Problem Sheet 5

Due for Submission: 11/11 Friday

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1. Let $M \in \mathcal{M}_0^{\mathrm{loc}}$ be a continuous local martingale vanishing at $t = 0$.

(1) Recall that H_0^2 is the space of L^2-bounded continuous martingales vanishing at $t = 0$. Show that $M \in H_0^2$ if and only if $\mathbb{E}[\langle M\rangle_\infty] < \infty$, where $\langle M\rangle_\infty \triangleq \lim_{t\to\infty}\langle M\rangle_t$.

(2) Show that $\langle M\rangle_t$ is deterministic (i.e. there exists a function $f : [0,\infty) \to \mathbb{R}^1$, such that with probability one, $\langle M\rangle_t(\omega) = f(t)$ for all $t \geqslant 0$) if and only if M_t is a Gaussian martingale, in the sense that it is a martingale and $(M_{t_1}, \cdots, M_{t_n})$ is Gaussian distributed in $\mathbb{R}^n$ for every $0 \leqslant t_1 < \cdots < t_n$. In this case, M_t has independent increments.

(3) Show that there exists a measurable set $\widetilde{\Omega} \in \mathcal{F}$, such that $\mathbb{P}(\widetilde{\Omega}) = 1$ and

$$\begin{aligned} \widetilde{\Omega}\bigcap\{\langle M\rangle_\infty < \infty\} &= \widetilde{\Omega}\bigcap\left\{\lim_{t\to\infty} M_t \text{ exists finitely}\right\}, \\ \widetilde{\Omega}\bigcap\{\langle M\rangle_\infty = \infty\} &= \widetilde{\Omega}\bigcap\left\{\limsup_{t\to\infty} M_t = \infty,\ \liminf_{t\to\infty} M_t = -\infty\right\}. \end{aligned}$$

Problem 2. Let B_t be the three dimensional Brownian motion with $\{\mathcal{F}_t^B\}$ being its augmented natural filtration. Define $X_t \triangleq 1/|B_{1+t}|$.

(1) Show that X_t is a continuous $\{\mathcal{F}_{1+t}^B\}$-local martingale which is uniformly bounded in L^2 (and hence uniformly integrable) but it is not an $\{\mathcal{F}_{1+t}^B\}$-martingale.

(2) Show that if a uniformly integrable continuous submartingale Y_t has a Doob-Meyer decomposition, it has to be of class (D) in the sense that $\{Y_\tau : \tau \text{ is a finite stopping time}\}$ is uniformly integrable. By showing that X_t is not of class (D), conclude that X_t does not have a Doob-Meyer decomposition.

Problem 3. This problem is the stochastic counterpart of Fubini's theorem. Give up if you don't like this question–it is hard and boring. I have to include it because we need to use it in the lecture notes when we study local times and I don't want to waste time proving it in class.

(1) A set $\Gamma \subseteq [0,\infty) \times \Omega$ is called *progressive* if the stochastic process $\mathbf{1}_\Gamma(t,\omega)$ is progressively measurable. Show that the family $\mathcal{P}$ of progressive sets forms a sub-σ-algebra of $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$, and a stochastic process X is progressively measurable if and only if it is measurable with respect to $\mathcal{P}$.

(2) Let $\Phi = \{\Phi^a : a \in \mathbb{R}^1\}$ be a family of real valued stochastic processes parametrized by $a \in \mathbb{R}^1$. Viewed as a random variable on $\mathbb{R}^1 \times [0,\infty) \times \Omega$, suppose that Φ is uniformly bounded and $\mathcal{B}(\mathbb{R}^1) \otimes \mathcal{P}$-measurable. Let X_t be a continuous semimartingale. Show that there exists a $\mathcal{B}(\mathbb{R}^1) \otimes \mathcal{P}$-measurable

$$\begin{aligned} Y : \mathbb{R}^1 \times [0,\infty) \times \Omega &\to \mathbb{R}^1, \\ (a,t,\omega) &\mapsto Y_t^a(\omega), \end{aligned}$$

such that for every $a \in \mathbb{R}^1$, Y^a and $I^X(\Phi^a)$ are indistinguishable as stochastic processes in t, and for every finite measure μ on $(\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$, with probability one, we have

$$\int_{\mathbb{R}^1} Y_t^a \mu(da) = \int_0^t \left(\int_{\mathbb{R}^1} \Phi_s^a \mu(da) \right) dX_s, \quad \forall t \geqslant 0.$$

Problem 4. Let B_t be an $\{\mathcal{F}_t\}$-Brownian motion defined on a filtered probability space which satisfies the usual conditions. Let μ_t and σ_t be two uniformly bounded, $\{\mathcal{F}_t\}$-progressively measurable processes.

