\rho,$$

and conditioned on $\mathcal{F}_\rho$, $W_{\langle L \rangle_t}$ is a Gaussian random variable with mean zero and variance $\langle L \rangle_t$, from the result of part (iii), we conclude that

$$\mathrm{ch}_t(\lambda) = \mathbb{E}\left[\mathrm{e}^{-\frac{1}{2}\lambda^2 \langle L \rangle_t}\right] = \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{8}\int_0^t \rho_s ds\right)\right].$$

(v) According to part (ii), we arrive at

$$\mathrm{ch}_t(\lambda) = \left(\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{8}\int_0^t b_s^2 ds\right)\right]\right)^2 = \frac{1}{\cosh(\lambda t/2)}, \qquad (e)$$

at least when λ is small. But it is easy to see that the functions

$$\Phi(z) \triangleq \mathbb{E}[\mathrm{e}^{zL_t}], \quad \Psi(z) \triangleq \frac{1}{\cosh(-izt/2)},$$

are holomorphic on the common domain $U \subseteq \mathbb{C}$ which contains the whole imaginary axis. According to the identity theorem in complex analysis, we conclude that (e) holds for all $\lambda \in \mathbb{R}^1$.