Problem Sheet 4

Due for Submission: 10/17 Monday

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1. Let B_t be a d-dimensional Brownian motion.

(1) Show that $X_t \triangleq O \cdot B_t$ and $Y_t \triangleq \langle \mu, B_t \rangle$ are both Brownian motions, where O is a $d \times d$ orthogonal matrix (i.e. $O^T O = \mathrm{I}_d$) and μ is a unit vector in $\mathbb{R}^d$.

(2) Given $s < u < t$, compute $\mathbb{E}[B_u | B_s, B_t]$.

Problem 2. Let B_t be a one dimensional Brownian motion.

(1) Show that

$$X_t \triangleq \begin{cases} tB_{\frac{1}{t}}, & t > 0; \\ 0, & t = 0, \end{cases}$$

is a Browninan motion.

(2) Show that with probability one, there exist two sequences of positive times $s_n \downarrow 0$, $t_n \downarrow 0$, such that $B_{s_n} < 0$ and $B_{t_n} > 0$ for every n.

(3) Show that with probabiliy one, B is not differentiable at $t = 0$, and hence conclude that with probability one, $t \mapsto B_t(\omega)$ is almost everywhere non-differentiable.

Problem 3. Let B_t be an $\{\mathcal{F}_t\}$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let σ, τ be two finite $\{\mathcal{F}_t\}$-stopping times such that $\tau \in \mathcal{F}_\sigma$ and $\sigma \leqslant \tau$. Show that for any bounded Borel measurable function f,

$$\mathbb{E}[f(B_\tau) | \mathcal{F}_\sigma] = P_t f(x) \,|_{t=\tau-\sigma,\ x=B_\sigma} \ .$$

Is it true that $B_\tau - B_\sigma$ and $\mathcal{F}_\sigma$ are independent?

Problem 4. Let B_t be a one dimensional Brownian motion with $\{\mathcal{F}_t^B\}$ being its augmented natural filtration. Construct an $\{\mathcal{F}_t^B\}$-stopping time τ explicitly which satisfies the Skorokhod embedding theorem for the uniform distribution on the set $\{-2, -1, 0, 1, 2\}$. Draw a picture to illustrate the construction as well.

Problem 5. Let B_t be a 2-dimensional Brownian motion starting at $i = (0, 1)$ with $\{\mathcal{F}_t^B\}$ being its augmented natural filtration.

(1) Show that for each $\lambda \in \mathbb{R}^1$, the process $X_t^\lambda \triangleq \mathrm{e}^{\lambda i \cdot B_t}$ is an $\{\mathcal{F}_t^B\}$-martingale, where the multiplication in the exponential function is the complex multiplication.

(2) Let τ be the hitting time of the real axis by B_t. Show that $B_\tau \in \mathbb{R}^1$ is Cauchy distributed (i.e. $\mathbb{P}(B_\tau \in dx) = (\pi(1 + x^2))^{-1},\ x \in \mathbb{R}^1$).

Problem 6. Let $X_t(w) \triangleq w_t$ be the coordinate process on W^1 and let $\{\mathcal{F}_t\}$ be the natural filtration of X_t. Denote $\mathbb{P}^{x,c}$ on $(W^1, \mathcal{B}(W^1))$ as the law of a one dimensional Brownian motion starting at x with drift c.

(1) Show that when restricted on each $\mathcal{F}_t$, $\mathbb{P}^{x,c}$ is absolutely continuous with respect to $\mathbb{P}^{x,0}$, with density given by

$$\left.\frac{d\mathbb{P}^{x,c}}{d\mathbb{P}^{x,0}}\right| = \mathrm{e}^{c(X_t - x) - \frac{1}{2}c^2 t}.$$

(2) Define $S_t = \max_{0 \leqslant s \leqslant t} X_s$. Compute $\mathbb{P}^{0,c}(S_t \in dx,\ X_t \in dy)$ $(x \geqslant 0,\ x \geqslant y)$.

Problem 7. (1) Let B_t be a d-dimensional Brownian motion with $\{\mathcal{F}_t^B\}$ being its augmented natural filtration.

(i) Given $\theta \in \mathbb{R}^d$, define $\mathrm{e}_\theta(x) = \mathrm{e}^{i\langle x, \theta\rangle}$ for $x \in \mathbb{R}^d$. Show that the process

$$\mathrm{e}_\theta(B_t) - 1 - \frac{1}{2}\int_0^t (\Delta \mathrm{e}_\theta)(B_s)ds, \quad t \geqslant 0,$$

is an $\{\mathcal{F}_t^B\}$-martingale.

(ii) Let f be a smooth function on $\mathbb{R}^d$ with compact support. Taking as granted the fact that there exists a rapidly decreasing function ϕ on $\mathbb{R}^d$ (i.e. $\sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial^\beta \phi(x)\right| < \infty$ for all α, β), such that

$$f(x) = \int_{\mathbb{R}^d} \mathrm{e}^{i\langle x,\theta\rangle}\phi(\theta)d\theta, \quad \forall x \in \mathbb{R}^d,$$

show that the process

$$f(B_t) - f(0) - \frac{1}{2}\int_0^t (\Delta f)(B_s)ds, \quad t \geqslant 0,$$

is an $\{\mathcal{F}_t^B\}$-martingale.

(2) Let $X_t(w) \triangleq w_t$ be the coordinate process on W^d. Denote $\mathbb{P}_d^x$ on $(W^d, \mathcal{B}(W^d))$ as the law of a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$.

(i) Show that $f(x) \triangleq \log|x|$ (in dimension $d = 2$) and $f(x) = |x|^{2-d}$ (in dimension $d \geqslant 3$) are harmonic on $\mathbb{R}^d \backslash \{0\}$ (i.e. $\Delta f(x) = 0$ for every $x \neq 0$).

(ii) Let $0 < a < |x| < b$. Define τ_a (respectively, τ_b) to be the hitting time of the sphere $|y| = a$ (respectively, $|y| = b$) by X_t. By choosing suitable f in (1), (ii), compute $\mathbb{P}_d^x(\tau_a < \tau_b)$ and $\mathbb{P}_d^x(\tau_a < \infty)$ in all dimensions $d \geqslant 2$.

(iii) Let U be a non-empty, bounded open subset of $\mathbb{R}^d$. Define $\sigma = \sup\{t \geqslant 0 :\ X_t \in U\}$. Show that $\mathbb{P}_d^0(\sigma = \infty) = 1$ in dimension $d = 2$, while $\mathbb{P}_d^0(\sigma < \infty) = 1$ in dimension $d \geqslant 3$. Therefore, the Brownian motion is *neighbourhood-recurrent* in dimension $d = 2$ and *neighbourhood-transient* in dimension $d \geqslant 3$.

(iv) For every dimension $d \geqslant 2$, show that $\mathbb{P}_d^0(\sigma_y < \infty) = 0$ for every $y \in \mathbb{R}^d$, where $\sigma_y \triangleq \inf\{t > 0 :\ X_t = y\}$. Therefore, the Brownian motion is *point-recurrent* only in dimension one.

Remark. It can be shown that in dimension $d = 2$, with probability one, there exists $y \in \mathbb{R}^2$ such that the set $\{t \geqslant 0 :\ X_t = y\}$ has cardinality of the continuum.