Notes on Differentiable Manifolds and de Rham Cohomology

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Contents

1 Fundamentals of differentiable manifolds

1.1 Definition of differentiable manifold.

In the present notes, we will be in the realm of differentiable manifolds. Intuitively, a differentiable manifold locally looks like a piece of Euclidean space, but globally those pieces are glued together in a smooth manner.

Let M be a non-empty set.

Definition 1.1. An (*n-dimensional*) *coordinate chart* on M is a non-empty subset U of M together with a bijection $\varphi:\ U \to \varphi(U) \subset \mathbb{R}^n$ onto some open subset $\varphi(U)$ of $\mathbb{R}^n$. An (*n-dimensional*) *atlas* on M is a family of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that

(1) M is covered by $\{U_\alpha\}_{\alpha\in A}$;
(2) for any $\alpha, \beta \in A$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$;
(3) (C^∞ compatibility) for any $\alpha, \beta \in A$, if $U_\alpha \cap U_\beta \neq \emptyset$, then the map

$$\varphi_\beta \circ \varphi_\alpha^{-1}:\ \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is C^∞ (has continuous partial derivatives of all order) with C^∞ inverse.

The previous definition illustrates what the space M looks like locally and globally in the mathematical way, if it is equipped with an atlas. This is very close to the description of a differentiable manifold. However, the precise definition should be independent of the choice of atlas (the way of parametrizing M) in some sense.

Definition 1.2. Two atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are *compatible* if their union is again an atlas.

Compatibility is nothing but an additional requirement that

$$\psi_i \circ \varphi_\alpha^{-1}:\ \varphi_\alpha(U_\alpha \cap V_i) \to \psi_i(U_\alpha \cap V_i)$$

is C^∞ with C^∞ inverse provided $U_\alpha \cap V_i \neq \emptyset$. Compatibility is obviously an equivalence relation on the set of atlases.

Now we can give the definition of differentiable manifold.

Definition 1.3. A $differential$ *structure* on M is an equivalence class of atlases. A space M equipped with a differential structure is called a $differentiable$ $manifold$. The dimension of the differential structure (i.e., of any atlas in the equivalence class) is called the *dimension* of the differentiable manifold M.

Since we are always working with differentiable manifolds in this note, from now on a differentiable manifold will be simply called a manifold.

Remark 1.1. By Definition 1.3, to specify a differential structure on a space M which makes M a manifold, it suffices to assign an atlas on M.

Now we give some examples of manifolds, which will serve as fundamental spaces we are going to study in this note.

Example 1.1. The Euclidean space $\mathbb{R}^n$. We can choose an atlas which consists of only one coordinate chart $U = \mathbb{R}^n$, and $\varphi = \text{id}$. Then $\mathbb{R}^n$ becomes an n-dimensional manifold. Similarly for open subsets of $\mathbb{R}^n$. When we talk about the open subsets of $\mathbb{R}^n$, we always use this canonical atlas.

Example 1.2. The unit circle

$$S^1 = \{(x^0, x^1) \in \mathbb{R}^2:\ (x^0)^2 + (x^1)^2 = 1\}.$$

We can choose four coordinate charts

$$\begin{array}{rcl} U & = & \{(x^0,x^1)\in S^1:\ x^1>0\},\ \varphi_U((x^0,x^1))=x^0;\\ D & = & \{(x^0,x^1)\in S^1:\ x^1<0\},\ \varphi_D((x^0,x^1))=x^0;\\ L & = & \{(x^0,x^1)\in S^1:\ x^0<0\},\ \varphi_L((x^0,x^1))=x^1;\\ R & = & \{(x^0,x^1)\in S^1:\ x^0>0\},\ \varphi_R((x^0,x^1))=x^1, \end{array}$$

each of which is parametrized by a copy of the open interval $(-1,1)$. It is easy to check that these coordinate charts together form an atlas of S^1. Then S^1 becomes a one dimensional manifold. Similarly, the n-sphere

$$S^n=\{(x^0,x^1\cdots,x^n)\in\mathbb{R}^{n+1}:\ (x^0)^2+(x^1)^2+\cdots+(x^n)^2=1\}$$

is an n-dimensional manifold by specifying an atlas consisting of $2n$ coordinate charts, each of which is parametrized by the open unit disk in $\mathbb{R}^n$.

Example 1.3. The product manifold. Let M and N be manifolds of dimension m and n with atlases $\{(U_\alpha,\varphi_\alpha)\}$ and $\{(V_i,\psi_i)\}$ defining their differential structures respectively. Then $M\times N$ is a manifold of dimension $m+n$ with differential structure given by the atlas $\{(U_\alpha\times V_i,\zeta_{\alpha,i})\}$, where $\zeta_{\alpha,i}$ is the map

$$\begin{array}{rcl} \zeta_{\alpha,i}:\ U_\alpha\times V_i & \to & \varphi_\alpha(U_\alpha)\times\psi_i(V_i)\subset\mathbb{R}^m\times\mathbb{R}^n,\\ (x,y) & \mapsto & (\varphi_\alpha(x),\psi_i(y)). \end{array}$$

A particular example is the n-dimensional torus $T^n=S^1\times\cdots\times S^1$.

Example 1.4. The n-dimensional real projective space $\mathbb{R}P^n$: the space of one dimensional linear subspaces (real lines through the origin) of $\mathbb{R}^{n+1}$. More precisely, define an equivalence relation " $\sim$ " on $\mathbb{R}^{n+1}\backslash\{0\}$ by

$$x\sim y\iff y=\lambda x \text{ for some nonzero } \lambda\in\mathbb{R}.$$

Then $\mathbb{R}P^n$ is defined to be the space of $\sim$-equivalence classes. For $i=0,\cdots,n$, let

$$U_i=\{[x]:\ x=(x^0,\cdots,x^n)\in\mathbb{R}^{n+1}\backslash\{0\} \text{ with } x^i\neq 0\},$$

where $[x]$ denotes the $\sim$-equivalence class of x, and define the map

$$\begin{array}{rcl} \varphi_i:\ U_i & \to & \mathbb{R}^n,\\ {[x]} & \mapsto & ({}_i\xi^0,\cdots,{}_i\xi^{i-1},{}_i\xi^{i+1},\cdots,{}_i\xi^n), \end{array}$$

where ${}_i\xi^j=x^j/x^i$ for $j\neq i$. Note that φ_i is well-defined (independent of the choice of representatives in $[x]$). Moreover, φ_i is a bijection so that (U_i,φ_i) is a coordinate chart on $\mathbb{R}P^n$. Finally, the change of coordinates on $U_i\cap U_j$ $(i\neq j)$ is given by

$$\begin{cases} {}_j\xi^h=\frac{{}_i\xi^h}{{}_i\xi^j}, & h\neq i,j;\\ {}_j\xi^i=\frac{1}{{}_i\xi^j}, \end{cases}$$

which is clearly C^∞ with C^∞ inverse. Therefore, $\{(U_i,\varphi_i)\}_{i=0,\cdots,n}$ is an atlas on $\mathbb{R}P^n$, which makes $\mathbb{R}P^n$ an n-dimensional manifold.

Example 1.5. The n-dimensional complex projective space $\mathbb{C}P^n$. Similar to the real projective space, $\mathbb{C}P^n$ is defined to be the space of one dimensional complex linear subspaces of $\mathbb{C}^{n+1}$, namely, the space of $\sim$-equivalence classes on $\mathbb{C}^{n+1}\backslash\{0\}$ where

$$z \sim w \iff w = \lambda z \text{ for some nonzero } \lambda \in \mathbb{C}.$$

As in the case of $\mathbb{R}P^n$, we can construct an atlas consisting of $n+1$ coordinate charts parametrized by a copy of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This makes $\mathbb{C}P^n$ a $2n$-dimensional manifold. In fact, $\mathbb{C}P^n$ is a *complex* manifold of dimension n since it is easy to see that the change of coordinates is always holomorphic with holomorphic inverse.

There will be more examples of manifolds arising from C^∞ maps between manifolds, matrix groups and fiber bundles etc., as we shall see later on.

The differential structure of a manifold M induces a canonical topology on M, called the *manifold topology*, defined as follows.

Take an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ in the differential structure of M. Then for any $U \subset M$, U is defined to be open if and only if

$$\varphi_\alpha(U \cap U_\alpha) \text{ is open in } \varphi_\alpha(U_\alpha) \text{ for any } \alpha \in A.$$

We leave it as an exercise to check that this defines a topology on M, and it is independent of the choice of atlases in the given differential structure.

Proposition 1.1. *Under the manifold topology, for any $\alpha \in A$, U_α is open, and $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism.*

Proof. The fact that U_α is open follows directly from the definition of atlas. The fact that φ_α maps open sets in U_α to open sets in $\varphi_\alpha(U_\alpha)$ follows from the openness of U_α in M and the definition of open sets in M. To see that φ_α is continuous, let W be an open subset of $\varphi_\alpha(U_\alpha)$, and

$$U = \varphi_\alpha^{-1}(W) \subset U_\alpha.$$

Then for any $\beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{aligned}
\varphi_\beta(U \cap U_\beta) &= \varphi_\beta(U \cap U_\alpha \cap U_\beta) \\
&= \varphi_\beta \circ \varphi_\alpha^{-1}(\varphi_\alpha(U \cap U_\alpha \cap U_\beta)) \\
&= \varphi_\beta \circ \varphi_\alpha^{-1}(W \cap \varphi_\alpha(U_\alpha \cap U_\beta)).
\end{aligned}$$

Since $U \cap U_\beta$ is open in M, we know that $\varphi_\alpha(U \cap U_\beta \cap U_\alpha)$ is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$. By the definition of atlas,

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism and hence a homeomorphism. Therefore, $\varphi_\beta(U \cap U_\beta)$ is open in $\varphi_\beta(U_\alpha \cap U_\beta)$, and hence open in $\varphi_\beta(U_\beta)$. $\square$

So far there is not much about the manifold topology and geometry we can say for a manifold M, and little analysis we can do on M if we don't impose additional conditions on the manifold topology. Throughout the rest of this note, we will make the following assumption on the manifold topology:

- the manifold topology is Hausdorff and has a countable base of open sets (second countability axiom).

It is easy to verify that all examples given before satisfy this assumption (in Example 1.3, assume that M and N both satisfy this assumption). Moreover, the manifold topology of S^n coincides with the relative topology as a subset of $\mathbb{R}^{n+1}$, and the manifold topology of $\mathbb{R}P^n$ ($\mathbb{C}P^n$, respectively) coincides with the quotient topology on $(\mathbb{R}^{n+1}\backslash\{0\})/\sim$ $((\mathbb{C}^{n+1}\backslash\{0\})/\sim$, respectively).

Remark 1.2. In some textbooks (for example [2], [5]), the definition of differentiable manifold starts from a Hausdorff space M with a countable topological basis. In this situation, a coordinate chart is required to be a homeomorphism from a non-empty open subset of M onto some open subset of $\mathbb{R}^n$. Once a differential structure is given, it is not hard to show that the manifold topology is the same as the original topology of M.

From now on, when we talk about topological properties, we always use the manifold topology.

As in the case of Euclidean spaces, it makes perfect sense to talk about C^∞ functions on a manifold and C^∞ maps between manifolds.

Definition 1.4. Let M be a manifold. A function $f: M \to \mathbb{R}$ is said to be C^∞ *at* $p \in M$ if

$$f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}$$

is C^∞ at $\varphi(p)$ (there exists some open neighborhood W of $\varphi(p)$ in $\varphi(U)$, such that $f\varphi^{-1}$ has continuous partial derivatives of all order at any point in W) for some coordinate chart (U, φ) around p. f is said to be C^∞ if it is C^∞ at every point of M. The space of C^∞ functions on M is denoted by $C^\infty(M)$.

A map $F: M \to N$ between two manifolds M, N is said to be C^∞ *at* $p \in M$ if there exists coordinate charts (U, φ) around p and (V, ψ) around $f(p)$ such that $F(U) \subset V$ and

$$\psi \circ F \circ \varphi^{-1}: \varphi(U) \to \psi(V)$$

is C^∞ at $\varphi(p)$. F is said to be C^∞ if it is C^∞ at every point of M. Two manifolds M and N are said to be $diffeomorphic$ if there exists some bijection $F: M \to N$ such that F, F^{-1} are both C^∞.

Remark 1.3. The previous definition of smoothness at p does not depend on the choice of coordinate charts.

It is easy to show that C^∞ functions and C^∞ maps are always continuous with respect to the manifold topology.

Example 1.6. An important example of C^∞ functions is a *bump function* on a manifold M. More precisely, for any open sets $U, V \subset M$ with $\overline{U}$ compact and $\overline{U} \subset V$, there exists some $f \in C^\infty(M)$, such that

$$f(x) = \begin{cases} 1, & x \in \overline{U}; \\ 0, & x \notin V. \end{cases}$$

The precise construction is in Problem Sheet 1.

Example 1.7. In Example 1.3, the natural projections

$$\pi_1 : \ M \times N \to M, \ \pi_2 : \ M \times N \to N$$

given by

$$\pi_1(x, y) = x, \ \pi_2(x, y) = y$$

are C^∞. In Example 1.4 and 1.5, the corresponding quotient maps are C^∞.

Example 1.8. An important example of C^∞ maps is a *smooth curve* on a manifold M: a C^∞ map from some open interval $I \subset \mathbb{R}$ to M.

Remark 1.4. It is possible that a space M has two distinct differential structures which are diffeomorphic (so they induce the same manifold topology). It is also possible that M has two distinct non-diffeomorphic differential structures with the same manifold topology (the Milnor's exotic sphere).

Now we present a fundamental tool in the study of manifolds: partition of unity. It is particularly important when we develop analysis on a manifold.

Definition 1.5. A collection $\{A_\alpha\}$ of subsets of a manifold M is called *locally finite* if for any $p \in M$, there exists some neighborhood U of p, such that $U \cap A_\alpha \neq \emptyset$ for only finitely many α.

Definition 1.6. A *partition of unity* on M is a family $\{\varphi_i\}_{i \in I}$ of C^∞ functions such that

(1) for any $i \in I$, $0 \leqslant \varphi_i \leqslant 1$;

(2) the collection of supports $\{\text{supp}\varphi_i = \overline{\{x \in M : \ \varphi_i(x) \neq 0\}}\}_{i \in I}$ is locally finite;

(3) for any $p \in M$, $\sum_{i \in I} \varphi_i(p) = 1$ (this is in fact a finite sum according to (2)).

The following theorem is the existence of partitions of unity on a manifold M. One can refer to [5] for its proof.

Theorem 1.1. *Let M be a manifold and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then*

(1) there exists a countable partition of unity $\{\varphi_i : i = 1, 2, \cdots\}$ such that for any $i \geqslant 1$, suppφ_i is compact and there exists some $\alpha \in A$ with supp$\varphi_i \subset U_\alpha$ (this is called subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$);

(2) there exists a partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ such that at most countably many φ_α are not identically zero and for any $\alpha \in A$, supp$\varphi_\alpha \subset U_\alpha$ (this is called subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$ with the same index).

1.2 The tangent space

On a smooth curve or surface sitting inside $\mathbb{R}^3$, one can define tangent lines or tangent planes in a natural way. However, although geometrically intuitive such definition is extrinsic and cannot be carried to an arbitrary manifold. We need an intrinsic description of tangent vectors and tangent spaces. The idea is to regard a tangent vector as taking directional derivatives or as a linear derivation of locally C^∞ functions. These two approaches are in fact equivalent.

We start with the notion of cotangent space.

Let M be an n-dimensional manifold.

Fix $p \in M$. Let C_p^∞ be the space of C^∞ functions defined on some open neighborhood of p. It makes sense to talk about addition and multiplication on C_p^∞ (for example, if f, g are defined on U, V respectively, then $f+g$, fg are defined on $U \cap V$). Introduce an equivalence relation " $\sim$ " on C_p^∞ by

$$f \sim g \iff f = g \text{ on some open neighborhood of } p.$$

Definition 1.7. The space of $\sim$-equivalence classes, denoted by $\mathcal{F}_p$, is called the space of C^∞ *germs* at p.

By acting on representatives, $\mathcal{F}_p$ is an (infinite dimensional) algebra over $\mathbb{R}$. Moreover, from the definition it makes sense to talk about the evaluation of a C^∞ germ $[f]$ at p as $f(p)$.

If we think of a tangent vector as taking directional derivatives, reasonably it should act on C^∞ germs linearly. Moreover, from the Euclidean case we know that two functions with the same first order partial derivatives at p have the same directional derivatives at p along any direction. Therefore, a tangent vector, regarded as taking directional derivatives, should be defined as a linear functional on a quotient space over $\mathcal{F}_p$ in which two C^∞ germs with the same first order partial derivatives at p are identified.

Let $\mathcal{H}_p$ be the set of C^∞ germs $[f] \in \mathcal{F}_p$ such that for some coordinate chart (U, φ) around p,

$$\frac{\partial}{\partial x^i}|_{\varphi(p)} f\varphi^{-1}(x^1, \cdots, x^n) = 0, \text{ for all } i = 1, \cdots, n.$$

By the chain rule of calculus, it is easy to see that such definition is independent of the choice of coordinate charts around p. $\mathcal{H}_p$ is a linear subspace of $\mathcal{F}_p$.

Definition 1.8. The quotient space $\mathcal{F}_p/\mathcal{H}_p$ is called the *cotangent space* of M at p, denoted by T_p^*M. An element of T_p^*M, which is an $\mathcal{H}_p$-equivalence class $[f] + \mathcal{H}_p$ over $\mathcal{F}_p$, is called a *cotangent vector* at p, denoted by $(df)_p$.

Theorem 1.2. *Let $f^1, \cdots, f^s \in C_p^\infty$, and $F(y^1, \cdots, y^s)$ be a C^∞ function on some open neighborhood of $(f^1(p), \cdots, f^s(p)) \subset \mathbb{R}^s$. Then $f = F(f^1, \cdots, f^s) \in C_p^\infty$, and*

$$(df)_p = \sum_{i=1}^{s} \frac{\partial F}{\partial y^i}|_{(f^1(p), \cdots, f^s(p))} (df^i)_p.$$

Proof. By the definition of $\mathcal{H}_p$ and T_p^*M, it suffices to show that for some coordinate chart (U,φ) around p, $f\circ\varphi^{-1}$ and $\sum_{i=1}^s \frac{\partial F}{\partial y^i}|_{(f^1(p),\cdots,f^s(p))} f^i\circ\varphi^{-1}$ have the same first order partial derivatives at $\varphi(p)$. But this is obvious from the chain rule of calculus. □

A immediate corollary of Theorem 1.2 is:

Corollary 1.1. *For any $f,g\in C_p^\infty$ and $\alpha\in\mathbb{R}$,*
(1) $d(f+g)_p=(df)_p+(dg)_p$;
(2) $d(\alpha f)_p=\alpha(df)_p$;
(3) $d(fg)_p=f(p)(dg)_p+g(p)(df)_p$.

