Problem Sheet 6

Due for Submission: 12/07 Wednesday

Problem 1. Show that the following SDEs are all exact. Solve them explicitly with the given initial data. Here B_t is a one dimensional Brownian motion.

(1) The stochastic harmonic oscillator model:

$$\begin{cases} dX_t = Y_t dt, \\ m dY_t = -kX_t dt - cY_t dt + \sigma dB_t, \end{cases}$$

where m, k, c, σ are positive constants. Initial data is arbitrary.

(2) The stochastic RLC circuit model:

$$\begin{cases} dX_t = Y_t dt, \\ L dY_t = -RY_t - \frac{1}{C} X_t + G(t) + \alpha dB_t, \end{cases}$$

where R, C, L, α are positive constants and $G(t)$ is a given deterministic function. Initial data is arbitrary.

(3) The stochastic population growth model:

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t,$$

where r, K, β are positive constants. Initial data is $X_0 = x > 0$.

Problem 2. Let B_t be a one dimensional Brownian motion on a filtered Probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ which satisfies the usual conditions.

(1) Define $X_t \triangleq B_t - tB_1$ $(0 \leqslant t \leqslant 1)$. Show that X_t is a Gaussian process. Compute its mean and covariance function $\rho(s,t) \triangleq \mathbb{E}[X_s X_t]$ $(0 \leqslant s, t \leqslant 1)$.

(2) Find the solution Y_t $(0 \leqslant t < 1)$ to the SDE

$$\begin{cases} dY_t = dB_t - \frac{Y_t}{1-t} dt, \quad 0 \leqslant t < 1, \\ Y_0 = 0. \end{cases}$$

Show that Y_t has the same law as X_t $(0 \leqslant t < 1)$. In particular, $\lim_{t \uparrow 1} Y_t = 0$ almost surely and we can define $Y_1 \triangleq 0$. This defines a process Y_t $(0 \leqslant t \leqslant 1)$ which has the same law as X_t $(0 \leqslant t \leqslant 1)$.

(3) Show that

$$\mathbb{P}\left(\sup_{0 \leqslant t \leqslant 1} Y_t \geqslant x \right) = \mathrm{e}^{-2x^2}, \quad x \geqslant 0.$$

Problem 3. Consider the one dimensional SDE

$$dY_t = 3Y_t^2 dt - 2|Y_t|^{\frac{3}{2}} dB_t.$$

(1) Show that this SDE is exact (in the context with possible explosion).

(2) Show that if $Y_0 \geqslant 0$, then $Y_t \geqslant 0$ for all t up to its explosion time e.

(3) Suppose that $Y_0 = 1$. Compute $\mathbb{P}(e > t)$ for $t \geqslant 0$. Conclude that $\mathbb{P}(e < \infty) = 1$ but $\mathbb{E}[e] = \infty$.

Problem 4. (1) Let H, G be continuous semimartingales with $\langle H, G\rangle = 0$ and $H_0 = 0$. Show that if

$$Z_t \triangleq \mathcal{E}_t^G \int_0^t (\mathcal{E}_s^G)^{-1} dH_s,$$

where $\mathcal{E}_t^G$ is the stochastic exponential of G defined by

$$\mathcal{E}_t^G \triangleq \mathrm{e}^{X_t - \frac{1}{2}\langle X\rangle_t},$$

then Z_t satisfies

$$Z_t = H_t + \int_0^t Z_s dG_s.$$

(2) Consider the following two SDEs on $\mathbb{R}^1$:

$$dX_t^i = \sigma(t, X_t)dB_t + b^i(X_t)dt, \quad i = 1, 2,$$

where $\sigma : [0,\infty) \times \mathbb{R}^1 \to \mathbb{R}^1$, $b^i : \mathbb{R}^1 \to \mathbb{R}^1$ are bounded continuous, σ is Lipschitz continuous and one of b^1, b^2 is Lipschitz continuous. Suppose further that $b^1 < b^2$ everywhere. Let X_t^i be a solution to the above SDE with $i = 1, 2$ respectively, defined on the same filtered probability space with the same Brownian motion, such that $X_0^1 \leqslant X_0^2$ almost surely. By putting $Z_t = X_t^2 - X_t^1$, and choosing a suitable positive bounded variation process H_t and a continuous semimartingale G_t in the first part of the question, show that

$$\mathbb{P}(X_t^1 < X_t^2 \ \ \forall t \geqslant 0) = 1.$$

Give an example to show that if σ is not Lipschitz continuous, then the conclusion can false even b^1, b^2 are Lipschitz continuous.