Course Plan: 21880–Stochastic Calculus

Instructor: Xi Geng
Fall 2016

1 Course description

Roughly speaking, in a classical sense stochastic calculus means the calculus for Brownian motion, or more generally, the calculus for semimartingales. A central motivation of stochastic calculus is the following. In classical analysis, the differential equation

$$dx_t = \mu(x_t)dt \tag{1}$$

describes the time evolution of a deterministic nonlinear system. However, it is generally interesting and important to ask the question about what happens if a random perturbation is presented in the description of system (1). Formally the resulting system can be written as

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \tag{2}$$

where dB_t represents some kind of random perturbation. The ultimate goal of this course is to give rigorous mathematical meaning to the random system (2) and to study important properties of its solutions.

We start our journey by a short review on discrete probability theory in a measure-theoretic flavor, followed by the basic notions of continuous time stochastic processes. Here a core concept which is new to us is a mathematical way of describing information up to certain amount of time which can possibly be random. Keeping track of information plays a fundamental role in the theory. After that, we introduce the powerful tool of martingale methods, which leads to an elegant intrinsic construction of stochastic integrals. As a fundamental example in the theory, we then spend some time studying the Brownian motion in a very much martingale flavor. The rest of the course will be devoted to the study of stochastic integration and differential equations. If it has to be one, a core result of the theory will be Itô's formula, and a core technique of our study will be martingale methods.

2 Course syllabus

2.1 Review of probability theory

- Conditional expectations
- Uniform integrability
- The Borel-Cantelli lemma

- The law of large numbers and the central limit theorem
- Weak convergence of probability measures

2.2 Generalities on continuous time stochastic processes

- Construction of stochastic processes: Kolmogorov's extension theorem
- Kolmogorov's continuity theorem
- Filtrations and stopping times

2.3 Continuous time martingales

- The martingale transform–discrete stochastic integration
- The martingale convergence theorems
- Doob's optional sampling theorems
- Doob's martingale inequalities
- Doob's regularization theorems
- The Doob-Meyer decomposition theorem

2.4 Brownian motion

- Invariance properties of Brownian motion
- The strong Markov property
- The reflection principle
- The Skorokhod embedding theorem
- The Donsker invariance principle
- Passage time distributions: Laplace transforms and densities
- Sample path properties: oscillations, irregularity and the p-variation of Brownian motion

2.5 Stochastic Integration

- Square-integrable martingales
- Construction of stochastic integrals
- Itô's formula
- The Burkholder-Davis-Gundy inequality
- Lévy's characterization of Brownian motion

- Continuous local martingales as time-changed Brownian motions
- Martingale representation theorems
- The Cameron-Martin-Girsanov transformation
- The Brownian local time and Tanaka's formula

2.6 Stochastic differential equations and diffusion processes

- Definitions of existence and uniqueness
- Existence results
- Uniqueness results
- Two probabilistic methods: transformation of drift and time change
- Diffusion processes
- Examples: linear diffusions and Bessel diffusions
- Connections with partial differential equations: the Dirichlet problem and the Feynman-Kac formula

3 Homework and grading

3.1 Homework

There will be 6 problem sheets in total, one for each section in the syllabus. Due dates for submission are stated on the problem sheets. $3 \times 80\mathrm{min}$ classes will be devoted to the discussion of solutions.

You are encouraged to discuss with your classmates in order to solve the problems. But solutions should be written down by yourself. Problems that are marked by a "$\star$" will be used in the lecture notes, hence you are strongly encouraged to have a full understanding on these problems.

3.2 Grading

There will not be a mid-term exam. Final Grade = Homework (40%)+Final exam (60%).

4 Office hours

Office hours will be Fridays 3:30–4:30 pm every week in Room 7109. But you are very welcome to send me emails via *xigeng2015@gmail.com* at any time if you have questions.

5 References

For the review of discrete probability theory:

[1] K.L. Chung, *A course in probability theory*, 3rd edition, Elsevier, 2010.

[2] D. Williams, *Probability with martingales*, Cambridge University Press, 1991.

For stochastic calculus:

[3] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes,* 2nd edition, North-Holland, 1989.

[4] I.A. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus,* 2nd edition, Springer-Verlag,1991.

[5] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd edition, Springer, 2004.

[6] L.C.G. Rogers and D. Williams, *Diffusions, Markov processes and martingales*, Volume 1 and 2, 2nd edition, John Wiley & Sons, 1994.