Another important corollary of Theorem 1.2 is the following.

Corollary 1.2. *T_p^*M is an n-dimensional vector space, where n is the dimension of M. Moreover, for any coordinate chart (U,φ) around p,*

$$\{(dx^1)_p,\cdots,(dx^n)_p\}$$

is a basis of T_p^*M (called the natural basis under (U,φ)), and for any $f\in C_p^\infty$,

$$(df)_p=\sum_{i=1}^n \frac{\partial f\varphi^{-1}}{\partial x^i}|_{\varphi(p)}(dx^i)_p, \tag{1.1}$$

where $x^i\in C_p^\infty$ is the i-th coordinate function under (U,φ).

Proof. Fix a coordinate chart (U,φ) around p. For any $f\in C_p^\infty$, we may write

$$f=f\circ\varphi^{-1}(x^1,\cdots,x^n),$$

where we regard $x^i\in C_p^\infty$ as coordinate functions defined on U. It follows from Theorem 1.2 that

$$(df)_p=\sum_{i=1}^n \frac{\partial f\circ\varphi^{-1}}{\partial x^i}|_{\varphi(p)}(dx^i)_p.$$

Therefore, T_p^*M is spanned by $\{(dx^1)_p,\cdots,(dx^n)_p\}$.

Moreover, assume that

$$\sum_{i=1}^n \lambda_i(dx^i)_p=0 \quad (\text{in } T_p^*M)$$

for some $\lambda_i\in\mathbb{R}$. By definition this means that under (U,φ), $\sum_{i=1}^n\lambda_i x^i$ and the zero function have the same first order partial derivatives at $\varphi(p)$. Note that here $x^1,\cdots,x^n$ are variables of the function $\sum_{i=1}^n\lambda_i x^i$ defined on $\varphi(U)$. By taking partial derivatives, it follows immediately that

$$\lambda_i=0,\ \text{for all } i.$$

Therefore, $(dx^1)_p,\cdots,(dx^n)_p$ are linearly independent. □

Now we are able to give the definition of tangent space.

Definition 1.9. The dual space of T_p^*M is called the *tangent space* of M at p, denoted by T_pM. Elements of T_pM are called *tangent vectors* at p.

Let (U, φ) be a coordinate chart around p. We use

$$\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^n}|_p\}$$

to denote the dual basis of $\{(dx^1)_p, \cdots, (dx^n)_p\}$. This is the natural basis of $T_p(M)$ under (U, φ). Moreover, for any $f \in C_p^\infty$, we have

$$\langle\frac{\partial}{\partial x^i}|_p, (df)_p\rangle = \sum_{j=1}^n \frac{\partial f \circ \varphi^{-1}}{\partial x^j}|_{\varphi(p)}\langle\frac{\partial}{\partial x^i}|_p, (dx^j)_p\rangle = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}|_{\varphi(p)}.$$

Therefore, $\frac{\partial}{\partial x^i}|_p$ can be regarded as the i-th partial derivative operator when acting on C^∞ germs.

In general, for any tangent vector $v \in T_pM$, v can be equivalently viewed as the differential operator taking directional derivatives along the "direction" v. This point will become clearer in Problem 5 of Problem Sheet 1, where the tangent space T_pM is identified to be the space of equivalence classes of smooth curves through p representing "directions".

Example 1.9. For any point $x \in \mathbb{R}^n$, $T_x\mathbb{R}^n$ is canonically identified as $\mathbb{R}^n$ since $\mathbb{R}^n$ is parametrized by one natural coordinate chart and $\{\frac{\partial}{\partial x^1}|_x, \cdots, \frac{\partial}{\partial x^n}|_x\}$ is the universal natural basis of $T_x\mathbb{R}^n$ for all $x \in \mathbb{R}^n$.

Remark 1.5. In the rest of the notes, sometimes we may use $(U; x^i)$ to denote a coordinate chart in order to emphasize the coordinates, or simply use U. We may also frequently drop the notion of φ when doing calculation in a coordinate chart (U, φ). So for example, we may simply write (1.1)

$$(df)_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p)(dx^i)_p.$$

Equivalently, we can define tangent vector as linear derivation over $\mathcal{F}_p$ (the space of C^∞ germs).

Theorem 1.3. *Let $\widetilde{T}_pM$ to be the vector space of linear derivations over $\mathcal{F}_p$, i.e., the space of linear functionals $X: \mathcal{F}_p \to \mathbb{R}$ such that*

$$X([f] \cdot [g]) = f(p)X([g]) + g(p)X([f]).$$

Define the map

$$\Phi: \widetilde{T}_pM \to T_pM$$

by

$$\langle\Phi(X), (df)_p\rangle := X([f]), \ (df)_p \in T_p^*M,$$

where $[f]$ is any representative of $(df)_p$. Then Φ is well-defined, and it is a linear isomorphism.

Proof. To see that Φ is well-defined, it suffices to prove: for any $X \in \tilde{T}_p M$, if $[f] \in \mathcal{H}_p$, then $X([f]) = 0$.

In fact, assume that $[f] \in \mathcal{H}_p$ and take some representative $f \in C_p^\infty$. Choose a convex coordinate chart (U, φ) on which f is defined (this is always possible by shrinking an arbitrary coordinate chart). Then by definition we know that

$$\frac{\partial f}{\partial x^i}(p) = 0, \ \forall i = 1, \cdots, n.$$

For any $x \in U$, by convexity and the fundamental theorem of calculus, we have

$$f(x) = f(a) + \sum_{i=1}^{n} \int_0^1 \frac{\partial f}{\partial x^i}((1-t)p + tx)(x^i - p^i)dt,$$

where $(p^1, \cdots, p^n)$ denotes the coordinate of p under (U, φ). Now for any $i = 1, \cdots, n$, define $g_i, h^i \in C_p^\infty$ by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}((1-t)p + tx)dt, \ h^i(x) = x^i - p^i.$$

It follows that

$$f(x) = f(a) + \sum_{i=1}^{n} g_i(x) h^i(x),$$

and

$$g_i(p) = h^i(p) = 0, \ \forall i = 1, \cdots, n.$$

On the other hand, since

$$X([1]) = X([1] \cdot [1]) = 2X([1])$$

and

$$X([c]) = X(c[1]) = cX([1]), \ \forall c \in \mathbb{R},$$

we know that X annihilates constant germs. Therefore, by linearity and the derivation property, we have

$$X([f]) = 0.$$

The linearity and injectivity of Φ follows directly from definition.

To see that Φ is surjective, let $v \in T_p M$, and define

$$X : \ \mathcal{F}_p \to \mathbb{R}$$

by

$$X([f]) := \langle v, (df)_p \rangle,$$

Then by Corollary 1.1 we know that $X \in \tilde{T}_p M$. It is then obvious that

$$v = \Phi(X).$$

Therefore, Φ is a linear isomorphism. □

One might wonder how the natural basis of T_p^*M or T_pM transforms under change of coordinates. In fact, this is an easy consequence of Corollary 1.2.

Proposition 1.2. *If (U, φ) and (V, ψ) are two coordinate charts around p, then for any $i = 1, \cdots, n$,*

$$\begin{aligned} (dy^i)_p &= \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j}(p)(dx^j)_p, \\ \frac{\partial}{\partial y^i}|_p &= \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^i}(p)\frac{\partial}{\partial x^j}|_p, \end{aligned}$$

where $(x^1, \cdots, x^n)$ and $(y^1, \cdots, y^n)$ are coordinates under (U, φ) and (V, ψ) respectively.

Therefore, if $\alpha \in T_p^*M$ with

$$\alpha = \sum_{i=1}^{n} \lambda_i (dx^i)_p = \sum_{j=1}^{n} \mu_j (dy^j)_p,$$

then

$$\mu_j = \sum_{i=1}^{n} \lambda_i \frac{\partial x^i}{\partial y^j}, \ \forall j = 1, \cdots, n.$$

Similarly, if $v \in T_pM$ with

$$v = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_p = \sum_{j=1}^{n} b^j \frac{\partial}{\partial y^j}|_p,$$

then

$$b^j = \sum_{i=1}^{n} a^i \frac{\partial y^j}{\partial x^i}.$$

1.3 The differential of C^∞ map and submanifolds

The notion of cotangent and tangent spaces enables us to linearize a C^∞ map locally. This is crucial in the study of manifolds.

Definition 1.10. Let $F: M \to N$ be a C^∞ map between manifolds M, N. Fix $p \in M$ and let $q = F(p)$. The *pullback of cotangent vectors by F at p* is the linear map

$$F^*: T_q^*N \to T_p^*M$$

defined by

$$F^*((df)_q) = (d(f \circ F))_p.$$

The *differential of F* at p is the dual map of F^*, usually denoted by $(dF)_p$. More precisely,

$$\langle (dF)_p(v), \alpha \rangle = \langle v, F^*(\alpha) \rangle, \ \forall v \in T_pM, \alpha \in T_q^*N.$$

Assume that $\dim M = m$, $\dim N = n$. Under coordinate charts (U, φ) around p and (V, ψ) around q, F is represented by

$$y^\alpha = F^\alpha(x^1, \cdots, x^m), \ \alpha = 1, \cdots, n.$$

For any $f \in C_q^\infty$, we have

$$\frac{\partial f \circ F}{\partial x^i}(p) = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial y^\alpha}(q) \frac{\partial F^\alpha}{\partial x^i}(p), \text{ for all } i.$$

It is then easy to see that F^* is well-defined. Moreover, by Corollary 1.1,

$$F^*((dy^\alpha)_q) = \sum_{i=1}^{m} \frac{\partial F^\alpha}{\partial x^i}(p)(dx^i)_p, \text{ for all } \alpha,$$

and by duality,

$$(dF)_p(\frac{\partial}{\partial x^i}|_p) = \sum_{\alpha=1}^{n} \frac{\partial F^\alpha}{\partial x^i}(p)(\frac{\partial}{\partial y^\alpha}|_q), \text{ for all } i.$$

Therefore, under the natural basis, the matrices of F^* and $(dF)_p$ are just the Jacobian $(\frac{\partial F^\alpha}{\partial x^i}(p))$.

It follows immediately from the definition that if $F: M \to N$ and $G: N \to P$ are C^∞ maps between manifolds and $q = F(p)$, then

$$(G \circ F)^* = F^* \circ G^*,$$

and

$$(d(G \circ F))_p = (dG)_q \circ (dF)_p. \tag{1.2}$$

Example 1.10. Let $\gamma: I \to M$ be a smooth curve on M. For any $t \in I$,

$$(d\gamma)_t(\frac{\partial}{\partial t}|_t) \in T_{\gamma(t)}M$$

is called the *tangent vector* of γ at t.

Example 1.11. If $F: \mathbb{R}^m \to \mathbb{R}^n$, then $(dF)_x$ is given by the linear map

$$v \mapsto \frac{d}{d\varepsilon}|_{\varepsilon=0} F(x + \varepsilon v), \ v \in \mathbb{R}^m,$$

where we identify $T_x\mathbb{R}^m$ and $T_{F(x)}\mathbb{R}^n$ to be $\mathbb{R}^m$ and $\mathbb{R}^n$ in the canonical way, respectively.

Definition 1.11. If $(dF)_p$ is injective, dF is called *nonsingular* at p. Of course this definition implies that $\dim M \leqslant \dim N$.

If F is a diffeomorphism, then dF is an isomorphism everywhere from the chain rule (1.2) applied to $F^{-1} \circ F = \mathrm{id}_M$. Conversely, by the inverse function theorem, we have the following local result.

Theorem 1.4. *Let $F: M \to N$ be a C^∞ map between two manifolds M, N of the same dimension, and let $p \in M$. If $(dF)_p$ is an isomorphism, then there exists open neighborhoods U of p and V of $q = f(p)$, such that $F(U) \subset V$ and $F|_U : U \to V$ is a diffeomorphism.*

Proof. Choose coordinate charts (U_0, φ) around p and (V_0, ψ) around q, such that $F(U_0) \subset V_0$. Then by assumption the Jacobian of F at p under these coordinate charts is nonsingular. It follows from the inverse function theorem that there exists open neighborhoods $\widetilde{U} \subset \varphi(U_0)$ of $\varphi(p)$ and $\widetilde{V} \subset \psi(V_0)$ of $\psi(q)$, such that

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \widetilde{U} \to \widetilde{V}$$

is a diffeomorphism. Take

$$U = \varphi^{-1}(\widetilde{U}), \; V = \psi^{-1}(\widetilde{V}),$$

then

$$F = \psi^{-1} \circ \widetilde{F} \circ \varphi : U \to V$$

is a diffeomorphism. □

Definition 1.12. Let $F: M \to N$ be a C^∞ map between two manifolds of the same dimension. If for any $p \in M$, there exists an open neighborhood U of p, such that $F(U)$ is open in N and

$$F|_U : U \to F(U)$$

is a diffeomorphism, then F is called a $local\ diffeomorphism$.

Remark 1.6. Theorem 1.4 tells us that if dF is an isomorphism everywhere, then F is a local diffeomorphism. However, F may fail to be a diffeomorphism. One can consider the map

$$F: \mathbb{R} \to S^1$$

given by

$$F(x) = e^{ix}.$$

However, if $F: M \to N$ is C^∞, bijective, and dF is nonsingular everywhere, then F is a diffeomorphism (we don't need to assume that M and N have the same dimension–it can be proved!). The proof of this result relies heavily on the second countability of the manifold topology. We will leave the challenging proof as an exercise (one can consult [5] for some hints).

Another application of the inverse function theorem is the following description of the local geometry of a C^∞ map $F: M \to N$ at some point p where dF is nonsingular.

Theorem 1.5. *Let M, N be two manifolds with $m = dimM < n = dimN$. If the differential of a C^∞ map $F: M \to N$ is nonsingular at $p \in M$, then* *there exists cubic coordinate charts* (U, φ) *centered at* p *and* (V, ψ) *centered at* $q = F(p)$*, such that*

$$F(U) = \psi^{-1}\{y \in \psi(V) : \ y^i = 0 \text{ for all } m+1 \leqslant i \leqslant n\} \subset V,$$

and for any $p' \in U$*,*

$$\begin{cases} y^i(F(p')) = x^i(p'), & 1 \leqslant i \leqslant m; \\ y^i(F(p')) = 0, & m+1 \leqslant i \leqslant n, \end{cases}$$

where x^i, y^i *are coordinates under* U, V *respectively.*

Proof. Choose coordinate charts (U_0, φ) around p and (V_0, ψ_0) around q, such that $F(U_0) \subset V_0$. Under these coordinate charts, the Jacobian of F at p has rank m. By a permutation of the coordinates under V_0, we may assume that the matrix

$$(\frac{\partial y^i}{\partial x^j}(p))_{1 \leqslant i,j \leqslant m}$$

is nonsingular, where under U_0, V_0,

$$y^i = F^i(x^1, \cdots, x^m), \ i = 1, \cdots, n.$$

Now define a C^∞ map

$$\Phi : \ \varphi(U_0) \times \mathbb{R}^{n-m} \to \psi_0(V_0)$$

by

$$(x^1, \cdots, x^m, w^1, \cdots, w^{n-m}) \mapsto (y^1, \cdots, y^m, w^1 + y^{m+1}, \cdots, w^{n-m} + y^n).$$

Then

$$\frac{\partial \Phi}{\partial (x, w)}((p, 0)) = \begin{pmatrix} (\frac{\partial y^i}{\partial x^j}(p))_{1 \leqslant i,j \leqslant m} & 0 \\ * & \mathrm{Id}_{n-m} \end{pmatrix},$$

which is again nonsingular. It follows from the inverse function theorem that we can find a cube $\widetilde{V}$ in $\varphi(U_0) \times \mathbb{R}^{n-m}$ centered at $(\varphi(p), 0)$ and some open neighborhood $\overline{V} \subset \psi_0(V_0)$ of $\psi(q)$, such that

$$\Phi|_{\widetilde{V}} : \ \widetilde{V} \to \overline{V}$$

is a diffeomorphism. Finally, the cubic coordinate charts

$$(U = \{x \in \varphi(U_0) : \ (x, 0) \in \widetilde{V}\}, \ \varphi)$$

and

$$(V = \psi_0^{-1} \circ \Phi(\widetilde{V}), \ \psi = \Phi^{-1} \circ \psi_0)$$

are desired. □

Theorem 1.5 tells us that if $(dF)_p$ is nonsingular, then we can parametrize p and $F(p)$ by cubes U and V such that $F(U)$ is a slice of V. In particular, F is injective near p.

It should be pointed out that if we look at the whole image $F(M)$ inside the cube V, it may far from being a slice or a union of slices, even in the case of submanifold.

Definition 1.13. Let $\varphi: M \to N$ be a C^∞ map between manifolds M, N.

(1) If $(d\varphi)$ is nonsingular everywhere, then φ is called an *immersion*.

(2) If φ is an injective immersion, then (M, φ) is called a *submanifold* of N.

(3) If (M, φ) is a submanifold of N, and $\varphi: M \to \varphi(M)$ is a homeomorphism (the topology of $\varphi(M) \subset N$ is the relative topology), then (M, φ) is called an *embedding*.

In the case of embedding, the local geometry becomes very simple: U, V can be chosen such that $\varphi(M) \cap V$ is the single slice $\varphi(U)$. In fact, we have the following result.

Theorem 1.6. *Let (M, φ) be an m-dimensional submanifold of an n-dimensional manifold N $(m < n)$. Then (M, φ) is an embedding if and only if for any $p \in M$, there exists cubic coordinate charts U centered at p and V centered at $q = \varphi(p)$ with properties described in Theorem 1.5 and $\varphi(U) = \varphi(M) \cap V$.*

Proof. "$\Rightarrow$". First of all, by Theorem 1.5, there exists cubic coordinate charts U_0 centered at p and V_0 centered at $q = F(p)$ with properties in the Theorem. Since (M, φ) is an embedding, $\varphi(U_0)$ is an open set in $\varphi(M)$, and hence there exists some open set $V_1 \subset V_0$, such that

$$\varphi(U_0) = \varphi(M) \cap V_1.$$

Now we can take some small cube $V \subset V_1$ centered at q and take U to be the slice $\varphi(U_0) \cap V$. U, V will satisfy the desired properties.

