Problem Sheet 3

Due for Submission: 09/30 Friday

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1 ($\star$)**.** (1) Suppose that $\{X_t, \mathcal{F}_t : t \geqslant 0\}$ is a right continuous supermartingale and τ is an $\{\mathcal{F}_t\}$-stopping time. Show that the stopped process $X_t^\tau \triangleq X_{\tau\wedge t}$ is a supermartingale both with respect to the filtrations $\{\mathcal{F}_t : t \geqslant 0\}$ and $\{\mathcal{F}_{\tau\wedge t} : t \geqslant 0\}$.

(2) Let $\{X_t : t \geqslant 0\}$ be an $\{\mathcal{F}_t\}$-adapted and right continuous stochastic process. Suppose that for any bounded $\{\mathcal{F}_t\}$-stopping times $\sigma \leqslant \tau$, X_σ, X_τ are integrable and $\mathbb{E}[X_\sigma] \leqslant \mathbb{E}[X_\tau]$. Show that $\{X_t, \mathcal{F}_t\}$ is a submartingale.

Problem 2. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t : t \geqslant 0\})$ be a filtered probability space which satisfies the usual conditions. Suppose that $\mathbb{Q}$ is another probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_t$ for every $t \geqslant 0$.

(1) Let M_t be a version of $d\mathbb{Q}/d\mathbb{P}$ when $\mathbb{P}, \mathbb{Q}$ are restricted on $\mathcal{F}_t$. Show that $\{M_t, \mathcal{F}_t\}$ is a martingale.

(2) Take a càdlàg modification of M_t and still denote it by M_t for simplicity. Show that $\{M_t\}$ is uniformly integrable if and only if $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_\infty$. In this case, we have:

(i) $M_\infty \triangleq \lim_{t\to\infty} M_t = d\mathbb{Q}/d\mathbb{P}$ when restricted on $\mathcal{F}_\infty$,

(ii) for every $\{\mathcal{F}_t\}$-stopping time τ, $\mathbb{Q} \ll \mathbb{P}$ when restricted on $\mathcal{F}_\tau$ and $M_\tau = d\mathbb{Q}/d\mathbb{P}$ on $\mathcal{F}_\tau$.

Problem 3 ($\star$)**.** Let $\{X_t, \mathcal{F}_t\}$ be a right continuous martingale which is bounded in L^p for some $p > 1$ (i.e. $\sup_{0\leqslant t<\infty} \mathbb{E}[|X_t|^p] < \infty$). Show that X_t converges to some X_∞ almost surely and in L^p.

Problem 4. (1) Show that $\log t \leqslant t/e$ for every $t > 0$, and conclude that

$$a \log^+ b \leqslant a \log^+ a + \frac{b}{e}$$

for every $a, b > 0$, where $\log^+ t = \max\{0, \log t\}$ $(t > 0)$.

(2) Suppose that $\{X_t, \mathcal{F}_t : t \geqslant 0\}$ is a non-negative and right continuous submartingale. Let $\rho : [0,\infty) \to \mathbb{R}$ be an increasing and right continuous function with $\rho(0) = 0$. Show that

$$\mathbb{E}[\rho(X_T^*)] \leqslant \mathbb{E}\left[X_T \int_0^{X_T^*} \lambda^{-1} d\rho(\lambda)\right], \quad \forall T > 0,$$

where $X_T^* \triangleq \sup_{t\in[0,T]} X_t$.

(3) By choosing $\rho(t) = (t-1)^+$ $(t \geqslant 0)$, show that

$$\mathbb{E}[X_T^*] \leqslant \frac{e}{e-1}(1 + \mathbb{E}[X_T \log^+ X_T]), \quad \forall T > 0.$$

Problem 5. Suppose that $\{X_t, \mathcal{F}_t : t \geqslant 0\}$ is a continuous martingale vanishing at $t = 0$ and

$$\sup_{t\geqslant 0} X_t(\omega) = \infty, \ \inf_{t\geqslant 0} X_t(\omega) = -\infty, \quad \forall \omega \in \Omega.$$

Define $\tau_0 = 0$, and $\tau_n = \inf\left\{t > \tau_{n-1} : |X_t - X_{\tau_{n-1}}| = 1\right\}$ $(n \geqslant 1)$. Show that τ_n are finite $\{\mathcal{F}_t\}$-stopping times. What is the distribution of the random sequence $\{X_{\tau_n} : n \geqslant 1\}$?

Problem 6. Let $\{X_t, \mathcal{F}_t\}$ be a continuous martingale which is uniformly integrable. Suppose that there exists a constant $M_X > 0$ such that

$$\mathbb{E}[|X_\infty - X_\tau||\mathcal{F}_\tau] \leqslant M_X \quad \text{a.s.},$$

for every $\{\mathcal{F}_t\}$-stopping time τ, where $X_\infty = \lim_{t\to\infty} X_t$ which exists almost surely and in L^1 according to uniform integrability. Let $X^* = \sup_{t\geqslant 0} |X_t|$.

(1) Show that for every $\lambda, \mu > 0$,

$$\mathbb{P}(X^* > \lambda + \mu) \leqslant \frac{M_X}{\mu}\mathbb{P}(X^* > \lambda).$$

(2) By using the result of (1), show that

$$\mathbb{P}(X^* > \lambda) \leqslant \mathrm{e}^{2-\frac{\lambda}{\mathrm{e}\cdot M_X}}, \quad \forall \lambda > 0.$$

In particular, $\mathrm{e}^{\alpha X^*}$ is integrable when $0 < \alpha < (\mathrm{e}M_X)^{-1}$, which also implies that $X^* \in L^p$ for every $p \geqslant 1$.

Problem 7. Let $\{X_t, \mathcal{F}_t : t \geqslant 0\}$ be a càdlàg submartingale over a filtered probability space which satisfies the usual conditions.

(1)($\star$) Suppose that X_t is non-negative, show that X_t is of class (DL). Suppose further that X_t is continuous, show that X_t is regular.

(2) Suppose that X_t is non-negative and uniformly integrable. Show that X_t is of *class (D)* in the sense that $\{X_\tau : \tau \in \mathcal{S}\}$ is uniformly integrable, where $\mathcal{S}$ is the set of finite $\{\mathcal{F}_t\}$-stopping times. Moreover, $A_\infty \triangleq \lim_{t\to\infty} A_t$ is integrable, where A_t is the natural increasing process in the Doob-Meyer decomposition of X_t.