Problem Sheet 2

Due for Submission: 09/16 Friday

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1. Recall that $(W^d, \mathcal{B}(W^d), \rho)$ is the continuous path space over $\mathbb{R}^d$. Let $\{\mathbb{P}_n\}$ be a sequence of probability measures on $(W^d, \mathcal{B}(\mathbb{R}^d))$. Show that $\mathbb{P}_n$ is weakly convergent if and only if the following conditions hold:

(i) the finite dimensional distributions of $\mathbb{P}_n$ are weakly convergent, i.e. for every $\mathfrak{t} = (t_1, \cdots, t_m)$ with $m \geqslant 1$, $t_1 < \cdots < t_m$, the sequence of probability measures

$$Q^n_{\mathfrak{t}}(\Gamma) \triangleq \mathbb{P}_n((w_{t_1}, \cdots, w_{t_m}) \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^{d\times m}),$$

on $(\mathbb{R}^{d\times m}, \mathcal{B}(\mathbb{R}^{d\times m}))$ is weakly convergent;

(ii) the family $\{\mathbb{P}_n\}$ is tight.

Problem 2 ($\star$). Consider the path space $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}\left((\mathbb{R}^d)^{[0,\infty)}\right)\right)$. Let $X_t(w) = w_t$ be the canonical coordinate process.

(1) By using Kolmogorov's extension theorem, show that there exists a unique probability measure $\mathbb{P}$ on $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}\left((\mathbb{R}^d)^{[0,\infty)}\right)\right)$, such that under $\mathbb{P}$,

(i) $X_0 = 0$ almost surely,

(ii) for every $0 \leqslant s < t$, $X_t - X_s$ is normally distributed with mean zero and covariance matrix $(t-s)\mathrm{I}_d$, where I_d is the $d \times d$ identity matrix;

(iii) for every $0 \leqslant t_1 < \cdots < t_n$, the increments $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

(2) Show that there exists a continuous modification $\widetilde{X}_t$ of X_t on $[0,\infty)$, such that for every $0 < \gamma < 1/2$, with probability one, $\widetilde{X}_t$ has γ-Hölder continuous sample paths on every finite interval $[0, T]$.

(3) Let $\widetilde{X}_t$ be the continuous modification of X_t given in (2). Show that with probability one,

$$\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{|\widetilde{X}_t - \widetilde{X}_s|}{|t-s|^{\frac{1}{2}}} = \infty \text{ and } \sup_{\substack{s,t\in[0,\infty)\\ s\neq t}} \frac{|\widetilde{X}_t - \widetilde{X}_s|}{|t-s|^{\gamma}} = \infty,$$

for every $T < \infty$ and $\gamma \in (0, 1/2)$.

Remark. The process $\widetilde{X}_t$ constructed in this problem is called a *d-dimensional pre-Brownian motion*.

Problem 3. (1) Give an example to show that we cannot allow $\beta = 0$ in Kolmogorov's continuity theorem (i.e. Theorem 2.3).

(2) Give an example to show that, under the assumptions in Kolmogorov's continuity theorem, we cannot strengthen the result to conclude that there exists a $\mathbb{P}$-null set outside which every sample path of X is continuous. What if we assume further that every sample path of X is right continuous with left limits?

(3) By adapting the proof of Kolmogorov's continuity theorem, prove the following result.

Let $X_t^{(n)}$ be a sequence of d-dimensional stochastic processes with continuous sample paths such that:

(i) there exist positive constants M and γ, such that

$$\mathbb{E}\left[\left|X_0^{(n)}\right|^\gamma\right] \leqslant M$$

for every n;

(ii) there exist positive constants α, β and M_k for $k \in \mathbb{N}$, such that

$$\mathbb{E}\left[\left|X_t^{(n)} - X_s^{(n)}\right|^\alpha\right] \leqslant M_k |t-s|^{1+\beta}$$

for every n, k and $s, t \in [0, k]$.

Then the sequence of probability measures $\mathbb{P}_n$ on $(W^d, \mathcal{B}(W^d))$ induced by $X_t^{(n)}$ is tight.

Problem 4. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ be a filtered probability space.

(1) Let τ be an $\{\mathcal{F}_t\}$-stopping time and σ be a random time such that $\sigma \geqslant \tau$. Suppose that σ is $\mathcal{F}_\tau$-measurable. Show that σ is an $\{\mathcal{F}_t\}$-stopping time.

(2) Suppose further that $\{\mathcal{F}_t\}$ is right continuous.

(i) Let $\{\tau_n\}$ be a decreasing sequence of $\{\mathcal{F}_t\}$-stopping times, and define $\tau = \lim_{n\to\infty} \tau_n$. Show that τ is an $\{\mathcal{F}_t\}$-stopping time and $\mathcal{F}_\tau = \cap_{n=1}^\infty \mathcal{F}_{\tau_n}$.

(ii)($\star$) Let σ be an $\{\mathcal{F}_t\}$-stopping time. For $t \geqslant 0$, define $\mathcal{G}_t = \mathcal{F}_{\sigma+t}$. Suppose that τ is a $\{\mathcal{G}_t\}$-stopping time. Show that $\sigma + \tau$ is an $\{\mathcal{F}_t\}$-stopping time.

Problem 5. Let X_t be a stochastic process on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with independent increments, i.e. $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent whenever $0 < t_0 < t_1 < \cdots < t_n$. Suppose further that X_t has right continuous sample paths.

(1) For $t \geqslant 0$, show that $\mathcal{U}_t \triangleq \sigma(X_{t+u} - X_t : \ u \geqslant 0)$ and $\mathcal{G}_{t+}^X$ are independent, where $\{\mathcal{G}_t^X\}$ is the natural filtration of X.

(2)($\star$) Let $\mathcal{F}_t^X = \sigma\left(\mathcal{G}_t^X, \mathcal{N}\right)$ be the augmented natural filtration of X_t, where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets. Show that $\{\mathcal{F}_t^X\}$ is right continuous.

Problem 6. Let $(\Omega, \mathcal{F}) = (W^d, \mathcal{B}(W^d))$. Define $X_t(w) = w_t$ to be the coordinate process on $(\Omega, \mathcal{F})$, and $\mathcal{F}_t^X \triangleq \sigma(X_s : \ 0 \leqslant s \leqslant t)$ to be the natural filtration of X_t. Suppose that τ is an $\{\mathcal{F}_t^X\}$-stopping time. Show that

$$\mathcal{F}_\tau^X = \sigma(X_{\tau\wedge t} : \ t \geqslant 0).$$