"$\Leftarrow$". For any $p \in M$, by assumption we know that there exists some coordinate chart $(V_0; y^\alpha)$ around $q = \varphi(p)$, such that $\varphi(M) \cap V_0$ is characterized by the equations

$$\{y \in V_0 : \ y^{m+1} = \cdots = y^n = 0\}.$$

To see $\varphi^{-1} : \ \varphi(M) \to M$ is continuous, it suffices to show that for any coordinate chart $(U; x^i)$ of p, there exists some open neighborhood V of q, such that

$$\varphi^{-1}(\varphi(M) \cap V) \subset U. \tag{1.3}$$

By the continuity of φ we may further assume $\varphi(U) \subset V_0$. It follows that under U and V_0, φ is given by

$$y^\alpha = \begin{cases} \varphi^\alpha(x^1, \cdots, x^m), & 1 \leqslant \alpha \leqslant m; \\ 0, & m+1 \leqslant \alpha \leqslant n. \end{cases}$$

Since $(d\varphi)_p$ is nonsingular,

$$(\frac{\partial y^\alpha}{\partial x^i}(p))_{1\leqslant\alpha,i\leqslant m}$$

is nonsingular. By the inverse function theorem,

$$y^\alpha = \varphi^\alpha(x^1,\cdots,x^m),\ \alpha = 1,\cdots,m,$$

has a C^∞ inverse near q on the hyperspace $\varphi(M)\cap V_0$, which is exactly the map φ^{-1}. In particular, we can choose some small cube $V\subset V_0$ centered at q, such that (1.3) holds. □

So far we have discussed the local geometry of submanifolds, but haven't mentioned anything about the global topology.

Assume that (M,φ) is a submanifold of N, then $\varphi(M)\subset N$ has a natural differential structure induced by $\varphi: M\to\varphi(M)$, such that $(\varphi(M),i)$ is a submanifold of N, where i is the inclusion. However, the manifold topology of $\varphi(M)$ may not be the same as the relative topology. The case of embedding is exactly the case when these two topologies coincide.

Now let A be a nonempty subset of a manifold M. The following is a uniqueness theorem for the differential structure on A, related to its global topology, such that (A,i) is a submanifold of M, where i is the inclusion. We are not going to proof this result, one can refer to [5] for a sketch.

Theorem 1.7. *(1) If there exists two differential structures on A, such that (A,i) is a submanifold of M and they induce the same manifold topology on A, then these two differential structures are the same.*

(2) If there exists a differential structure on A, such that (A,i) is a submanifold of M and the induced manifold topology is the relative topology, then this is the unique manifold structure on A such that (A,i) is a submanifold of M. In this case, (A,i) is an embedding.

Many important examples of manifolds arise from the implicit function theorem. Intuitively, if we solve n equations with m variables ($m>n$), then we obtain a manifold of dimension $m-n$. But the solvability of these equations requires some kind of nondegeneracy on the differential of the system.

Theorem 1.8. *Let $F: M\to N$ be a C^∞ map between manifolds M,N of dimension m,n respectively ($m>n$). Let $q\in N$ and assume that for any $p\in F^{-1}(q)\neq\emptyset$, $(dF)_p$ is surjective. Then $P=F^{-1}(q)$ has a unique differential structure such that (P,i) is a submanifold of M, where i is the inclusion. Moreover, under this differential structure*

$$\dim P = m-n,$$

and (P,i) is an embedding.

Proof. We are going to construct a differential structure on P of dimension $m-n$, such that (P, i) is a submanifold of M and the induced manifold topology is the relative topology. It then follows from the second part of Theorem 1.7 that this is the unique differential structure on P such that (P, i) is a submanifold of M, and it is in fact an embedding.

The construction is again benefited from the inverse function theorem. Fix a coordinate chart $(V; y^\alpha)$ around q. For any $p \in P$, choose a coordinate chart $(U_p'; x^i)$ around p such that $F(U_p') \subset V$. Since $(dF)_p$ is surjective, by a permutation of the x coordinates we may assume that

$$(\frac{\partial y^\alpha}{\partial x^i}(p))_{1\leqslant \alpha, i \leqslant n}$$

is nonsingular, where

$$y^\alpha = F^\alpha(x^1, \cdots, x^m),\ 1 \leqslant \alpha \leqslant n.$$

Define $\overline{F}:\ U_p' \to V \times \mathbb{R}^{m-n}$ by

$$\overline{F}(x^1, \cdots, x^n, x^{n+1}, \cdots, x^m) = (y^1, \cdots, y^n, x^{n+1}, \cdots, x^m).$$

It follows that the Jacobian of $\overline{F}$ is nonsingular at p. By the inverse function theorem, we can choose some cube (U_p, φ) centered at $\overline{F}(x^1, \cdots, x^m)$ to parametrize $p \in M$ locally. Moreover, under such parametrization, $P \cap U_p$ is exactly the $(m-n)$-dimensional slice

$$\{(y^1, \cdots, y^n, x^{n+1}, \cdots, x^m) \in U_p :\ y^\alpha = y^\alpha(q) \text{ for all } 1 \leqslant \alpha \leqslant n\}$$

of the cube U_p. Since (U_p, φ) is a coordinate chart around p in M, by restriction it is obvious that under the relative topology on P, $(P \cap U_p, \varphi|_{P\cap U_p})$ is also a coordinate chart around p in P (in the sense of Remark 1.2). Moreover, since $\{U_p\}_{p\in P}$ is C^∞ compatible on M, by restriction it follows immediately that $\{P \cap U_p\}_{p\in P}$ is an atlas on P. Therefore, from Remark 1.2 it defines a differential structure on P of dimension $m-n$, whose induced manifold topology coincides with the original topology, namely, the relative topology. Under such differential structure, for any $p \in P$, under U_p the inclusion i is just the inclusion of the slice $P \cap U_p$ into the cube U_p. Therefore, di is nonsingular at p, and hence (P, i) is a submanifold of M. $\square$

Remark 1.7. From the proof of Theorem 1.8, it is not hard to see that for any $p \in P$,

$$(di)_p(T_pP) = \mathrm{Ker}(dF)_p \subset T_pM.$$

Example 1.12. Consider $F:\ \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$F(x^0, x^1, \cdots, x^n) = \sum_{i=0}^{n} (x^i)^2,$$

then for any $x \in F^{-1}(1)$, the Jacobian of F at x is

$$2(x^0, \cdots, x^n)^T,$$

which is of rank 1. Therefore, $(dF)_x$ is surjective. By Theorem 1.8, $(F^{-1}(1) = S^n, i)$ is a submanifold of $\mathbb{R}^{n+1}$. By Theorem 1.7 this differential structure is the same as the one introduced in Example 1.2, since they are both submanifolds of $\mathbb{R}^{n+1}$ with the same manifold topology–the relative topology.

Example 1.13. Let $GL(n;\mathbb{R})$ be the group of $n \times n$ real invertible matrices. This is an open subset of $\mathrm{Mat}(n;\mathbb{R}) \cong \mathbb{R}^{n^2}$, the space of $n \times n$ real matrices. Therefore, $GL(n;\mathbb{R})$ is an n^2-dimensional manifold. Let $O(n)$ be the group of orthogonal matrices of order n, i.e.,

$$A \in O(n) \iff A^T A = I.$$

We are going to show that $(O(n), i)$ is an $\frac{n(n-1)}{2}$-dimensional submanifold of $GL(n;\mathbb{R})$. Define

$$F: \ GL(n;\mathbb{R}) \to \mathrm{Sym}(n)$$

by

$$F(A) = A^T A,$$

where $\mathrm{Sym}(n)$ denotes the $\frac{n(n+1)}{2}$-dimensional vector space of $n \times n$ real symmetric matrices. Then for any $A \in F^{-1}(I)$, by Example 1.11,

$$(dF)_A: \ \mathrm{Mat}(n;\mathbb{R}) \to \mathrm{Sym}(n)$$

is given by

$$(dF)_A(K) \quad = \quad \frac{d}{d\varepsilon}|_{\varepsilon=0} F(A + \varepsilon K) = A^T K + K^T A.$$

For any $S \in \mathrm{Sym}(n)$, let $K = \frac{AS}{2}$, then we have

$$(dF)_A(K) = S.$$

Therefore, $(dF)_A$ is surjective. By Theorem 1.8, $(O(n), i)$ is as submanifold of $GL(n;\mathbb{R})$ of dimension $\frac{n(n-1)}{2}$.

1.4 Vector fields and the tangent bundle

We have defined tangent vector at one point. If for any point on the manifold, we assign a tangent vector at this point, then we obtain a vector field.

Definition 1.14. Let M be a manifold. A *vector field* X on M is a collection $\{X_p\}_{p\in M}$ of tangent vectors such that

$$X_p \in T_pM, \ \forall p \in M.$$

Let X be a vector field. For any $p \in M$, X_p can be regarded as a linear derivation on the space of C^∞ germs at p. Therefore, for $f \in C^\infty(M)$,

$$(Xf)(p) = X_p([f]),\ p \in M,$$

defines a real function on M.

Definition 1.15. If for any $f \in C^\infty(M)$, $Xf \in C^\infty(M)$, then X is called a *smooth* vector field. The space of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

On the other hand, for any coordinate chart $(U; x^i)$, $X|_U$ can be expressed by

$$X_x = \sum_{i=1}^n a^i(x)\frac{\partial}{\partial x^i}$$

for some real functions $a^i(x)$ defined on U.

Theorem 1.9. *X is a smooth vector field on M if and only if for any coordinate chart $(U; x^i)$, $a^i(x) \in C^\infty(U)$ for all $i = 1, \cdots, n$.*

Proof. "$\Rightarrow$". For any coordinate chart $(U; x^i)$, the coordinate functions $x^i \in C^\infty(U)$. For fixed $p \in U$, we may use a bump function (see Example 1.6) to extend x^i from some open neighborhood of p to a C^∞ function $\widetilde{x}^i$ defined on M. It follows that

$$a^i(x) = X_x\widetilde{x}^i$$

near p. Therefore, $a^i(x)$ is smooth at p.

"$\Leftarrow$". Let $f \in C^\infty(M)$. For any $p \in M$, choose a coordinate chart $(U; x^i)$ around p. It follow that under U, Xf is given by

$$(Xf)(x) = \sum_{i=1}^n a^i(x)\frac{\partial f}{\partial x^i}(x),\ x \in U.$$

By assumption, we know that Xf is smooth at p. □

Let X be a smooth vector field on M. Then the induced linear operator $X: C^\infty(M) \to C^\infty(M)$ satisfies the derivation property:

$$X(fg) = fXg + gXf,\ \forall f, g \in C^\infty(M). \tag{1.4}$$

We are going to show that, any such linear operator X arises from a smooth vector field. To see that, we first show that X is a local operator, namely, if $f = 0$ on some open set U, then $Xf = 0$ on U. Indeed, for any $p \in U$, choose some open neighborhood W of p with $\overline{W}$ compact and $\overline{W} \subset U$, let $h \in C^\infty(M)$ be such that

$$h(q) = \begin{cases} 1, & q \in \overline{W} \\ 0, & q \notin U. \end{cases}$$

It follows that $hf \equiv 0$ on M, and by (1.4) we have

$$0 = X(hf)(p) = h(p)(Xf)(p) + f(p)(Xh)(p) = (Xf)(p).$$

Therefore, $Xf = 0$ on U. Now for any $p \in M$, and $[f] \in \mathcal{F}_p$, define

$$X_p([f]) = (X\widetilde{f})(p),$$

where $\widetilde{f} \in C^\infty(M)$ is an extension of a representative $f \in [f]$ from an open neighborhood of p to the manifold M. It follows from the previous discussion that X_p is well-defined and $X_p \in T_pM$ (in the sense of Theorem 1.3). Moreover, $\{X_p\}_{p\in M}$ defines a smooth vector field on M such that the induced linear operator on $C^\infty(M)$ is X.

By using the language of (infinite dimensional) vector space, this correspondence is a linear isomorphism.

The space $\mathfrak{X}(M)$ of smooth vector fields on M carries a product structure which turns $\mathfrak{X}(M)$ into a Lie algebra.

Definition 1.16. Let X, Y be two smooth vector fields on M. Their *Lie bracket* $[X, Y]$, is define to be the linear operator

$$\begin{aligned} [X,Y]:\ C^\infty(M) &\to C^\infty(M), \\ f &\mapsto XYf - YXf. \end{aligned}$$

From the definition it is immediate that $[X, Y]$ satisfies the derivation property (1.4). Therefore, by the previous discussion we know that $[X, Y]$ is again a smooth vector field.

Moreover, we have the following easy algebraic properties about the Lie bracket. It indicates that $(\mathfrak{X}(M), +, [\cdot,\cdot])$ is a Lie algebra.

Proposition 1.3. *Let $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Then*

(1) $[X, Y] = -[Y, X]$;

(2) $[X + Y, Z] = [X, Z] + [Y, Z]$;

(3) $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$;

(4) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Here fX is the smooth vector field $\{f(p)X_p\}_{p\in M}$, or equivalently,

$$(fX)(\varphi) = f(X\varphi),\ \ \varphi \in C^\infty(M).$$

Proof. Straight forward by definition. □

Under a coordinate chart $(U; x^i)$, from the Definition 1.16 it is obvious that the coordinate vector fields $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$ satisfy:

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0,\ \ \forall i, j = 1, \cdots, n.$$

Therefore, for smooth vector fields X, Y on U, if

$$X_x = \sum_{i=1}^n a^i(x)\frac{\partial}{\partial x^i},\ \ Y_x = \sum_{j=1}^n b^j(x)\frac{\partial}{\partial x^j},\ \ x \in U,$$

by Proposition 1.3 $[X,Y]$ is given by

$$[X,Y]_x = \sum_{i=1}^{n} c^i(x)\frac{\partial}{\partial x^i},$$

where

$$c^i(x) = \sum_{j=1}^{n}(a^j(x)\frac{\partial b^i}{\partial x^j}(x) - b^j(x)\frac{\partial a^i}{\partial x^j}(x)),\ i = 1,\cdots,n.$$

It is natural and convenient to look at vector fields from the view of the tangent bundle. The notion of fibre bundles is very important for us to understand global properties on the geometry and topology of a manifold. Later on we will come to the study of fiber bundles in the context of de Rham cohomology.

Now we construct the tangent bundle TM over a manifold M. The construction is geometrically intuitive.

Let

$$TM = \cup_{p\in M} T_pM$$

be the set of tangent vectors at any point on M. We are going to introduce a canonical differential structure on TM induced by the one on M, such that TM is a manifold of dimension $2n$.

Fix an atlas $\{(U_\alpha,\varphi_\alpha):\ \alpha\in A\}$ in the differential structure of M. For any $\alpha\in A$, we define the map

$$\Phi_\alpha:\ E_\alpha = \cup_{p\in U_\alpha} T_pM \to \varphi_\alpha(U_\alpha)\times\mathbb{R}^n$$

by

$$v \mapsto (x^1,\cdots,x^n,a^1,\cdots,a^n),$$

where $(x^1,\cdots,x^n)$ is the coordinates of $p\in U_\alpha$ such that $v\in T_pM$, and $(a^1,\cdots,a^n)$ is the coordinates of v under the natural basis $\{\frac{\partial}{\partial x^1}|_p,\cdots,\frac{\partial}{\partial x^n}|_p\}$ of T_pM. Φ_α is a bijection because φ_α is bijective and each T_pM $(p\in U_\alpha)$ is canonically isomorphic to $\mathbb{R}^n$ under the natural basis. Now we are going to show that $\{(E_\alpha,\Phi_\alpha):\ \alpha\in A\}$ defines an atlas on TM. In fact, for any $\alpha,\beta\in A$, $\Phi_\alpha(E_\alpha\cap E_\beta)$ is the open set

$$\varphi_\alpha(U_\alpha\cap U_\beta)\times\mathbb{R}^n\subset\mathbb{R}^{2n},$$

and the change of coordinates

$$\Phi_\beta\circ\Phi_\alpha^{-1}:\ \varphi_\alpha(U_\alpha\cap U_\beta)\times\mathbb{R}^n\to\varphi_\beta(U_\alpha\times U_\beta)\times\mathbb{R}^n$$

is given by

$$y^i \ = \ (\varphi_\beta\circ\varphi_\alpha^{-1})^i(x^1,\cdots,x^n),\ b^i = \sum_{j=1}^{n} a^j\frac{\partial y^i}{\partial x^j},\ i=1,\cdots,n.$$

This is obviously C^∞ with C^∞ inverse since $\{(U_\alpha,\varphi_\alpha):\ \alpha\in A\}$ is an atlas on M and $(\frac{\partial y^i}{\partial x^j})_{1\leqslant i,j\leqslant n}$ is nonsingular everywhere in $\varphi_\alpha(U_\alpha\cap U_\beta)$. Therefore,

$\{(E_\alpha, \Phi_\alpha) : \alpha \in A\}$ is an atlas on TM, which defines a differential structure on TM of dimension $2n$. It is not hard to see that this differential structure is independent of the choice of $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ on M. Moreover, from the construction we can show that the manifold topology of TM is Hausdorff and has a countable base of open sets. This is left as an exercise.

Let $\pi : TM \to M$ be the natural projection, i.e., $\pi(v) = p$ if $v \in T_pM$. It follows that π is a surjective C^∞ map. For any $p \in M$, the vector space $\pi^{-1}(p) = T_pM$ is called the *fiber* of TM over p.

Definition 1.17. A *section* of TM is a C^∞ map $X : M \to TM$ such that

$$\pi \circ X = \mathrm{id}_M.$$

By definition, a section of TM is just a smooth vector field on X. Since each fiber of TM is a vector space (tangent space), the zero section is well-defined. Moreover, by using bump functions we can construct nontrivial sections of TM. However, TM does not necessarily have an everywhere nonzero section! Equivalently, not every manifold M has a non-vanishing smooth vector field. This issue is closely related to the topology of the manifold. For example, later on we will prove that the n-sphere S^n has a non-vanishing smooth vector field if and only if n is odd, (this is known as the Hairy Ball Theorem) while the n-dimensional torus (or more generally, every Lie group) has a non-vanishing smooth vector field.

