Math 21880: Stochastic Calculus

Fall 2016

12/12/2016
Time: 1:00 pm to 4:00 pm

NAME (PRINT) : ________________

NOTES:

- This is a closed book exam.
- This exam has 4 questions. Each question is worth 25 points for a total of 100.

Question 1. (1) What are the definitions of filtered probability space, stopping time, and continuous time submartingale? State the optional sampling theorem for right continuous submartiangales in the case of bounded stopping times.

(2) Let $\{X_t, \mathcal{F}_t : \ t \geqslant 0\}$ be a right continuous submartingale and let τ be an $\{\mathcal{F}_t\}$-stopping time. Show that the process $\{X_{\tau\wedge t}, \mathcal{F}_t : \ t \geqslant 0\}$ is a submartingale.

(3) (i) Consider the function

$$u(\lambda, x) \triangleq \mathrm{e}^{\lambda x - \frac{1}{2}\lambda^2}, \quad \lambda, x \in \mathbb{R}^1.$$

By writing $u(\lambda, x) = \mathrm{e}^{\frac{1}{2}x^2} \cdot \mathrm{e}^{-\frac{1}{2}(x-\lambda)^2}$, show that

$$u(\lambda, x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x),$$

where $H_n(x)$ satisfies

$$H_n(x) = (-1)^n \mathrm{e}^{\frac{1}{2}x^2} \frac{d}{d^n x} \mathrm{e}^{-\frac{1}{2}x^2}.$$

(ii) Let B_t be a one dimensional Brownian motion with natural filtration $\{\mathcal{F}_t^B\}$. Show that for each $\lambda \in \mathbb{R}^1$, the process

$$\mathcal{E}_t^\lambda \triangleq \mathrm{e}^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \quad t \geqslant 0,$$

is an $\{\mathcal{F}_t^B\}$-martingale. Conclude that for each $n \geqslant 1$, the process

$$M_t^{(n)} \triangleq t^{\frac{n}{2}} H_n\left(\frac{B_t}{\sqrt{t}}\right), \quad t \geqslant 0,$$

is an $\{\mathcal{F}_t^B\}$-martingale.

(iii) Given $r > 0$, define

$$\tau_r \triangleq \inf\{t \geqslant 0 : \ B_t \notin (-r, r)\}.$$

By considering $H_2(x)$ and $H_4(x)$, compute $\mathbb{E}[\tau_r]$ and $\mathbb{E}[\tau_r^2]$.

Question 2. (1) State Itô's formula and Girsanov's theorem under appropriate framework.

(2) Let B_t be a one dimensional $\{\mathcal{F}_t\}$-Brownian motion defined on a filtered probability space. Let μ_t and σ_t be two uniformly bounded, $\{\mathcal{F}_t\}$-progressively measurable processes in $\mathbb{R}^1$.

(i) By using Itô's formula only, show that the unique continuous semimartingale X_t which satisfies

$$X_t = 1 + \int_0^t X_s \mu_s ds + \int_0^t X_s \sigma_s dB_s, \quad t \geqslant 0,$$

is given by

$$X_t = \exp\left(\int_0^t \sigma_s dB_s + \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right), \quad t \geqslant 0.$$

(ii) Assume further that $\sigma \geqslant C$ for some constant $C > 0$. Given $T > 0$, construct a probability measure $\widetilde{\mathbb{P}}_T$ on $\mathcal{F}_T$ which is equivalent to $\mathbb{P}$, such that $\{X_t, \mathcal{F}_t : \ 0 \leqslant t \leqslant T\}$ is a continuous local martingale under $\widetilde{\mathbb{P}}_T$.

(3) Let $X_t = (X_t^1, X_t^2)$ be the unique solution to the SDE

$$\begin{cases} dX_t^1 = 2dt + dB_t^1 + dB_t^2, & 0 \leqslant t \leqslant T; \\ dX_t^2 = 4dt + dB_t^1 - dB_t^2, & 0 \leqslant t \leqslant T; \\ X_0 = x \in \mathbb{R}^2, \end{cases}$$

on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t : \ 0 \leqslant t \leqslant T\})$. Find a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, under which $\{X_t, \mathcal{F}_t\}$ is a martingale.