(1) By using Itô's formula, find a continuous semimartingale X_t explicitly, such that

$$X_t = 1 + \int_0^t X_s \mu_s ds + \int_0^t X_s \sigma_s dB_s, \quad t \geqslant 0.$$

By using Itô's formula again, show that such X_t is unique.

(2) Assume further that $\sigma \geqslant C$ for some constant $C > 0$. Given $T > 0$, construct a probability measure $\widetilde{\mathbb{P}}_T$, equivalent to $\mathbb{P}$, under which $\{X_t, \mathcal{F}_t : \ 0 \leqslant t \leqslant T\}$ is a continuous local martingale.

Problem 5. Let B_t be a one dimensional Brownian motion and let $\{\mathcal{F}_t^B\}$ be the augmented natural filtration.

(1) Fix $T > 0$. For $\xi = B_T^2$ and B_T^3, find the unique progressively measurable process Φ on $[0, T]$ with $\mathbb{E}\left[\int_0^T \Phi_t^2 dt\right] < \infty$, such that $\xi = \mathbb{E}[\xi] + \int_0^T \Phi_t dB_t$.

(2) Construct a process $\Phi \in L^2_{\text{loc}}(B)$ with $\int_0^\infty \Phi_t^2 dt < \infty$ almost surely (so $\int_0^\infty \Phi_t dB_t$ is well defined), such that $\int_0^\infty \Phi_t dB_t = 0$ but with probability one, $0 < \int_0^\infty \Phi_t^2 dt < \infty$.

(3) Consider $S_1 \triangleq \max_{0 \leqslant t \leqslant 1} B_t$. By writing $\mathbb{E}[S_1 | \mathcal{F}_t^B]$ as a function of (t, S_t, B_t), find the unique progressively measurable process Φ on $[0, 1]$ with $\mathbb{E}\left[\int_0^1 \Phi_t^2 dt\right] < \infty$, such that $S_1 = \mathbb{E}[S_1] + \int_0^1 \Phi_t dB_t$.

Problem 6. (1) Let B_t be the d-dimensional Brownian motion. Define $\tau \triangleq \inf\{t \geqslant 0 : \ |B_t| = 1\}$. What is the distribution of B_τ? Show that B_τ and τ are independent.

(2) Let $c \in \mathbb{R}^d$ and define $X_t \triangleq B_t + ct$ to be the d-dimensional Brownian motion with drift vector c. Define τ in the same way as before but for the process X_t. By using Girsanov's theorem under a suitable framework, show that X_τ and τ are independent.

Problem 7. Let B be a one dimensional Brownian motion and let l be its local time at 0.

(1) Let $X_t = B_t + c l_t$ where $c \in \mathbb{R}^1$. Define L^a to be the local time at a of X. Show that for every $T > 0$ and $k \geqslant 1$, there exists some constant $C_{T,k}$ such that

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |L_t^a - L_t^b|^{2k}\right] \leqslant C_{T,k} |a - b|^k.$$

Conclude that $\{L_t^a : \ a \in \mathbb{R}^1, t \geqslant 0\}$ has a modification which is locally γ-Hölder continuous in a uniformly on every finite t-interval for every $\gamma \in (0, 1/2)$.

(2) Let $\lambda, \mu > 0$ with $\lambda \neq \mu$. After taking the modification given by Theorem 5.18 in the lecture notes, show that the local time L_t^a of the continuous semimartingale $X_t \triangleq \lambda B_t^+ - \mu B_t^-$ is discontinuous at $a = 0$. Compute this jump (at any given $t > 0$).