In exactly the same way, we can construct the cotangent bundle T^*M. For any $f \in C^\infty(M)$, since $(df)_p \in T_p^*M$ for every p, f induces a section of T^*M. This section, denoted by df, is called the *differential of f*. However, not every section of T^*M arises from a C^∞ function in this way.

2 Differential forms and integration on manifolds

2.1 Tensor products

The analysis of tensor fields, in particular, of differential forms on a manifold is fundamental in the study of differential geometry. To understand the concept of tensor fields, we first need to study tensor products.

Definition 2.1. Let V and W be two real vector spaces. The *tensor product* of V and W is a real vector space T together with a bilinear map (i.e., linear in each component)

$$\varphi : V \times W \to V \otimes W,$$

such that the following *universal property* holds: if Z is a real vector space and $f : V \times W \to Z$ is a bilinear map, then there exists a unique linear map $g : T \to Z$, such that

$$f = g \circ \varphi.$$

The tensor product is unique up to unique isomorphism. More precisely, if (T', φ') is another real vector space with a bilinear map satisfying the previous universal property, then there exists a unique isomorphism $i: T \to T'$, such that

$$\varphi' = i \circ \varphi.$$

We are not going to prove this algebraic result–it is just a manipulation of the definition. We will use $(V \otimes W, \otimes)$ to denote the tensor product of V and W with the bilinear map (up to unique isomorphism).

Remark 2.1. It is the properties of the tensor product which are more important than what it is (as we use the properties of real numbers very naturally, we seldom remind ourselves that they are equivalence classes of Cauchy sequences of rationals).

We are going to construct the tensor product explicitly, in a relatively simple and conceivable way. From now on we assume that V and W are finite-dimensional real vector spaces.

For convenience we first construct $V^* \otimes W^*$, where V^* and W^* are the dual space of V and W respectively.

Let $L(V, W)$ be the space of bilinear functionals $f: V \times W \to \mathbb{R}$. For $v^* \in V^*$ and $w^* \in W^*$, define

$$v^* \otimes w^* : V \times W \to \mathbb{R}$$

by

$$v^* \otimes w^*(v, w) = v^*(v)w^*(w), \ (v, w) \in V \times W.$$

It is easy to see that $v^* \otimes w^* \in L(V, W)$. Moreover, $\otimes$ is bilinear from $V^* \times W^*$ to $L(V, W)$. $V^* \otimes W^*$ is defined to be the subspace of $L(V, W)$ spanned by the elements of the form $v^* \otimes w^*$, where $v^* \in V^*, w^* \in W^*$.

It should be noticed that not all the elements in $V^* \otimes W^*$ are of the form $v^* \otimes w^*$. For example, $v^* \otimes v^* + w^* \otimes w^*$ is usually not a monomial.

Fix a basis $\{a_i : i = 1, \cdots, m\}$ of V and $\{b_\alpha : \alpha = 1, \cdots, n\}$ of W respectively. Then for any v^*, w^*, under the corresponding dual basis $\{a^{*i}\}$ and $\{b^{*\alpha}\}$, we have the expression

$$v^* = \sum_{i=1}^{m} v^*(a_i)a^{*i}, \ w^* = \sum_{\alpha=1}^{n} w^*(b_\alpha)b^{*\alpha}.$$

Therefore, from the bilinearity of $\otimes$ we know that $V^* \otimes W^*$ is spanned by $\{a^{*i} \otimes b^{*\alpha} : 1 \leqslant i \leqslant m, 1 \leqslant \alpha \leqslant n\}$. Moreover, this is a basis of $V^* \otimes W^*$ since it is obvious that they are linearly independent.

Theorem 2.1. *$V^* \otimes W^* = L(V, W)$. Moreover, $V^* \otimes W^*$ together with the bilinear map $\otimes: V^* \times W^* \to V^* \otimes W^*$ satisfies the universal property introduced before, and hence $(V^* \otimes W^*, \otimes)$ is the tensor product of V^* and W^*.*

Proof. For any $f \in L(V, W)$, we can write

$$f = \sum_{i,\alpha} f(a_i, b_\alpha) a^{*i} \otimes b^{*\alpha}.$$

Therefore, $V^* \otimes W^* = L(V, W)$.

To prove the universal property, assume that Z is a real vector space and $f : V^* \times W^* \to Z$ is a bilinear map. Define g by specifying its values on the basis:

$$g(a^{*i} \otimes b^{*\alpha}) = f(a^{*i}, b^{*\alpha}), \text{ for all } i \text{ and } \alpha. \tag{2.1}$$

Then for any $v^* \in V^*, w^* \in W^*$, we have

$$\begin{aligned} g \circ \otimes(v^*, w^*) &= g(v^* \otimes w^*) \\ &= \sum_{i,\alpha} v^*(a_i) w^*(b_\alpha) g(a^{*i} \otimes b^{*\alpha}) \\ &= \sum_{i,\alpha} v^*(a_i) w^*(b_\alpha) f(a^{*i}, b^{*\alpha}) \\ &= f(v^*, w^*). \end{aligned}$$

Moreover, g is uniquely determined by f since it is determined by (2.1) on the basis $\{a^{*i} \otimes b^{*\alpha}\}$, if it satisfies

$$f = g \circ \otimes.$$

Therefore, $(V^* \otimes W^*, \otimes)$ satisfies the universal property of tensor product. □

Since V can be regarded as the dual space of V^* (and the same for W), we can define $V \otimes W$ in exactly the same way. It turns out that

$$V \otimes W = L(V^*, W^*)$$

with basis $\{a_i \otimes b_\alpha : 1 \leqslant i \leqslant m, 1 \leqslant \alpha \leqslant n\}$ and the dimension of $V \otimes W$ is again mn.

$V^* \otimes W^*$ is the dual space of $V \otimes W$ in a natural way, if we define

$$v^* \otimes w^*(v \otimes w) = v^*(v) w^*(w). \tag{2.2}$$

It follows that $\{a^{*i} \otimes b^{*\alpha}\}$ is the dual basis of $\{a_i \otimes b_\alpha\}$.

Now assume that V, W, Z are finite-dimensional real vector spaces. Similarly, we can define $V \otimes W \otimes Z$ as the linear subspace of $L(V^*, W^*, Z^*)$ spanned by the elements of the form $v \otimes w \otimes z$, where $L(V^*, W^*, Z^*)$ is the space of trilinear functionals $f : V^* \times W^* \times Z^* \to \mathbb{R}$. It is not hard to show that

$$V \otimes W \otimes Z = L(V^*, W^*, Z^*),$$

and $(V \otimes W) \otimes Z$, $V \otimes (W \otimes Z)$ are both isomorphic to $V \otimes W \otimes Z$. We will leave these algebraic details as an exercise. More generally, for finite-dimensional real vector spaces $V_1, \cdots, V_n$, we can define their tensor product $V_1 \otimes \cdots \otimes V_n$. It turns out that $(V_1 \otimes \cdots \otimes V_n, \otimes \cdots \otimes)$ is the unique vector space and unique n-linear map (up to unique isomorphism) satisfying the universal property: if Z is a real vector space and $f: V_1 \times \cdots \times V_n \to Z$ is an n-linear map, then there exists a unique linear map $g: V_1 \otimes \cdots \otimes V_n \to Z$ such that

$$f = g \circ (\otimes \cdots \otimes).$$

If $\{e_i^{(l)}: 1 \leqslant i \leqslant k_l\}$ is a basis of V_l, then $\{e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}: 1 \leqslant i_l \leqslant k_l, 1 \leqslant l \leqslant n\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$.

Now let V be an n-dimensional real vector space.

Definition 2.2. Elements in the tensor product

$$V_s^r = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}$$

are called an (r,s)-*type tensors*. In particular, elements in V_0^r are called *contravariant* tensors of degree r, and elements in V_s^0 are called *covariant* tensors of degree s. Usually we use $T^r(V)$ to denote V_0^r, and use $T^s(V^*)$ to denote V_s^0. Conventionally, $T^0(V) = T^0(V^*) = \mathbb{R}$.

From the previous explicit construction of tensor product, we know that

$$V_s^r = L(V^*, \cdots, V^*, V, \cdots, V). \tag{2.3}$$

Moreover, if $\{e_i: i = 1, \cdots, n\}$ is a basis of V, then

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_s}: 1 \leqslant i_1, \cdots, i_r, j_1, \cdots, j_s \leqslant n\} \tag{2.4}$$

is a basis of V_s^r. If x is an (r,s)-type tensor, then x can be uniquely expressed as

$$x = \sum_{\substack{1 \leqslant i_1, \cdots, i_r \leqslant n \\ 1 \leqslant j_1, \cdots, j_s \leqslant n}} x_{j_1 \cdots j_s}^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_s}, \tag{2.5}$$

where by (2.3) and the duality (2.2),

$$\begin{aligned} x_{j_1 \cdots j_s}^{i_1 \cdots i_r} &= x(e^{*i_1}, \cdots, e^{*i_r}, e_{j_1}, \cdots, e_{j_s}) \\ &= e^{*i_1} \otimes \cdots \otimes e^{*i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s}(x). \end{aligned}$$

$(x_{j_1 \cdots j_s}^{i_1 \cdots i_r})_{\substack{1 \leqslant i_1, \cdots, i_r \leqslant n \\ 1 \leqslant j_1, \cdots, j_s \leqslant n}}$ are called the *coordinates* of x under the basis (2.4).

It is usually convenient to introduce the *Einstein summation convention*. More precisely, if some index appears as a superscript and a subscript at a single expression, then they are automatically summed over the domain of the index. For example, we can write (2.5) as

$$x = x_{j_1 \cdots j_s}^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_s}.$$

In the rest of the notes we will always use the Einstein summation convention, unless we want to emphasize the domain of the index.

Now a natural question is what the change of coordinates formula looks like for (r, s)-type tensors. This is not hard to derive from the multi-linearity structure of tensors.

Proposition 2.1. *Let $\{\overline{e}_i : \ i = 1, \cdots, n\}$ be another basis of V, and let (a_i^j) be the transition matrix between the two basis, i.e.,*

$$\overline{e}_i = a_i^j e_j.$$

Then for any (r, s)-type tensor $x \in V_s^r$, the change of coordinates formula between the corresponding two basis of V_s^r is given by

$$\overline{x}^{i_1 \cdots i_r}_{j_1 \cdots j_s} = x^{k_1 \cdots k_r}_{l_1 \cdots l_s} \beta^{i_1}_{k_1} \cdots \beta^{i_r}_{k_r} a^{l_1}_{j_1} \cdots a^{l_s}_{j_s}, \tag{2.6}$$

where (β_j^i) is the inverse of (a_j^i).

Proof. By the multi-linearity of tensors we have

$$\begin{aligned} x &= x^{k_1 \cdots k_r}_{l_1 \cdots l_s} e_{k_1} \otimes \cdots \otimes e_{k_r} \otimes e^{*l_1} \otimes \cdots \otimes e^{*l_s} \\ &= x^{k_1 \cdots k_r}_{l_1 \cdots l_s} (\beta^{i_1}_{k_1} \overline{e}_{i_1}) \otimes \cdots \otimes (\beta^{i_r}_{k_r} \overline{e}_{i_r}) \otimes (a^{l_1}_{j_1} \overline{e}^{*j_1}) \otimes \cdots \otimes (a^{l_s}_{j_s} \overline{e}^{*j_s}) \\ &= x^{k_1 \cdots k_r}_{l_1 \cdots l_s} \beta^{i_1}_{k_1} \cdots \beta^{i_r}_{k_r} a^{l_1}_{j_1} \cdots a^{l_s}_{j_s} \overline{e}_{i_1} \otimes \cdots \otimes \overline{e}_{i_r} \otimes \overline{e}^{*j_1} \otimes \cdots \otimes \overline{e}^{*j_s}. \end{aligned}$$

Therefore, the change of coordinates formula (2.6) follows immediately. □

Remark 2.2. From the formula (2.6) we can see why we call the "V" components as contravariant and the "V^*" components as covariant: the change of coordinates for the "V" part (the super indexes) is given by the inverse of the transition matrix (a_j^i), while the change of coordinates for the "V^*" part (the sub indexes) is given by the original transition matrix.

One can define the tensor product of tensors of different type. Let $x \in V_{s_1}^{r_1}$ and $y \in V_{s_2}^{r_2}$, then $x \otimes y$ is defined to be the $(r_1 + r_2, s_1 + s_2)$-type tensor given by

$$\begin{aligned} &x \otimes y(v^{*1}, \cdots, v^{*r_1+r_2}, v_1, \cdots, v_{s_1+s_2}) \\ =&x(v^{*1}, \cdots, v^{*r_1}, v_1, \cdots, v_{s_1}) y(v^{*r_1+1}, \cdots, v^{*r_1+r_2}, v_{s_1+1}, \cdots, v_{s_1+s_2}). \end{aligned}$$

Under a basis the coordinates of $x \otimes y$ is given by

$$(x \otimes y)^{i_1 \cdots i_{r_1+r_2}}_{j_1 \cdots j_{s_1+s_2}} = x^{i_1 \cdots i_{r_1}}_{j_1 \cdots j_{s_1}} y^{i_{r_1+1} \cdots i_{r_1+r_2}}_{j_{s_1+1} \cdots j_{s_1+s_2}}.$$

It is easy to see that such tensor product operator is bilinear and associative.

Consider the direct sum of real vector spaces

$$T(V) = \oplus_{r \geqslant 0} T^r(V).$$

Elements of $T(V)$ is of the form

$$x = \sum_{r \geqslant 0} x_r,$$

where $x_r \in T^r(V)$ and there are all but finitely many nonzero terms in the sum. Besides the linear structure, $T(V)$ carries a product structure naturally induced by $\otimes : T^{r_1}(V) \times T^{r_2}(V) \to T^{r_1+r_2}(V)$, which is defined previously. More precisely,

$$x \otimes y = \sum_{r \geqslant 0} \sum_{r_1+r_2=r} x_{r_1} \otimes y_{r_2}. \tag{2.7}$$

It turns out that $T(V)$ is an algebra over $\mathbb{R}$, which is called the *tensor algebra over* V.

Let $\mathcal{S}_r$ be the permutation group of order r ($r \geqslant 1$). Recall that the *sign* of a permutation $\sigma \in \mathcal{S}_r$, denoted by $\mathrm{sgn}(\sigma)$, is the parity of the number of inversions for σ, i.e., of pairs (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$. For any $\sigma \in \mathcal{S}_r$, we can define a linear transformation $\sigma : T^r(V) \to T^r(V)$ by

$$(\sigma x)(v^{*1}, \cdots, v^{*r}) = x(v^{*\sigma(1)}, \cdots, v^{*\sigma(r)}),$$

where $x \in T^r(V)$ and $(v^{*1}, \cdots, v^{*r}) \in V^* \times \cdots \times V^*$. It is easy to see that when acting on a monomial $v_1 \otimes \cdots \otimes v_r$,

$$\sigma(v_1 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

Definition 2.3. A contravariant tensor $x \in T^r(V)$ is called *symmetric* if

$$\sigma x = x, \ \forall \sigma \in \mathcal{S}_r.$$

x is called *antisymmetric* if

$$\sigma x = \mathrm{sgn}(\sigma) \cdot x, \ \forall \sigma \in \mathcal{S}_r.$$

The space of symmetric (antisymmetric, respectively) tensors in $T^r(V)$ is denoted by $P^r(V)$ ($\Lambda^r(V)$, respectively).

By definition, it is easy to see that x is symmetric if and only if under some basis $\{e_i : i = 1, \cdots, n\}$ of V, the coordinates of x are symmetric:

$$x^{i_1 \cdots i_r} = x^{i_{\sigma(1)} \cdots i_{\sigma(r)}}, \text{ for all } i_1, \cdots, i_r \text{ and for all } \sigma \in \mathcal{S}_r.$$

Similar result holds for the antisymmetric case.

The spaces $P^r(V)$ and $\Lambda^r(V)$ can be obtained by using the symmetrization and antisymmetrization operators.

Define the linear operators $S_r, A_r : T^r(V) \to T^r(V)$ by

$$S_r(x) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \sigma x, \ A_r(x) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \mathrm{sgn}(\sigma) \cdot \sigma x. \tag{2.8}$$

Then we have the following result.

Proposition 2.2. *$P^r(V) = S_r(T^r(V))$ and $\Lambda^r(V) = A_r(T^r(V))$.*

Proof. We only consider the antisymmetric case. $\Lambda^r(V) \subset A_r(T^r(V))$ is trivial since for any $x \in \Lambda^r(V)$,

$$A_r(x) = x.$$

Now assume that

$$y = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \cdot \sigma x \in A_r(T^r(V)),$$

then for any $\tau \in \mathcal{S}_r$,

$$\begin{aligned} \tau y &= \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \cdot \tau\sigma x \\ &= \operatorname{sgn}(\tau) \cdot \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\tau\sigma) \cdot \tau\sigma x \\ &= \operatorname{sgn}(\tau) \cdot y. \end{aligned}$$

Therefore, $y \in \Lambda^r(V)$. □

Another convenient way of looking at antisymmetric tensors in $T^r(V)$ is to see how it acts on V^*. More precisely, we have the following result.

Proposition 2.3. *Let $x \in T^r(V) = L(V^*, \cdots, V^*)$. Then $x \in \Lambda^r(V)$ if and only if x is an alternating r-linear functional in $L(V^*, \cdots, V^*)$, i.e.,*

$$x(v^{*1}, \cdots, v^{*r}) = 0$$

whenever $v^{*i} = v^{*j}$ for some $i \neq j$.

Proof. x is an alternating r-linear functional if and only if for any $(v^{*1}, \cdots, v^{*r}) \in V^* \times \cdots \times V^*$ and $1 \leqslant i < j \leqslant r$,

$$x(v^{*1}, \cdots, v^{*i}, \cdots, v^{*j}, \cdots, v^{*n}) = -x(v^{*1}, \cdots, v^{*j}, \cdots, v^{*i}, \cdots, v^{*n}).$$

Since any permutation $\sigma \in \mathcal{S}_r$ can be written as a product of finitely many transpositions (2-element exchanges), it is also equivalent to the fact that for any $\sigma \in \mathcal{S}_r$,

$$x(v^{*\sigma(1)}, \cdots, v^{*\sigma(r)}) = \operatorname{sgn}(\sigma) \cdot x(v^{*1}, \cdots, v^{*r}),\ \forall (v^{*1}, \cdots, v^{*}) \in V^* \times \cdots \times V^*.$$

□

Remark 2.3. We can also define symmetric and antisymmetric covariant tensors in $T^r(V^*)$ is a similar way. The previous results can be easily formulated and proved in the covariant case as well.