Question 3. (1) What does it mean by saying that the SDE

$$dX_t = \alpha(t, X)dB_t + \beta(t, X)dt$$

is exact? What are the definitions of weak solution and pathwise uniqueness?

(2) Let B_t be a one dimensional Brownian motion.

(i) Define

$$\tau_1 \triangleq \inf\{t \geqslant 0 : \ B_t = 1\}.$$

By using the reflection principle of Brownian motion, compute the probability density of τ_1.

(ii) By considering $Y_t \triangleq X_t^{-1/2}$, solve the SDE

$$\begin{cases} dX_t = 3X_t^2 dt - 2|X_t|^{\frac{3}{2}} dB_t, \\ X_0 = 1, \end{cases}$$

explicitly on any given set-up. Conclude that $\mathbb{P}(e < \infty) = 1$ and $\mathbb{E}[e] = \infty$, where e is the explosion time of X.

(3) Solve the one dimensional SDE

$$\begin{cases} dX_t = (\alpha X_t^{1+\theta} + \beta X_t)dt + \gamma X_t dB_t, \\ X_0 = x > 0, \end{cases}$$

explicitly on any given set-up, where $\alpha, \beta, \gamma \in \mathbb{R}^1$ and $\theta > 0$. Conclude that if $\alpha \leqslant 0$, then the solution does not explode in finite time.

Question 4. (1) Let $\gamma_t = (x_t, y_t)$ be a C^1-path. What is the geometric meaning of the integral

$$\frac{1}{2}\int_0^t (x_s dy_s - y_s dx_s)?$$

(2) Let $B_t = (B_t^1, B_t^2)$ be a two dimensional Brownian motion. Define the martingale

$$L_t \triangleq \frac{1}{2}\int_0^t (B_s^1 dB_s^2 - B_s^2 dB_s^1), \quad t \geqslant 0.$$

L_t is known as Lévy's stochastic area process. The three dimensional process (B_t, L_t) is called the Brownian motion on the Heisenberg group.

This part of the question is devoted to computing the characteristic function

$$\mathrm{ch}_t(\lambda) \triangleq \mathbb{E}[\mathrm{e}^{i\lambda L_t}], \quad \lambda \in \mathbb{R}^1,$$

of L_t for fixed $t > 0$.

(i) Given $\alpha > 0$ and $t > 0$, find the unique solution $\{g(s) : \ 0 \leqslant s \leqslant t\}$ to the ODE

$$\begin{cases} \frac{1}{2}g'(s) + \frac{1}{2}g^2(s) = \alpha, \quad 0 \leqslant s \leqslant t, \\ g(t) = 0. \end{cases}$$

(ii) Let b_t be a one dimensional Brownian motion. By using the result in part (i) and Itô's formula for some appropriate function $F(s,x)$, compute the expectation

$$\mathbb{E}\left[\exp\left(-\alpha\int_0^t b_s^2 ds\right)\right]$$

for given $\alpha > 0$ and $t > 0$. [You may assume that α is small enough so that $\mathbb{E}[\exp(\alpha\int_0^t b_s^2 ds)]$ is finite.]

(iii) Let C_t be the time-change associated with $\langle L\rangle_t$. Show that the process $\{W_t \triangleq L_{C_t} : t \geqslant 0\}$ is a Brownian motion independent of the process $\{\rho_t \triangleq (B_t^1)^2 + (B_t^2)^2 : t \geqslant 0\}$.

(iv) By using the result in part (iii), show that

$$\mathrm{ch}_t(\lambda) = \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{8}\int_0^t \rho_s ds\right)\right], \quad \forall \lambda \in \mathbb{R}^1.$$

(v) Combining the previous results, conclude that

$$\mathrm{ch}_t(\lambda) = \frac{1}{\cosh(\lambda t/2)}, \quad \lambda \in \mathbb{R}^1,$$

where $\cosh(x) = (\mathrm{e}^x + \mathrm{e}^{-x})/2$ is the hyperbolic cosine function.