Problem Sheet 1

Not Required for Submission

You don't need to review the whole subject of probability theory from textbooks in order to do these exercises–it would be more than sufficient if you understand the lecture notes for this part and could still remember some basic real analysis.

You are encouraged to discuss with your classmates whenever you find it helpful.

Problem 1. (1) Establish the following identities for conditional expectations. We use X, Y to denote integrable random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}, \mathcal{H}$ to denote sub-σ-alegras of $\mathcal{F}$.

(i)($\star$) Suppose that X is bounded. Show that $\mathbb{E}[X\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]]$.

(ii)($\star$) Let $f(x, y)$ be a bounded measurable function on $\mathbb{R}^2$. Suppose that X is $\mathcal{G}$-measurable, and Y and $\mathcal{G}$ are independent. Then

$$\mathbb{E}[f(X, Y)|\mathcal{G}] = \varphi(X),$$

where $\varphi(x) \triangleq \mathbb{E}[f(x, Y)]$ for $x \in \mathbb{R}^1$.

(iii) Suppose that $\sigma(\sigma(X), \mathcal{G})$ and $\mathcal{H}$ are independent ($\sigma(X)$ denotes the σ-algebra generated by X). Then

$$\mathbb{E}[X|\mathcal{G}, \mathcal{H}] = \mathbb{E}[X|\mathcal{G}].$$

(2) Let X, Y be two integrable random variables which satisfy

$$\mathbb{E}[X|Y] = Y,\ \mathbb{E}[Y|X] = X,\ \text{a.s.}$$

Show that $\mathbb{P}(X = Y) = 1$.

Problem 2. (1) ($\star$)Suppose that X is an integrable random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{G}_i : i \in \mathcal{I}\}$ is a family of sub-σ-algebras of $\mathcal{F}$. Show that $\{\mathbb{E}[X|\mathcal{G}_i] : i \in \mathcal{I}\}$ is uniformly integrable.

(2) Let $\{X_t : t \in T\}$ be a family of random variables. Suppose that there exists a non-negative Borel-measurable function φ on $[0, \infty)$ such that $\lim_{x\to\infty} \varphi(x)/x = \infty$ and $\sup_{t\in T} \mathbb{E}[\varphi(|X_t|)] < \infty$. Show that $\{X_t : t \in T\}$ is uniformly integrable. In particular, a family of random variables uniformly bounded in L^p ($p > 1$) is uniformly integrable.

Problem 3. Let $\{X_n : n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with exponential distribution:

$$\mathbb{P}(X_n > x) = \mathrm{e}^{-x},\ x \geqslant 0.$$

(1) Compute $\mathbb{P}(X_n > \alpha \log n \text{ for infinitely many } n)$ where $\alpha > 0$ is an arbitrary constant.

(2) Let $L = \limsup_{n\to\infty}(X_n/\log n)$. Show that $L = 1$ almost surely.

(3) Let $M_n = \max_{1\leqslant i\leqslant n} X_i - \log n$. Show that M_n is weakly convergent. What is the weak limiting distribution of M_n?

Problem 4. Prove the equivalence of the first two statements in Theorem 1.6.

Problem 5. Let $\mathbb{P}_n$ be the normal distribution $\mathcal{N}(\mu_n, \sigma_n^2)$ on $\mathbb{R}^1$, where $\mu_n \in \mathbb{R}^1$ and σ_n^2 is nonnegative.

(1) Show that the family $\{\mathbb{P}_n\}$ is tight if and only if the sequences $\{\mu_n\}$ and $\{\sigma_n^2\}$ are bounded.

(2) Show that $\mathbb{P}_n$ is weakly convergent if and only if the sequences $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ for some μ and σ^2. In this case, the weak limit of $\mathbb{P}_n$ is $\mathcal{N}(\mu, \sigma^2)$.