2.2 The exterior algebra

Here we study the space of antisymmetric contravariant tensors in detail. Its algebraic structure is fundamental in the study of differential forms.

Let V be an n-dimensional real vector space.

Definition 2.4. $\Lambda^r(V)$ is called the *r-th exterior power of V*. Elements in $\Lambda^r(V)$ are called (*exterior*) r-vectors. Conventionally $\Lambda^0(V) = \mathbb{R}$.

Of course we can multiply exterior vectors of different degree simply by viewing them as elements in the tensor algebra $T(V)$ and using the algebraic structure of $T(V)$. However, such multiplication of exterior vectors does not necessarily give us exterior vectors again. An essential point of introducing exterior vectors is that we can define an exterior product on exterior vectors intrinsically.

Recall that A_r is the antisymmetrization operator defined in (2.8).

Definition 2.5. Let $\xi \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. The *exterior product* $\xi \wedge \eta$ of ξ and η is defined to be

$$\xi \wedge \eta = \frac{(k+l)!}{k!l!} A_{k+l}(\xi \otimes \eta) \in \Lambda^{k+l}(V).$$

The exterior product has the following basic properties.

Proposition 2.4. *The exterior product*

$$\wedge : \ \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$$

is bilinear and associative. Moreover, for any $\xi \in \Lambda^k(V), \eta \in \Lambda^l(V)$,

$$\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi.$$

Proof. Bilinearity is an immediate consequence of the linearity of the antisymmetrization operator and the bilinearity of the tensor product operator.

For associativity, by definition we have

$$\begin{aligned}
&(\xi \wedge \eta) \wedge \zeta(v^{*1}, \cdots, v^{*k+l+h}) \qquad (2.9)\\
=&\frac{1}{(k+l)!h!} \sum_{\sigma \in \mathcal{S}_{k+l+h}} \operatorname{sgn}(\sigma) \cdot (\xi \wedge \eta)(v^{*\sigma(1)}, \cdots, v^{*\sigma(k+l)})\\
&\cdot \zeta(v^{*\sigma(k+l+1)}, \cdots, v^{*\sigma(k+l+h)}) \qquad (2.10)\\
=&\frac{1}{(k+l)!h!} \frac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l+h}} \sum_{\tau \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \cdot \xi(v^{*\sigma\tau(1)}, \cdots, v^{*\sigma\tau(k)})\\
&\cdot \eta(v^{*\sigma\tau(k+1)}, \cdots, v^{*\sigma\tau(k+l)}) \cdot \zeta(v^{*\sigma(k+l+1)}, \cdots, v^{*\sigma(k+l+h)}).
\end{aligned}$$

But for each $\tau \in \mathcal{S}_{k+l}$, τ can be regarded as a permutation in $\mathcal{S}_{k+l+h}$ where τ leaves the last h factors invariant. Therefore, by exchanging the two sums we can write (2.9) as

$$
\begin{aligned}
&(\xi \wedge \eta) \wedge \zeta(v^{*1}, \cdots, v^{*k+l+h}) \\
=&\frac{1}{(k+l)!h!} \frac{1}{k!l!} \sum_{\tau \in \mathcal{S}_{k+l}} \sum_{\sigma \in \mathcal{S}_{k+l+h}} \operatorname{sgn}(\sigma\tau) \cdot (\xi \otimes \eta \otimes \zeta)(v^{*\sigma\tau(1)}, \cdots, v^{*\sigma\tau(k+l+h)}).
\end{aligned}
$$

Now it is easy to see that for each $\tau \in \mathcal{S}_{k+l}$,

$$
\begin{aligned}
&\sum_{\sigma \in \mathcal{S}_{k+l+h}} \operatorname{sgn}(\sigma\tau) \cdot (\xi \otimes \eta \otimes \zeta)(v^{*\sigma\tau(1)}, \cdots, v^{*\sigma\tau(k+l+h)}) \\
=&(k+l+h)! A_{k+l+h}(\xi \otimes \eta \otimes \zeta)(v^{*1}, \cdots, v^{*k+l+h}).
\end{aligned}
$$

Therefore we have

$$
(\xi \wedge \eta) \wedge \zeta(v^{*1}, \cdots, v^{*k+l+h}) = \frac{(k+l+h)!}{k!l!h!} A_{k+l+h}(\xi \otimes \eta \otimes \zeta)(v^{*1}, \cdots, v^{*k+l+h}).
$$

Similarly,

$$
\xi \wedge (\eta \wedge \zeta)(v^{*1}, \cdots, v^{*k+l+h}) = \frac{(k+l+h)!}{k!l!h!} A_{k+l+h}(\xi \otimes \eta \otimes \zeta)(v^{*1}, \cdots, v^{*k+l+h}).
$$

Finally, for the antisymmetry, first notice that

$$
\begin{aligned}
&\xi \wedge \eta(v^{*1}, \cdots, v^{*k+l}) \\
=&\frac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma) \xi(v^{*\sigma(1)}, \cdots, v^{*\sigma(k)}) \eta(v^{*\sigma(k+1)}, \cdots, v^{*\sigma(k+l)}).
\end{aligned}
$$

Let $\tau \in \mathcal{S}_{k+l}$ be the permutation

$$
\tau(i) = \begin{cases} k+i, & 1 \leqslant i \leqslant l; \\ i-l, & l+1 \leqslant i \leqslant k+l. \end{cases}
$$

It follows that $\operatorname{sgn}(\tau) = (-1)^{kl}$, and

$$
\begin{aligned}
\xi \wedge \eta(v^{*1}, \cdots, v^{*k+l}) &= \frac{(-1)^{kl}}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma\tau) \eta(v^{*\sigma\tau(1)}, \cdots, v^{*\sigma\tau(l)}) \\
&\quad \cdot \xi(v^{*\sigma\tau(l+1)}, \cdots, v^{*\sigma\tau(k+l)}) \\
&= \frac{(-1)^{kl}}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma\tau) \eta \otimes \xi(v^{*\sigma\tau(1)}, \cdots, v^{*\sigma\tau(k+l)}) \\
&= (-1)^{kl} \eta \wedge \xi(v^{*1}, \cdots, v^{*k+l}).
\end{aligned}
$$

□

From the proof of Proposition 2.4, for any $v_1, \cdots, v_r \in V$, we have

$$v_1 \wedge \cdots \wedge v_r = r! A_r(v_1 \otimes \cdots \otimes v_r). \tag{2.11}$$

Therefore, for any $v^{*1}, \cdots, v^{*r} \in V^*$, we have

$$\begin{aligned} v_1 \wedge \cdots \wedge v_r(v^{*1}, \cdots, v^{*r}) &= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) v^{*\sigma(1)}(v_1) \cdots v^{*\sigma(r)}(v_r) \\ &= \det(v^{*i}(v_j))_{1 \leqslant i,j \leqslant r}. \end{aligned} \tag{2.12}$$

(2.12) is called the *evaluation formula* for $v_1 \wedge \cdots \wedge v_r$.

Now assume that $\{e_i : 1 \leqslant i \leqslant n\}$ is a basis of V. We are going to construct a basis of $\Lambda^r(V)$ in terms of $\{e_i\}$.

First of all, for any $r > n$, if $x \in \Lambda^r(V) \subset T^r(V)$, then

$$x = x^{i_1 \cdots i_r} e_{i_1} \otimes \cdots e_{i_r}.$$

Applying A_r on both sides, we have

$$\begin{aligned} x &= x^{i_1 \cdots i_r} A_r(e_{i_1} \otimes \cdots \otimes e_{i_r}) \\ &= \frac{x^{i_1 \cdots i_r}}{r!} e_{i_1} \wedge \cdots \wedge e_{i_r}. \end{aligned}$$

Since $r > n$, at least two of those e_{i_j} on the R.H.S. are the same. By Proposition 2.4, we have $x = 0$. Therefore, $\Lambda^r(V) = 0$.

On the other hand, for any $1 \leqslant r \leqslant n$, we are going to show that $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$ form a basis of $\Lambda^r(V)$. In fact, the same reason as before shows that $\Lambda^r(V)$ is spanned by $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$. To see they are linearly independent, let

$$\sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} a^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = 0. \tag{2.13}$$

For each $1 \leqslant j_1 < \cdots < j_r \leqslant 1$, let $k_1 < \cdots < k_{n-r}$ be such that

$$\{j_1, \cdots, j_r, k_1, \cdots, k_{n-r}\} = \{1, \cdots, n\}.$$

By taking exterior product with $e_{k_1} \wedge \cdots \wedge e_{k_{n-r}}$ on both sides of (2.13), we have

$$\pm a^{j_1 \cdots j_r} e_1 \wedge \cdots \wedge e_n = 0.$$

By acting on $(e^{*1}, \cdots, e^{*n})$ where $\{e^{*i} : 1 \leqslant i \leqslant n\}$ is the dual basis of $\{e_i\}$, it follows from (2.12) that

$$a^{j_1 \cdots j_r} = 0.$$

Therefore, $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$ are linearly independent, and hence they form a basis of $\Lambda^r(V)$. It is easy to see that

$$\dim \Lambda^r(V) = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}.$$

Similar to the case of tensor algebra, we can define the exterior algebra over V. Let $\Lambda(V)$ be the direct sum

$$\Lambda(V) = \sum_{r \geqslant 0} \Lambda^r(V).$$

This is in fact a finite direct sum from degree 0 to degree $n = \dim V$. $\Lambda(V)$ carries a canonical product defined in the same way as in (2.7), but replacing "$\otimes$" by the exterior product "$\wedge$". Under the basis $\{e_i\}$ of V, $\Lambda(V)$ is a 2^n-dimensional algebra over $\mathbb{R}$ with basis

$$\{1, e_{i_1} \wedge \cdots \wedge e_{i_r} : \ 1 \leqslant r \leqslant n, 1 \leqslant i_1 < \cdots < i_r \leqslant n\}.$$

$\Lambda(V)$ is called the *exterior algebra over V*.

Since V^* is also a finite dimensional real vector space, we can also construct the exterior algebra over V^* in the same way.

There is a natural pairing of $\Lambda^r(V^*)$ and $\Lambda^r(V)$ such that $\Lambda^r(V^*)$ is the dual space of $\Lambda^r(V)$. On monomials such paring is defined to be

$$\langle v^{*1} \wedge \cdots \wedge v^{*r}, v_1 \wedge \cdots \wedge v_r \rangle = \det(v^{*i}(v_j))_{1 \leqslant i,j \leqslant r}. \tag{2.14}$$

We leave it as an exercise to see that this pairing is well-defined. In particular, under the basis $\{e_i\}$ of V, $\{e^{*i_1} \wedge \cdots \wedge e^{*i_r} : \ 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$ and $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : \ 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$ are dual to each other. In this manner $\Lambda(V^*)$ becomes the dual space of $\Lambda(V)$.

Remark 2.4. Since $\Lambda^r(V) \subset T^r(V)$ and $\Lambda^r(V^*) \subset T^r(V^*)$, by using the pairing (2.2) of tensor products we have another pairing of $\Lambda^r(V^*)$ and $\Lambda^r(V)$. These two pairings defer by a constant factor:

$$\varphi(\xi) = r!\langle \varphi, \xi \rangle, \ \forall \varphi \in \Lambda^r(V^*), \xi \in \Lambda^r(V).$$

A linear map $F : \ V \to W$ between two finite-dimensional real vector spaces V and W induces the *pullback* of exterior (covariant) vectors by F. More precisely, for each $r \geqslant 1$, define

$$F^* : \ \Lambda^r(W^*) \to \Lambda^r(V^*)$$

by

$$(F^*\varphi)(v_1, \cdots, v_r) = \varphi(F(v_1), \cdots, F(v_r)), \ \varphi \in \Lambda^r(W^*), (v_1, \cdots, v_r) \in V \times \cdots \times V.$$

When $r = 0$, we define F^* to be the identity map from $\mathbb{R}$ to $\mathbb{R}$. In particular, when $r = 1$, F^* is just the dual map of F. From Proposition 2.3, it is easy to see that $F^*\varphi \in \Lambda^r(V^*)$. Moreover, we have the following result.

Proposition 2.5. $F^* : \ \Lambda(W^*) \to \Lambda(V^*)$ *is an algebra homomorphism.*

Proof. The only less trivial part is to prove that for any $\varphi \in \Lambda^k(W^*), \psi \in \Lambda^l(W^*)$,

$$F^*(\varphi \wedge \psi) = F^*\varphi \wedge F^*\psi. \tag{2.15}$$

In fact, by definition we have

$$\begin{aligned} F^*(\varphi \wedge \psi)(v_1, \cdots, v_{k+l}) &= \varphi \wedge \psi(F(v_1), \cdots, F(v_{k+l})) \\ &= \frac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma) \cdot \varphi(F(v_{\sigma(1)}), \cdots, F(v_{\sigma(k)})) \\ &\quad \cdot \psi(F(v_{\sigma(k+1)}), \cdots, F(v_{\sigma(k+l)})) \\ &= \frac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sgn}(\sigma) \cdot (F^*\varphi)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &\quad \cdot (F^*\psi)(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}) \\ &= F^*\varphi \wedge F^*\psi(v_1, \cdots, v_{k+l}). \end{aligned}$$

Therefore (2.15) holds. □

If $F: V \to W$ and $G: W \to Z$ are linear maps between finite dimensional real vector spaces, then by definition we have

$$(G \circ F)^* = F^* \circ G^*. \tag{2.16}$$

2.3 Differential forms and the exterior differentiation

The analysis of (r, s)-type tensor fields and differential forms is closely related to global geometric and topological features of a manifold. (r, s)-type tensor fields and differential forms are sections of the (r, s)-type tensor bundle and the exterior algebra bundle over a manifold. The construction of these bundles are very similar to the case of the tangent bundle.

Let M be a manifold of dimension n.

For $p \in M$, let

$$T^r_s(p) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{r} \otimes \underbrace{T^*_pM \otimes \cdots \otimes T^*_pM}_{s}.$$

$T^r_s(p)$ is a real vector space of dimension n^{r+s}. Define

$$T^r_s(M) = \cup_{p \in M} T^r_s(p).$$

Similar to the construction of the tangent bundle, $T^r_s(M)$ has a canonical differential structure induced by the one on M, such that it becomes a manifold of dimension $n + n^{r+s}$. More precisely, let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an atlas in the differential structure on M. For each $p \in U_\alpha$, $T^r_s(p)$ has a canonical basis given by

$$\{\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes dx^{j_s} : \ 1 \leqslant i_1, \cdots, i_r, j_1, \cdots, j_s \leqslant n\}$$

evaluated at p. The associated coordinate chart

$$\Phi_\alpha^{(r,s)}: \ E_\alpha^{(r,s)} = \cup_{p\in U_\alpha} T_s^r(p) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^{n^{r+s}} \subset \mathbb{R}^n \times \mathbb{R}^{n^{r+s}}$$

is given by

$$\Phi_\alpha^{(r,s)}(\xi) = (x^i, a_{j_1\cdots j_s}^{i_1\cdots i_r})_{1\leqslant i,i_1,\cdots,i_r,j_1,\cdots,j_s\leqslant n},$$

where x^i are the coordinates of the base point of ξ and $a_{j_1\cdots j_s}^{i_1\cdots i_r}$ are the coordinates of ξ under the natural basis of $T_s^r(p)$. By Proposition 2.1 one can easily write down the change of coordinates formula between two charts in terms of the Jacobian of the change of coordinates on M. In particular, along the fiber $T_s^r(p)$ the change of coordinates is a linear isomorphism. It follows by the same argument as in the case of the tangent bundle that $T_s^r(M)$ is a manifold of dimension $n + n^{r+s}$.

Definition 2.6. $T_s^r(M)$ is called the (r,s)-*type tensor bundle over* M. An (r,s)-*type tensor field on* M is a section of $T_s^r(M)$, i.e., a C^∞ map $\xi: \ M \to T_s^r(M)$ such that $\pi \circ \xi = \mathrm{id}_M$ where π is the natural projection map.

When $r = 1, s = 0$, $T_s^r(M)$ is simply the tangent bundle, and when $r = 0, s = 1$ this is the cotangent bundle. By using the tensor product defined in (2.7) along each fiber, we are able to form the tensor product of an (r_1, s_1)-type tensor field and an (r_2, s_2)-type tensor field, which yields an (r_1+r_2, s_1+s_2)-type tensor field.

In exactly the same way, we can construct the *exterior* r-*bundle*

$$\Lambda^r(M) = \cup_{p\in M}\Lambda^r(T_p^*M)$$

and the *exterior algebra bundle*

$$\Lambda(M) = \cup_{p\in M}\Lambda(T_p^*M)$$

over M. Under a coordinate chart (U, x^i) of M, the natural basis of $\Lambda^r(T_p^*M)$ for $p \in U$ is given by

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_r}: \ 1 \leqslant i_1 < \cdots < i_r \leqslant n\}$$

evaluated at p. It turns out that $\Lambda^r(M)$ ($\Lambda(M)$, respectively) is a manifold of dimension $n + \begin{pmatrix} n \\ r \end{pmatrix}$ ($n + 2^n$, respectively).

Definition 2.7. A section of the exterior r-bundle $\Lambda^r(M)$ is called a (*smooth*) r-*form on* M. A *differential form on* M is a section of the exterior algebra bundle $\Lambda(M)$. The space of r-forms (differential forms, respectively) is denoted by $\Omega^r(M)$ ($\Omega(M)$, respectively).

By definition, $\Omega^0(M)$ is simply the space $C^\infty(M)$ of smooth functions on M. Moreover, a differential form ω can be uniquely written as

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^n,$$

where each ω^r is an r-form in $\Omega^r(M)$. There is a canonical exterior product structure on the real vector space $\Omega(M)$ induced by the exterior product "$\wedge$" on each fiber pointwisely. It turns out that $\Omega(M)$ is an infinite dimensional (graded) algebra over $\mathbb{R}$.

By the definition of the exterior algebra, an r-form ω can be regarded as a $(0, r)$-type (antisymmetric) tensor field. According to Proposition 2.3, ω acts on the space $\mathfrak{X}(M)$ of smooth vector fields on M pointwisely as an alternating r-linear map

$$\omega : \ \mathfrak{X}(M) \times \cdots \mathfrak{X}(M) \to C^\infty(M).$$

It is not hard to see that this is also an equivalent characterization of r-forms on M. This point is quite convenient and important in the calculation of differential forms. Under a coordinate chart (U, x^i), an r-form ω can be expressed as

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Like the case of functions, differential forms can be "differentiated" via the exterior differentiation. This is a fundamental concept in the analysis on manifolds.

Theorem 2.2. *There exists a unique linear operator*

$$d : \ \Omega(M) \to \Omega(M)$$

which satisfies the following properties.

(1) For any $r \geqslant 0$,

$$d(\Omega^r(M)) \subset \Omega^{r+1}(M).$$

(2) If ω_1 *is an* r*-form, then for any* $\omega_2 \in \Omega(M)$,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2.$$

(3) If $f \in C^\infty(M)$*, then* df *is differential of* f.

(4) If $f \in C^\infty(M)$*, then*

$$d^2 f = 0.$$

Proof. We prove the theorem by several steps.

(1) We show that if d exists, then it is a local operator. More precisely, for any open subset U, if $\omega_1, \omega_2 \in \Omega(M)$ with

$$\omega_1|_U = \omega_2|_U,$$

then

$$d\omega_1|_U = d\omega_2|_U.$$

By linearity, it suffices to show that: for any open subset U, if $\omega \in \Omega(M)$ vanishes on U, then $d\omega$ vanishes on U as well. In fact, for any $p \in U$, choose a bump function h near x with respect to (V, U) where $V \subset U$ is some open neighborhood of p whose closure is compact. It follows from property (2) that

$$d(h\omega) = dh \wedge \omega + h \wedge d\omega. \tag{2.17}$$

But $h\omega \equiv 0$, by property evaluating (2.17) at p we obtain that $d\omega(p) = 0$. Since $p \in U$ is arbitrary, we see that $d\omega = 0$ on U.

(2) If d exists, then for any open subset U, d induces a linear operator $d_U : \Omega(U) \to \Omega(U)$ satisfying (1) to (4). Explicitly, if $\omega \in \Omega(U)$, then we define

$$(d_U\omega)(p) = (d\widetilde{\omega})(p), \ p \in U,$$

where $\widetilde{\omega} \in \Omega(M)$ is an extension of ω near p. If we have another such extension $\overline{\omega}$, then $\widetilde{\omega} = \overline{\omega}$ in a neighborhood of p. By the locality of d, we know that

$$(d\widetilde{\omega})(p) = (d\overline{\omega})(p).$$

Therefore, d_U is well-defined. Moreover, it is easy to see that d_U satisfies properties (1) to (4). For example, to see (4) holds, notice that for any $f \in C^\infty(U)$ and $p \in U$,

$$(d_U^2 f)(p) = d(\widetilde{d_U f})(p),$$

where $\widetilde{d_U f} \in \Omega^1(M)$ is an extension of $d_U f$ near p. But if we choose an extension $\widetilde{f} \in C^\infty(M)$ of f near p, then $d\widetilde{f}$ is an extension of $d_U f$. It follows from property (4) for $\widetilde{f}$ that

$$(d_U^2 f)(p) = 0.$$

Since $p \in U$ is arbitrary, we know that $d_U^2 f = 0$. We call d_U the restriction of d on U.

(3) Let (U, x^i) be a coordinate chart on M. If $\omega \in \Omega(U)$ is given by

$$\omega = \sum_{r=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \tag{2.18}$$

then define

$$d_U\omega = \sum_{r=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} da_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}. \tag{2.19}$$

It is easy to see that $d_U : \Omega(U) \to \Omega(U)$ is a linear operator satisfying properties (1) to (4). By the previous step, we know that d_U can be restricted to any open subset of U, which is still given by (2.19). Now let (V, y^j) be another coordinate chart on M with $U \cap V \neq \emptyset$ and define $d_V : \Omega(V) \to \Omega(V)$ in the same way. Then the restriction of d_V on $U \cap V$, denoted by $d_V|_{U\cap V}$, satisfies properties (1) to (4). In particular, for $\omega \in \Omega(U \cap V)$ given by (2.18), we have

$$d_V|_{U\cap V}(\omega) = \sum_{r=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} da_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = d_U|_{U\cap V}(\omega),$$

since x^{i_l} are coordinate functions on $U \cap V$ and hence $d_V|_{U\cap V} dx^{i_l} = 0$.

(4) For $\omega \in \Omega(M)$, define $d\omega$ by

$$(d\omega)(p) = (d_U \omega|_U)(p)$$

for any coordinate chart (U, x^i) on M and any $p \in U$, where d_U is the linear operator defined in step (3). It follows from the discussion in step (3) that d is a globally well-defined linear operator satisfying properties (1) to (4).

(5) Let $d' : \Omega(M) \to \Omega(M)$ be a linear operator satisfying properties (1) to (4). Then for any $\omega \in \Omega(M)$ and any coordinate chart (U, x^i) on M, we have

$$(d'\omega)(p) = (d'_U \omega|_U)(p), \; p \in U,$$

where d'_U is the restriction of d' to U. Since d'_U satisfies properties (1) to (4), the same argument in step (3) shows that it is given exactly by (2.19), which is the same as the restriction of d on U. Therefore, $d' = d$.

Now the proof is complete. □

By the locality of the exterior differential d, computation on coordinate charts yields the following fact.

Proposition 2.6. $d^2 = 0$.

Example 2.1. Consider $M = \mathbb{R}^3$, and use (x, y, z) to denote the canonical coordinates of M.

(1) For a C^∞ function f,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

The coefficients of df form the gradient of f.

(2) For a one form

$$\omega = adx + bdy + cdz,$$

we have

$$\begin{aligned} d\omega &= (\frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz) \wedge dx + (\frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial y}dy + \frac{\partial b}{\partial z}dz) \wedge dy \\ &\quad +(\frac{\partial c}{\partial x}dx + \frac{\partial c}{\partial y}dy + \frac{\partial c}{\partial z}dz) \wedge dz \\ &= (\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})dy \wedge dz + (\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x})dz \wedge dx + (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy. \end{aligned}$$

The coefficients of $d\omega$ form the curl of the vector field (a, b, c).

(3) For a 2-form

$$\omega = ady \wedge dz + bdz \wedge dx + cdx \wedge dy,$$

we have

$$d\omega = (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z})dx \wedge dy \wedge dz.$$

The coefficient of $d\omega$ is the divergence of the vector field (a, b, c).

From the previous computation, Proposition 2.6 in the case of $M = \mathbb{R}^3$ takes the following classical form: for any smooth function f and smooth vector field X on $\mathbb{R}^3$,

$$\begin{cases} \mathrm{curl}(\mathrm{grad} f) = 0; \\ \mathrm{div}(\mathrm{curl} X) = 0. \end{cases}$$

In the study of exterior algebra, we've seen that a linear map induces the pullback of exterior vectors. Since differential forms are pointwise exterior vectors over cotangent spaces, a C^∞ map between manifolds will induce the pullback of differential forms naturally.

More precisely, assume that $F: M \to N$ is a C^∞ map between manifolds M and N. We define the *pullback of differential forms by F* to be the linear map $F^*: \Omega(N) \to \Omega(M)$ given by

$$(F^*\omega)(p) = (dF)_p^*\omega(F(p)), \ p \in M,$$

where each $(dF)_p^*$ is the pullback by the linear map $(dF)_p: T_pM \to T_{F(p)}N$. If ω is a zero form (i.e., a C^∞ function), then by definition $F^*\omega$ is the composition of F and ω. If ω is a one form, $(F^*\omega)(p)$ is just the pullback of $\omega(F(p))$ by F at p in the sense of Definition 1.10.

To see the smoothness of $F^*\omega$, we can simply work it out explicitly in coordinate charts. Fix $p \in M$ and let $q = F(p)$. Choose some coordinate charts (U, x^α) around p and (V, y^i) around q such that $F(U) \subset V$. Under V we may write

$$\omega|_V = \sum_{r=0}^{n} \sum_{i_1 < \cdots < i_r} a_{i_1 \cdots i_r} dy^{i_1} \wedge \cdots \wedge dy^{i_r}.$$

It follows from Proposition 2.5 and the case for pulling back zero and one forms that

$$\begin{aligned} &(F^*\omega)|_U \\ =&\sum_{r=0}^{n} \sum_{i_1 < \cdots < i_r} (a_{i_1 \cdots i_r} \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_r} \\ =&\sum_{r=0}^{n} \sum_{i_1 < \cdots < i_r} (a_{i_1 \cdots i_r} \circ F) \sum_{\alpha_1, \cdots, \alpha_r = 1}^{m} \frac{\partial F^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial F^{i_r}}{\partial x^{\alpha_r}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_r}, \end{aligned} \tag{2.20}$$

where under U and V, F is expressed as

$$y^i = F^i(x^1, \cdots, x^m), \ i = 1, \cdots, n.$$

From this we can easily see that $F^*\omega$ is smooth in U, in particular, it is smooth at p. Since p is arbitrary, we know that $F^*\omega$ defines a differential form on M.

Equivalently, for an r-form ω on N and for any smooth vector fields $X_1, \cdots, X_r$ on M, as an alternating r-linear map

$$(F^*\omega)(X_1, \cdots, X_r) = \omega(dF(X_1), \cdots, dF(X_r)) \in C^\infty(M).$$

By pushing one step forward in the calculation in (2.20), we can easily show the following very important result.

Proposition 2.7. *F^* commutes with d. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \Omega(N) & \xrightarrow{F^*} & \Omega(M) \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\ \Omega(N) & \xrightarrow{F^*} & \Omega(M). \end{array}$$

Proof. Use the previous notation, we know that

$$(d\omega)|_V = \sum_{r=0}^{n} \sum_{i_1<\cdots<i_r} da_{i_1\cdots i_r} \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_r},$$

and therefore

$$(F^*d\omega)|_U = \sum_{r=0}^{n} \sum_{i_1<\cdots<i_r} d(a_{i_1\cdots i_r} \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_r}.$$

But from the first equality in (2.20) and the properties of d, we have

$$(dF^*\omega)|_U = \sum_{r=0}^{n} \sum_{i_1<\cdots<i_r} d(a_{i_1\cdots i_r} \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_r}.$$

Therefore, $F^*d\omega = dF^*\omega$ in U and in particular at p. Since p is arbitrary, we have $F^*d = dF^*$. □

Since the pullback of differential forms is defined pointwisely, from (2.16) we know immediately that if $F:\ M \to N$ and $G:\ N \to P$ are C^∞ maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^* :\ \Omega(P) \to \Omega(M). \tag{2.21}$$

2.4 One-parameter groups of diffeomorphisms and the Lie derivative

There is another natural way of thinking about smooth vector fields. If we know the direction of the tangent vector at each point on the manifold, in principle we should be able to follow a trajectory determined by the vector field to move along the manifold starting from any initial position. First proceeding for time t and then proceeding for time s is the same as proceeding for time $t+s$ without any stop. Moreover, as your starting position varies, the trajectory varies in a smooth manner. Mathematically, this is the relation between a one-parameter group of diffeomorphisms and its generating smooth vector field.

Let M be a manifold of dimension n.

Definition 2.8. A one-parameter group of diffeomorphisms on M is a C^∞ map

$$\varphi:\ M\times\mathbb{R}^1\to M$$

such that:

(1) for each $t\in\mathbb{R}^1$, $\varphi_t(\cdot)=\varphi(\cdot,t):\ M\to M$ is a diffeomorphism;

(2) $\varphi_0=\mathrm{id}_M$;

(3) $\varphi_{s+t}=\varphi_s\circ\varphi_t$.

A one-parameter group of diffeomorphisms on M specifies the global trajectories explicitly about how to move on M. At any point $p\in M$, by following the trajectory $\varphi_t(p)$ infinitesimally, we are able to determine the direction at p (a tangent vector). More precisely, let $X_p\in T_pM$ be the tangent vector of the smooth curve $\varphi_t(p)$ at $t=0$. This defines a vector field X on M. The smoothness of X follows easily from the smoothness of φ. For any $f\in C^\infty$, we have

$$(Xf)(p)=\frac{d}{dt}|_{t=0}f(\varphi_t(p)).$$

X is called *the generating vector field of* φ.

An interesting and important point is the converse: if we have a smooth vector field X, is there a one-parameter group of diffeomorphisms φ whose generating vector field is X?

In general, the answer is no. Just consider $M=(-\infty,0)$ and the unit speed vector field $X=\frac{\partial}{\partial x}$. For a particle starting at any $x\in M$, the vector field X will bring the particle to the boundary $x=0$ and "blow" it out of M in finite time (the trajectory is not defined globally).

However, a smooth vector field X can always be integrated to a *local* one-parameter group of diffeomorphisms.

Definition 2.9. A smooth curve $\gamma:\ (a,b)\to M$ is called *an integral curve of* X if for any $t\in(a,b)$,

$$(d\gamma)_t(\frac{\partial}{\partial t}|_t)=X_{\gamma(t)}.$$

In other words, the tangent vector of γ at t is $X_{\gamma(t)}$.

Locally we can always find an integral curve of X starting at a point p. Essentially this is just the problem of solving ordinary differential equations (ODEs).

Proposition 2.8. *For any* $p\in M$, *there exists some* $\varepsilon>0$ *and a smooth curve* $\gamma:\ (-\varepsilon,\varepsilon)\to M$, *such that* $\gamma(0)=p$ *and* γ *is an integral curve of* X.

Proof. Choose a coordinate chart (U,x^i) around p. Under U an integral curve $\gamma(t)=(\gamma^1(t),\cdots,\gamma^n(t))$ of X starting at p is determined by the ODE

$$\begin{cases}\frac{d\gamma^i}{dt}=X^i(\gamma(t)),\\ \gamma(0)=p,\end{cases}\tag{2.22}$$

where

$$X(x) = X^i(x)\frac{\partial}{\partial x^i}, \ x \in U.$$

According to ODE theory, there exists some $\varepsilon > 0$ and a unique solution $\gamma : (-\varepsilon, \varepsilon) \to M$ to (2.22). This gives an integral curve of X starting at p locally. □

By the local uniqueness in ODE theory, an integral curve of Xstarting at p can be uniquely extended to the maximal domain. Let $\mathcal{I}$ be the set of open intervals (a, b) such that $0 \in (a, b)$ and there exists some integral curve $\gamma : (a, b) \to M$ of X with $\gamma(0) = p$. The fact that $\mathcal{I}$ is non-empty follows from Proposition 2.8. Moreover, let $(a, b), (c, d) \in \mathcal{I}$ and γ_1, γ_2 be integral curves of X defined on (a, b) and (c, d) respectively, with $\gamma_1(0) = \gamma_2(0) = p$. Consider the set

$$T = \{t \in (a, b) \cap (c, d) : \ \gamma_1(t) = \gamma_2(t)\}.$$

By continuity it is obvious that T is closed in $(a, b) \cap (c, d)$. On the other hand, for any $t_0 \in T$, since γ_1 and γ_2 are both integral curves of X with $\gamma_1(t_0) = \gamma_2(t_0)$, under some coordinate chart around $\gamma_1(t_0)$, by the uniqueness part in ODE theory we know that γ_1 and γ_2 must coincide in some neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon)$ of t_0. Therefore, T is open in $(a, b) \cap (c, d)$. Since $0 \in T$, by the connectedness of $(a, b) \cap (c, d)$ we know that

$$T = (a, b) \cap (c, d).$$

In other words, $\gamma_1 = \gamma_2$ on $(a, b) \cap (c, d)$. Therefore, if we let

$$(a(p), b(p)) = \cup_{(a,b) \in \mathcal{I}}(a, b),$$

then we can define an integral curve $\gamma_p : \ (a(p), b(p)) \to M$ of X with $\gamma_p(0) = p$. Obviously, $(a(p), (b))$ is the maximal interval containing 0 on which we can define an integral curve of X starting at p. We shall call γ_p the *maximal* integral curve of X starting at p.

For any $t \in \mathbb{R}^1$, define

$$\mathcal{D}_t = \{p \in M : \ t \in (a(p), b(p))\}.$$

Intuitively, $\mathcal{D}_t$ is the set of points on M starting from which along the maximal integral curve one can proceed at least for time t. Obviously $\mathcal{D}_0 = M$. Moreover, it is possible that $\mathcal{D}_t$ is empty for some $t \neq 0$. However, by Proposition 2.8 we know that $\mathcal{D}_t \neq \emptyset$ when t is small and

$$M = \cup_{t>0}\mathcal{D}_t = \cup_{t<0}\mathcal{D}_t.$$

For $t \in \mathbb{R}^1$ and $p \in \mathcal{D}_t$, define

$$\varphi_t(p) = \gamma_p(t).$$

We are going to show that φ is a local one-parameter group of diffeomorphisms whose generating vector field is X. But we need to make it more precise since φ is not globally defined.

The most important point is the group property. For $t \in \mathbb{R}^1$ and $p \in \mathcal{D}_t$, consider the curve

$$s \mapsto \gamma_p(s+t).$$

This is an integral curve of X starting at $\gamma_p(t)$ defined for all $s \in (a(p)-t, b(p)-t)$. Therefore,

$$(a(p)-t, b(p)-t) \subset (a(\gamma_p(t)), b(\gamma_p(t))).$$

On the other hand, the curve

$$s \mapsto \gamma_{\gamma_p(t)}(s-t)$$

is an integral curve of X starting at p defined for all $s \in (a(\gamma_p(t))+t, b(\gamma_p(t))+t)$, and hence

$$(a(\gamma_p(t))+t, b(\gamma_p(t))+t) \subset (a(p), b(p)).$$

It follows that

$$(a(p)-t, b(p)-t) = (a(\gamma_p(t)), b(\gamma_p(t))). \tag{2.23}$$

Consequently, the domain of $\varphi_s \circ \varphi_t$, denoted by

$$\widetilde{\mathcal{D}}_{s,t} = \{p \in M : \ t \in (a(p), b(p)) \text{ and } s \in (a(\gamma_p(t)), b(\gamma_p(t)))\},$$

is contained in the domain $\mathcal{D}_{s+t}$ of φ_{s+t}. Moreover, by (2.23) we know that both $\varphi_{\cdot} \circ \varphi_t(p)$ and $\varphi_{\cdot+t}(p)$ can be regarded as the maximal integral curve of X starting at $\gamma_p(t)$. Therefore, we have

$$\varphi_s \circ \varphi_t(p) = \varphi_{s+t}(p), \ \forall p \in \widetilde{\mathcal{D}}_{s,t}. \tag{2.24}$$

It should be pointed out that in general $\widetilde{\mathcal{D}}_{s,t} \neq \mathcal{D}_{s+t}$, but they are equal if s and t have the same sign.

Now we show that each $\mathcal{D}_t$ (if not empty) is open and φ_t is a diffeomorphism from $\mathcal{D}_t$ to $\mathcal{D}_{-t}$ with inverse φ_{-t}. From (2.24) with $s=-t$ it is trivial to see that φ is a bijection with inverse φ_{-t}. The openness of $\mathcal{D}_t$ and the smoothness of φ_t follow from the following key result.

Proposition 2.9. *For any $p \in M$, there exists some $\varepsilon > 0$ and some open neighborhood V of p, such that the map*

$$(t, q) \mapsto \varphi_t(q) \tag{2.25}$$

is well-defined and C^∞ from $(-\varepsilon, \varepsilon) \times V$ to M.

Proof. This is a direct consequence of the local smooth dependence of solutions to ODEs on their initial values. More precisely, first take a coordinate chart U around p, then standard results in ODE theory tell us that there exists $\varepsilon > 0$ and some open neighborhood $V \subset U$ of p, such that for each $q \in V$, the ODE (2.22) with initial condition at q has a unique solution on $(-\varepsilon, \varepsilon)$ taking values in U, and the solution map (2.25) is C^∞ on $(-\varepsilon, \varepsilon) \times V$. □

Now let $t > 0$ and $p \in \mathcal{D}_t$ (the case of $t < 0$ is similar). Since $\gamma_p([0,t])$ is a compact set in M, by Proposition 2.9 and a standard covering argument, there exists some $\varepsilon > 0$ and some open set W containing $\gamma_p([0,t])$, such that the map (2.25) is well-defined and C^∞ on $(-\varepsilon, \varepsilon) \times W$. Choose a natural number n such that $\frac{t}{n} < \varepsilon$ and let $W_0 = W$. For $i = 1, \cdots, n$, define inductively that

$$\alpha_i = \varphi_{\frac{t}{n}}|_{W_{i-1}} \text{ and } W_i = \alpha_i^{-1}(W_{i-1}) \subset W_{i-1}.$$

It follows that W_n is an open neighborhood of p and W_n is contained in the domain of $\alpha_1 \circ \cdots \circ \alpha_n$. In particular, W_n is contained in the domain of $\varphi_{t/n} \circ \cdots \circ \varphi_{t/n}$. Therefore, from the group property (2.24), we have

$$\alpha_1 \circ \cdots \circ \alpha_n = \varphi_t \tag{2.26}$$

on W_n. In particular, $W_n \subset \mathcal{D}_t$ and hence $\mathcal{D}_t$ is open, and the smoothness of φ_t follows from (2.26) and the smoothness of each α_i. Therefore, for each $t \in \mathbb{R}^1$, φ_t is a diffeomorphism from $\mathcal{D}_t$ (if not empty) to $\mathcal{D}_{-t}$ with inverse φ_{-t}.

It is not hard to see that the domain of φ, denoted by

$$\mathcal{D} = \{(p,t) : \ p \in \mathcal{D}_t\} \subset M \times \mathbb{R}^1$$

is open and φ is C^∞ on $\mathcal{D}$.

In summary, we have now finished the proof that a smooth vector field X can always be integrated to a local one-parameter group φ of diffeomorphisms on M. From the proof we can see that the map φ is maximal, unique and canonical.

Definition 2.10. A smooth vector field X is *complete* if $\mathcal{D} = M \times \mathbb{R}^1$. In other words, for each $p \in M$, the integral curve of X starting at p is well-defined for all $t \in \mathbb{R}^1$.

If a smooth vector field X is complete, then it integrates to a one-parameter group of diffeomorphisms on M in the sense of Definition 2.8. As we've seen before ($M = (-\infty, 0)$ and $X = \frac{\partial}{\partial x}$), X is not necessarily complete. However, we have the following important result in the compact case. The proof is left as an exercise.

Theorem 2.3. *If M is compact, then every smooth vector field X on M is complete.*

By using the local one-parameter group of diffeomorphisms generated by a smooth vector field X, we are not only able to differential C^∞ functions along X (the directional derivative), but also able to make sense of differentiating vector fields and even tensor fields along X. This is know as the Lie derivative, which is a fundamental and very useful notion in differential geometry.

Let ξ be an (r,s)-type tensor field on M. For each $p \in M$, φ_t is well-defined when t is small. A natural idea of differentiating ξ along the direction X_p at p is to pull back the tensor field ξ to the fiber $T_s^r(p)$ along the integral curve γ_p, and differentiate in the classical sense on the vector space $T_s^r(p)$. Before giving the precise definition, let's first illustrate how to pull back ξ along γ_p. By using a coordinate chart around p and linearity, we may assume without loss of generality that

$$\xi = X_1 \otimes \cdots \otimes X_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s$$

near p. The pullback $\Phi_t^* \xi \in T_s^r(p)$ is then defined to be

$$\begin{aligned}(\Phi_t^* \xi)(p) &= (d\varphi_{-t})_{\varphi_t(p)}((X_1)_{\varphi_t(p)}) \otimes \cdots \otimes (d\varphi_{-t})_{\varphi_t(p)}((X_r)_{\varphi_t(p)}) \\ &\quad \otimes \varphi_t^*(\alpha^1_{\varphi_t(p)}) \otimes \cdots \otimes \varphi_t^*(\alpha^s_{\varphi_t(p)}),\end{aligned}$$

where $(d\varphi_{-t})_{\varphi_t(p)}$ is the differential of φ_{-t} at $\varphi_t(p)$ and φ_t^* is the pullback of cotangent vectors by φ_t at p. Note that $\Phi_t^* \xi$ is defined for any $p \in M$ and small t (depending on p).

Definition 2.11. The *Lie derivative of ξ with respect to X*, denoted by $L_X \xi$, is defined to be the (r, s)-type tensor field

$$(L_X \xi)(p) = \lim_{t \to 0} \frac{(\Phi_t^* \xi)(p) - \xi(p)}{t} \in T_s^r(p), \ p \in M. \tag{2.27}$$

We haven't proved that the limit in (2.27) exists and $L_X \xi$ is smooth in p. But this will become clear soon as we move on.

Since the pullback of a C^∞ function is just composition, by definition, when acting on $f \in C^\infty(M)$,

$$(L_X f)(p) = \lim_{t \to 0} \frac{f(\varphi_t(p)) - f(p)}{p} = \frac{d}{dt}|_{t=0} f(\varphi_t(p)) = (Xf)(p).$$

Therefore, $L_X f$ is well-defined and

$$L_X f = Xf \in C^\infty(M). \tag{2.28}$$

Now consider the most important situation: the Lie derivative $L_X Y$ of a smooth vector field Y. Let ψ be the local one-parameter group of diffeomorphisms generated by Y. By definition, for any $f \in C^\infty(M)$,

$$\begin{aligned}(L_X Y)(f)(p) &= \lim_{t \to 0} \frac{(d\varphi_{-t})_{\varphi_t(p)}(Y_{\varphi_t(p)}) - Y_p}{t} f \\ &= \lim_{t \to 0} \frac{Y_{\varphi_t(p)}(f \circ \varphi_{-t}) - Y_p f}{t} \\ &= \frac{d}{dt}|_{t=0} Y_{\varphi_t(p)}(f \circ \varphi_{-t}).\end{aligned}$$

Define a function

$$H(t, u) = f(\varphi_{-t}(\psi_u(\varphi_t(p)))),$$

this is well-defined and C^∞ in some open neighborhood of $(0, 0)$. Moreover, it is easy to see that

$$Y_{\varphi_t(p)}(f \circ \varphi_{-t}) = \frac{\partial H}{\partial u}|_{u=0}.$$

Therefore,

$$(L_X Y)(f)(p) = \frac{\partial^2 H}{\partial t \partial u}|_{(t,u)=(0,0)}.$$

If we let

$$K(s,u,t) = f(\varphi_s(\psi_u(\varphi_t(p)))),$$

then from the chain rule of calculus we have

$$\frac{\partial^2 H}{\partial t \partial u}|_{(t,u)=(0,0)} = -\frac{\partial^2 K}{\partial s \partial u}|_{(s,u,t)=(0,0,0)} + \frac{\partial^2 K}{\partial u \partial t}|_{(s,u,t)=(0,0,0)}.$$

But

$$\begin{aligned}
\frac{\partial^2 K}{\partial s \partial u}|_{(s,u,t)=(0,0,0)} &= \frac{\partial^2}{\partial s \partial u}|_{(s,u)=(0,0)} f(\varphi_s \psi_u(p)) \\
&= \frac{d}{du}|_{u=0} X_{\psi_u(p)} f \\
&= Y_p(Xf),
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{\partial^2 K}{\partial u \partial t}|_{(s,u,t)=(0,0,0)} &= \frac{\partial^2}{\partial u \partial t}|_{(u,t)=(0,0)} f(\psi_u(\varphi_t(p))) \\
&= \frac{d}{dt}|_{t=0} Y_{\varphi_t(p)} f \\
&= X_p(Yf).
\end{aligned}$$

Consequently, the limit in (2.27) exists and we have

$$(L_X Y)(f)(p) = X_p(Yf) - Y_p(Xf) = ([X,Y]f)(p).$$

This further shows that $L_X Y$ is smooth in p and in fact we have proved the following important result.

Proposition 2.10. *For any smooth vector fields X and Y, $L_X Y = [X,Y]$.*

Now let's consider the Lie derivative of a $(0,1)$-type tensor field α (i.e., a one-form). For any smooth vector field Y, by definition we have

$$\begin{aligned}
(L_X \alpha)(Y)(p) &= \lim_{t\to 0} \frac{\varphi_t^* \alpha_{\varphi_t(p)} - \alpha_p}{t}(Y_p) \\
&= \lim_{t\to 0} \frac{\alpha_{\varphi_t(p)}((d\varphi_t)_p(Y_p)) - \alpha_p(Y_p)}{t} \\
&= \lim_{t\to 0} \frac{\alpha_{\varphi_t(p)}((d\varphi_t)_p(Y_p)) - \alpha_{\varphi_t(p)}(Y_{\varphi_t(p)})}{t} \\
&\quad + \lim_{t\to 0} \frac{\alpha_{\varphi_t(p)}(Y_{\varphi_t(p)}) - \alpha_p(Y_p)}{t}. \qquad (2.29)
\end{aligned}$$

The first term on the R.H.S. of (2.29) equals

$$\begin{aligned}
\lim_{t\to 0}\alpha_{\varphi_t(p)}(\frac{(d\varphi_t)_p(Y_p)-Y_{\varphi_t(p)}}{t}) &= \lim_{t\to 0}\alpha_{\varphi_t(p)}((d\varphi_t)_p(\frac{Y_p-(d\varphi_{-t})_{\varphi_t(p)}Y_{\varphi_t(p)}}{t}))\\
&= \lim_{t\to 0}(\varphi_t^*\alpha_{\varphi_t(p)})\frac{Y_p-(d\varphi_{-t})_{\varphi_t(p)}Y_{\varphi_t(p)}}{t}\\
&= -\alpha(L_XY)(p).
\end{aligned}$$

The second term on the R.H.S. of (2.29) equals

$$\frac{d}{dt}|_{t=0}\alpha_{\varphi_t(p)}(Y_{\varphi_t(p)})=X_p(\alpha(Y)).$$

Therefore, we have

$$(L_X\alpha)(Y)=X(\alpha(Y))-\alpha(L_XY).$$

This immediately shows that $L_X\alpha$ is well-defined and smooth in p.

In general, if ξ is an (r,s)-type tensor field on M, locally it is a linear combination of monomials of the form $X_1\otimes\cdots\otimes X_r\otimes\alpha^1\otimes\cdots\otimes\alpha^s$. By the definition of Lie derivatives, it is easy to see that

$$\begin{aligned}
&L_X(X_1\otimes\cdots\otimes X_r\otimes\alpha^1\otimes\cdots\otimes\alpha^s)\\
=&\sum_{i=1}^r X_1\otimes\cdots\otimes L_XX_i\otimes\cdots\otimes X_r\otimes\alpha^1\otimes\cdots\otimes\alpha^s\\
&+\sum_{j=1}^s X_1\otimes\cdots\otimes X_r\otimes\alpha^1\otimes\cdots L_X\alpha^j\otimes\cdots\otimes\alpha^s,
\end{aligned}\tag{2.30}$$

which is well-defined and smooth locally by the previous discussion. Therefore, $L_X\xi$ is a well-defined (r,s)-type tensor field on M. Moreover, from (2.30) it is not hard to see that we've in fact proved the following result.

Proposition 2.11. *Let ξ,η be two tensor fields on M. Then*

(1) $L_X(\xi\otimes\eta)=L_X\xi\otimes\eta+\xi\otimes L_X\eta$.

(2) If ξ is of (r,s)-type, then for any $(0,1)$-type tensor fields $\alpha^1,\cdots,\alpha^r$ and any smooth vector fields $Y_1,\cdots,Y_s$,

$$\begin{aligned}
&(L_X\xi)(\alpha^1,\cdots,\alpha^r,Y_1,\cdots,Y_s)\\
=&X(\xi(\alpha^1,\cdots,\alpha^r,Y_1,\cdots,Y_s))-\sum_{i=1}^r\xi(\alpha^1,\cdots,L_X\alpha^i,\cdots,\alpha^r,Y_1,\cdots,Y_s)\\
&-\sum_{j=1}^s\xi(\alpha^1,\cdots,\alpha^r,Y_1,\cdots,L_XY_j,\cdots,Y_s).
\end{aligned}\tag{2.31}$$

It is easy to see that the second result in the proposition is just a reformulation of the first one (the Leibniz rule) in the dual setting.

The importance of the Lie derivative lies in the case when acting on differential forms. Since each r-form ω can be regarded as a $(0, r)$-type (antisymmetric) tensor field, $L_X\omega$ makes sense and by (2.11) it is not hard to see that

$$L_X(\omega_1 \wedge \omega_2) = L_X\omega_1 \wedge \omega_2 + \omega_1 \wedge L_X\omega_2 \tag{2.32}$$

holds. Equivalently, (2.31) also holds for differential forms.

Moreover, L_X commutes with the exterior differential d. This is because locally we have

$$\varphi_t^* d\omega = d\varphi_t^*\omega$$

according to Proposition 2.7. Therefore,

$$\begin{aligned} L_X d\omega &= \lim_{t\to 0} \frac{\varphi_t^* d\omega - d\omega}{t} \\ &= \lim_{t\to 0} \frac{d\varphi_t^*\omega - d\omega}{t} \\ &= dL_X\omega. \end{aligned} \tag{2.33}$$

Conversely, we can easily show that (2.28), (2.32) and (2.33) together characterize the Lie derivative L_X on differential forms uniquely. The reason is that these three rules tell us explicitly how to compute $L_X\omega$ locally for any differential form ω.

Before going further, we first introduce the interior product on differential forms.

Proposition 2.12. *Given a smooth vector field X on M, there exists a unique linear map $i_X : \Omega(M) \to \Omega(M)$ with the following properties.*

(1) For any $r \geqslant 0$,

$$i_X(\Omega^r(M)) \subset \Omega^{r-1}(M),$$

where $\Omega^{-1}(M) = \{0\}$.

(2) If ω_1 is an r-form, then for any $\omega_2 \in \Omega(M)$,

$$i_X(\omega_1 \wedge \omega_2) = i_X\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge i_X\omega_2.$$

(3) For any one form $\omega \in \Omega^1(M)$,

$$i_X(\omega) = \omega(X).$$

Proof. Like the exterior differential d, the three properties in the proposition tell us explicitly how to compute i_X locally. Therefore, the existence and uniqueness can be proved easily in the same spirit as proving the existence and uniqueness of d. $\square$

Definition 2.12. i_X is called the *interior product*.

Example 2.2. Consider $\alpha = dx \wedge dy$ and $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then

$$\begin{aligned} i_X\alpha &= i_X dx \wedge dy - dx \wedge i_X dy \\ &= (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})xdy - (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})ydx \\ &= xdy - ydx. \end{aligned}$$

It should be pointed out that i_X is not a differential operator, and it does not commute with d. Instead, it acts on differential forms pointwisely. More precisely, we can show that i_X is given explicitly in the following global way. For any r-form ω and smooth vector fields $Y_1, \cdots, Y_{r-1}$,

$$(i_X\omega)(Y_1, \cdots, Y_{r-1}) = \omega(X, Y_1, \cdots, Y_{r-1}). \tag{2.34}$$

(2.34) yields immediately that $i_X^2 = 0$.

Finally, we have the following important result. This is know as the *Cartan formula*.

Theorem 2.4. $L_X = d \circ i_X + i_X \circ d$.

Proof. It suffices to show that $d \circ i_X + i_X \circ d$ satisfies (2.28), (2.32) and (2.33), which can all be verified by definition easily. □

2.5 Integration on manifolds and Stokes' theorem

Now we are going to study integration of compactly supported n-forms over a regular domain on an oriented manifold. The most fundamental theorem in integration theory is Stokes' theorem, which unifies and generalizes the fundamental theorem of calculus in $\mathbb{R}^1$, Green's theorem in $\mathbb{R}^2$, Gauss's theorem in $\mathbb{R}^3$, and Stokes' theorem for 2-dimensional surfaces in $\mathbb{R}^3$ in a simple and elegant form.

Let M be an n-dimensional manifold.

Since we only consider integration on oriented manifolds, we first introduce the notion of orientation.

Definition 2.13. M is called *orientable* if there exists a non-vanishing n-form ω on M. An *orientaion* on M is an $\sim$-equivalence class of non-vanishing n-forms on M, where $\omega' \sim \omega$ if and only if $\omega' = f\omega$ with $f > 0$. M is called *oriented* if it is assigned by an orientation.

Example 2.3. $M = \mathbb{R}^n$ is orientable with the orientation given by $dx^1 \wedge \cdots \wedge dx^n$. For Euclidean space we will always use this canonical orientation.

If M is oriented, let ω be an n-form defining its orientation. Then for each $p \in M$, since $\omega(p) \in \Lambda^n(T_p^*M)$, we can define an orientation on T_pM in the sense of finite dimensional real vector spaces: a basis $\{v_1, \cdots, v_n\}$ of T_pM is positive oriented if and only if

$$\omega(p)(v_1, \cdots, v_n) > 0.$$

An orientation on M is then equivalent to a smooth choice of orientation (equivalence class of basis) on each tangent space. A coordinate chart (U, x^i) is called *oriented* if

$$\omega = a dx^1 \wedge \cdots \wedge dx^n$$

with $a > 0$ everywhere in U. An atlas $\{U_\alpha\}_{\alpha\in A}$ is *oriented* if each U_α is oriented. It is easy to see that the determinant of the Jacobian of change of coordinates in an oriented atlas is always positive. Conversely, we have the following useful result.

Proposition 2.13. *Let $\{U_\alpha\}_{\alpha\in A}$ be an atlas on M. If the determinant of the Jacobian of change of coordinates in this atlas is always positive (an atlas with this property is called orientation compatible), then M is orientable. Moreover, we can assign an orientation on M, such that $\{U_\alpha\}_{\alpha\in A}$ is an oriented atlas under this orientation.*

Proof. Choose a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to the atlas $\{U_\alpha\}_{\alpha\in A}$ with the same index. For each $\alpha \in A$, let x^i_α be the coordinates in U_α. Define

$$\omega = \sum_{\alpha\in A} \varphi_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha.$$

This is obviously well-defined by local finiteness. To see that ω is smooth, for $p \in M$, there exists some open neighborhood V of p, such that V intersections finitely many $\mathrm{supp}\varphi_\alpha$, say $\mathrm{supp}\varphi_{\alpha_1}, \cdots, \mathrm{supp}\varphi_{\alpha_m}$. Since each $\mathrm{supp}\varphi_{\alpha_i}$ is closed, without loss of generality we may assume that for each

$$p \in \mathrm{supp}\varphi_{\alpha_i} \subset U_{\alpha_i}, \ \forall i = 1, \cdots, m.$$

It follows that in an open neighborhood W of p,

$$\omega = \sum_{i=1}^{m} \varphi_{\alpha_i} dx^1_{\alpha_i} \wedge \cdots \wedge dx^n_{\alpha_i},$$

and $W \subset U_{\alpha_i}$ for all $i = 1, \cdots, m$. Therefore, ω is smooth at p. This at the same time shows that $\omega(p)$ is nonzero since if $p \in U_\alpha$, then

$$\omega(p) = (\sum_{i=1}^{m} \varphi_{\alpha_i} \det(\frac{\partial x^k_{\alpha_i}}{\partial x^l_\alpha}(p))_{1\leqslant k,l\leqslant n}) dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

with $\sum_{i=1}^m \varphi_{\alpha_i}(p) = 1$ and the determinants on the R.H.S. are all positive. Therefore, ω is a non-vanishing n-form and hence defines an orientation on M. The same reason shows that $\{U_\alpha\}_{\alpha\in A}$ is oriented under this orientation. □

Definition 2.14. A subset $D \subset M$ is called a *regular domain* if each point of M falls into one of the following three mutually exclusive types:

(A) there exists some open neighborhood U of p such that $U \subset M \backslash D$;

(B) there exists some open neighborhood U of p such that $U \subset D$;

(C) there exists some coordinate chart (U, φ) around p such that $\varphi(p) = 0$ and

$$\varphi(U \cap D) = \varphi(U) \cap \mathbb{H}^n,$$

where $\mathbb{H}^n$ is the upper half space in $\mathbb{R}^n$ defined by $\{(x^1, \cdots, x^n) : \ x^n \geqslant 0\}$.

It is an easy exercise to see that a regular domain D must be a closed subset of M. The set of type (B) points is denoted by $\mathrm{Int}(D)$, which is exactly the interior of D. The set of type (C) points is denoted by ∂D, which is exactly the boundary of D.

A trivial example of a regular domain is $D = M$, in which all points of M are of type (B). Moreover, for a regular domain D, $\partial D = \emptyset$ if and only in D is a union of connected components of M.

If the boundary ∂D of a regular domain D is nonempty, then it carries a canonical differential structure of dimension $n-1$ induced from M, such that $(\partial D, i)$ is an embedded submanifold of M, where i is the inclusion map. In fact, for any $p \in \partial D$, take a coordinate chart (U, φ) around p given in (C). It follows that

$$\varphi|_{U\cap\partial D} : \ U \cap \partial D \to \varphi(U \cap \partial D) = \varphi(U) \cap \partial\mathbb{H}^n$$

is a homeomorphism. If we regard $\varphi(U) \cap \partial\mathbb{H}^n$ as an open set in $\mathbb{R}^{n-1}$ via $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$, then $(U \cap \partial D, \varphi|_{U\cap\partial D})$ defines a coordinate chart around $p \in \partial D$ in the sense of Remark 1.2. By letting p vary, we obtain an atlas on ∂D since the change of coordinates between two coordinate charts arising from (C) restricts to a diffeomorphism between two open sets in $\partial\mathbb{H}^n$. It follows that ∂D is an $(n-1)$-dimensional manifold whose manifold topology is the relative topology. Moreover, it is trivial to see that $(\partial D, i)$ is a submanifold of M, and hence by Theorem 1.7 we know that $(\partial D, i)$ is an embedding.

From now on, assume that M is oriented, and let D be a regular domain on M.

Let ω be a compactly supported n-form on M. We are going to define the integration of ω over D. The idea is to localize to the Euclidean case by using the partition of unity.

First consider the case when $\mathrm{supp}\omega$ is contained in some oriented coordinate chart (U, x^i). Write

$$\omega = a dx^1 \wedge \cdots \wedge dx^n$$

under U. If $\mathrm{supp}\omega \cap D = \emptyset$, define

$$\int_D \omega = 0;$$

otherwise define

$$\int_D \omega = \int_{\varphi(U\cap D)} a dx^1 \cdots dx^n, \tag{2.35}$$

where φ is the coordinate map and the R.H.S. of 2.35 is the Lebesgue integral. This is well-defined since a is compactly supported in $\varphi(U)$. Now assume that (V, y^i) is another oriented coordinate chart containing the support of ω, and

$$\omega = b dy^1 \wedge \cdots \wedge dy^n.$$

Since $\text{supp}\omega \cap D \subset U \cap V$, by the change of variables formula for Lebesgue integrals, we have

$$\begin{aligned}\int_{\psi(V\cap D)} b dy^1 \cdots dy^n &= \int_{\psi(U\cap V\cap D)} b dy^1 \cdots dy^n \\ &= \int_{\varphi(U\cap V\cap D)} a dx^1 \cdots dx^n \\ &= \int_{\varphi(U\cap D)} a dx^1 \cdots dx^n.\end{aligned}$$

Therefore, the definition of $\int_D \omega$ is independent of such coordinate charts. It is easy to see that if ω_1, ω_2 are both compactly supported in an oriented coordinate chart U, then

$$\int_D (\omega_1 + \omega_2) = \int_D \omega_1 + \int_D \omega_2. \tag{2.36}$$

Now consider the general case. Let $\{U_\alpha\}_{\alpha\in A}$ be an oriented atlas on M, and choose a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$ with the same index. Define

$$\int_D \omega = \sum_{\alpha\in A} \int_D \varphi_\alpha \omega. \tag{2.37}$$

First of all, it is a finite sum since ω is compactly supported. Moreover, since $\varphi_\alpha\omega$ is compactly supported in U_α, each term is well-defined as before. Finally, if we have another such atlas $\{V_i\}_{i\in I}$ and a partition of unity $\{\psi_i\}_{i\in I}$ subordinate to this atlas with the same index, then by the previous linearity (2.36),

$$\begin{aligned}\sum_{i\in I} \int_D \psi_i \omega &= \sum_{i\in I} \int_D (\sum_{\alpha\in A} \varphi_\alpha) \psi_i \omega \\ &= \sum_{i,\alpha} \int_D \varphi_\alpha \psi_i \omega \\ &= \sum_{\alpha\in A} \int_D (\sum_{i\in I} \psi_i) \varphi_\alpha \omega \\ &= \sum_{\alpha\in A} \int_D \varphi_\alpha \omega,\end{aligned}$$

where each sum is a finite sum by compactness. Therefore, $\int_D \omega$ is well-defined.

Definition 2.15. The real number defined by (2.37) is called the *integral of ω over D*.

It follows immediately that $\int_D \cdot$ is linear on the space of compactly supported n-forms.

Now we are going to prove Stokes' theorem.

Let D be a regular domain on M. Informally, Stokes' theorem says that for a compactly supported $(n-1)$-form ω, the integral of $d\omega$ over D is the same as the integral of ω over ∂D.

To formulate the result more precisely, we first need an induced orientation on ∂D. For each $p \in \partial D$, choose an oriented coordinate chart (U_p, x^i) according to (C) of Definition 2.14. The Jacobian of change of coordinates restricted on ∂D takes the form

$$J = \begin{pmatrix} (\frac{\partial y^i}{\partial x^j}(x^1, \cdots, x^{n-1}, 0))_{1 \leqslant i,j \leqslant n-1} & * \\ 0 & \frac{\partial y^n}{\partial x^n}(x^1, \cdots, x^{n-1}, 0) \end{pmatrix}.$$

Since the coordinate charts are both oriented, we know that the determinant of J is positive. It follows again from (C) that

$$\frac{\partial y^n}{\partial x^n}(x^1, \cdots, x^{n-1}, 0) > 0,$$

and therefore

$$\det((\frac{\partial y^i}{\partial x^j}(x^1, \cdots, x^{n-1}, 0))_{1 \leqslant i,j \leqslant n-1}) > 0.$$

This shows that the atlas $\{U_p \cap \partial D\}_{p \in \partial D}$ on ∂D is orientation compatible. Therefore, by Proposition 2.13 it defines an orientation $[\omega]$ on ∂D. However, for the consideration of the sign consistency in Stokes' theorem, in the following we will use the orientation $-[\omega]$ instead of $[\omega]$ on ∂D. On the coordinate chart (U_p, x^i), this orientation is equivalent to

$$(-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Now we are in a position to state and prove Stokes' theorem. Recall that $(\partial D, i)$ is an embedded submanifold of M. If ω is a compactly supported differential form on M, then it is easy to see that the pullback $i^*\omega$ is compactly supported on ∂D.

Theorem 2.5. *Let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_D d\omega = \int_{\partial D} i^*\omega. \tag{2.38}$$

Proof. Choose an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on M, where each U_α arises from (A), (B) or (C) of Definition 2.14. Let $\{h_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to this atlas with the same index. From the definition of integration, we have

$$\int_D d\omega = \sum_{\alpha \in A} \int_D h_\alpha d\omega,$$

and

$$\int_{\partial D} i^*\omega = \sum_{\alpha \in A} \int_{\partial D} i^*(h_\alpha \omega).$$

Since $\sum_\alpha h_\alpha = 1$, we know that

$$\sum_{\alpha\in A} h_\alpha d\omega = \sum_{\alpha\in A} d(h_\alpha\omega).$$

Therefore,

$$\int_D d\omega = \sum_{\alpha\in A}\int_D d(h_\alpha\omega).$$

Now to prove (2.38), it suffices to show that

$$\int_D d(h_\alpha\omega) = \int_{\partial D} i^*(h_\alpha\omega).$$

Hence we only need to consider the special case when ω is compactly supported in some U_α.

If U_α arises from (A), then (2.38) is trivial.

If U_α arises from (B), write

$$\omega = \sum_{i=1}^n a_i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n$$

under U_α. Then

$$d\omega = (\sum_{i=1}^n(-1)^{i-1}\frac{\partial a_i}{\partial x^i})dx^1\wedge\cdots\wedge dx^n.$$

Since $\text{supp}\omega\subset U_\alpha\subset D$, it follows from Fubini's theorem and the fundamental theorem of calculus

$$\int_D d\omega = \sum_{i=1}^n(-1)^{i-1}\int_{\varphi_\alpha(U_\alpha)}\frac{\partial a_i}{\partial x^i}dx^1\cdots dx^n = 0.$$

On the other hand, it is easy to see that $i^*\omega = 0$. Therefore, (2.38) holds.

If U_α arises from (C), then by the same argument as before we have

$$\begin{aligned}&\int_D d\omega\\ =&\sum_{i=1}^n(-1)^{i-1}\int_{\varphi_\alpha(U_\alpha\cap D)=\varphi_\alpha(U_\alpha)\cap\mathbb{H}^n}\frac{\partial a_i}{\partial x^i}dx^1\cdots dx^n\\ =&(-1)^n\int_{\varphi_\alpha(U_\alpha\cap\partial D)=\varphi_\alpha(U_\alpha)\cap\partial\mathbb{H}^n} a_n(x^1,\cdots,x^{n-1},0)dx^1\cdots dx^{n-1}.\end{aligned}\tag{2.39}$$

On the other hand, if we use

$$(W,(u^1,\cdots,u^{n-1}) = ((-1)^n x^1, x^2,\cdots,x^{n-1}))$$

to parametrize ∂D locally, then $i^*\omega$ is compactly supported in this single coordinate chart which is compatible with our choice of orientation on ∂D, and it is given by

$$i^*\omega = a_n((-1)^n u^1, u^2, \cdots, u^{n-1})du^1 \wedge \cdots \wedge du^{n-1}.$$

Therefore, the R.H.S. of (2.39) is the same as

$$\int_W a_n((-1)^n u^1, u^2, \cdots, u^{n-1})du^1 \cdots du^{n-1},$$

which by definition is $\int_{\partial D} i^*\omega$. □

Remark 2.5. By using a cut off argument, if the regular domain D is compact, then we don't need to assume that the n-form ω is compactly supported and the integral of ω over D is well-defined. Moreover, in this case ∂D is a compact manifold of dimension $n-1$, and Stokes' theorem holds for any $(n-1)$-forms on M.

Now let's see how Stokes' theorem unifies the classical results in Riemann's integration theory in low dimensions. We first give a remark on the orientation on ∂D. For any $p \in \partial D$, we call a vector $v \in T_pM$ an *outer vector* to D if v is the tangent vector of some curve $\gamma : (-\delta, \delta) \to M$ through p such that $\gamma(t) \notin D$ for $0 < t < \delta$. A basis $\{v_1, \cdots, v_{n-1}\}$ of $T_p\partial D$ is define to be positive oriented if $\{v, v_1, \cdots, v_n\}$ is positive oriented with respect to the orientation on M. This definition is independent of the choice of outer vectors to D, and as p varies on ∂D this is a smooth specification of orientation on tangent spaces of ∂D. Therefore, it defines an orientation on ∂D. This is equivalent to the one we introduced before. We leave it as an exercise for the readers to work out the details.

Example 2.4. (1) $D = [a, b] \subset \mathbb{R}^1$.

The fundamental theorem of calculus takes the form

$$\int_{[a,b]} f'(x)dx = f(b) - f(a). \tag{2.40}$$

If we denote $\partial D = \{a\} \cup \{b\}$, then (2.40) is just Stokes' formula (2.38) where $\omega = f$.

(2) D is a regular bounded domain in $\mathbb{R}^2$.

The Green's theorem takes the form

$$\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_{\partial D} (Pdx + Qdy).$$

This is Stokes' formula if we let $\omega = Pdx + Qdy$.

(3) D is a regular bounded domain in $\mathbb{R}^3$.

Gauss's theorem takes the form

$$\int_D (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz = \int_{\partial D} (Pdydz + Qdzdx + Rdxdy).$$

This is again Stokes' formula if we let $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$.

(4) D is a regular compact domain in a 2-dimensional surface $M \subset \mathbb{R}^3$.

Stokes' theorem for surfaces takes the form

$$\int_D ((\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dzdx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy) = \int_{\partial D} (Pdx + Qdy + Rdz).$$

This is also Stokes' formula if we let $\omega = Pdx + Qdy + Rdz$.

The orientation on ∂D in each of the previous four cases is induced from the standard orientation on the corresponding Euclidean space, defined in the sense of the remark before this example.

Remark 2.6. It is possible to define integration of p-forms on singular p-chains (p-dimensional triangles on M), and Stokes' theorem also holds for this case. Moreover, Stokes' theorem gives a duality relation

$$\langle \partial c, \omega \rangle = \langle c, d\omega \rangle,$$

where c is a singular p-chain. This point of view is fundamental in the famous de Rham theorem, which asserts that the de Rham cohomology is isomorphic to the singular cohomology via integration.

3 The de Rham cohomology

3.1 The de Rham complex and the de Rham cohomology groups

In the rest of the course, we will be mainly focused on the study of a very important algebraic topological invariant over differentiable manifolds: the de Rham cohomology. This is a fundamental object carrying rich topological information of the manifold. Here we shall see how algebraic structures comes into the story and interacts with differential calculus in the study of topological issues.

To motivate our study, let's first look at a simple but nontrivial example.

For a C^∞ function f on $\mathbb{R}^2$, we know how to compute its gradient $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. One might ask the following natural question: if we are given a pair f_1, f_2 of C^∞ functions on $\mathbb{R}^2$, is there some $f \in C^\infty(\mathbb{R}^2)$ such that (f_1, f_2) is the gradient of f? Let's assume the existence of f for the moment, which means that

$$\frac{\partial f}{\partial x} = f_1, \ \frac{\partial f}{\partial y} = f_2.$$

Therefore, by differentiating again we have

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}. \tag{3.1}$$

In other words, (3.1) is a necessary condition of the existence of such f. Conversely, if f_1, f_2 satisfies (3.1), then the function

$$f(x, y) = \int_0^1 (xf_1(tx, ty) + yf_2(tx, ty))dt \in C^\infty(\mathbb{R}^2)$$

has gradient (f_1, f_2). In fact, we have

$$\frac{\partial f}{\partial x} = \int_0^1 (f_1(tx, ty) + tx\frac{\partial f_1}{\partial x}(tx, ty) + ty\frac{\partial f_2}{\partial x}(tx, ty))dt.$$

On the other hand,

$$\frac{d}{dt}(tf_1(tx, ty)) = f_1(tx, ty) + tx\frac{\partial f_1}{\partial x}(tx, ty) + ty\frac{\partial f_1}{\partial y}(tx, ty).$$

It follows from (3.1)

$$\frac{\partial f}{\partial x} = \int_0^1 \frac{d}{dt}(tf_1(tx, ty))dt = f_1(x, y).$$

Similarly, $\frac{\partial f}{\partial y} = f_2(x, y)$.

However, the situation becomes completely different if we are working on the punctured plane $\mathbb{R}^2 \backslash \{0\}$. Of course (3.1) is still necessary of the existence of f, but it may not be sufficient any more! Consider a pair of C^∞ functions on $\mathbb{R}^2 \backslash \{0\}$ defined by

$$(f_1, f_2) = (-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}).$$

It is easy to see that (3.1) holds. If (f_1, f_2) is the gradient of some $f \in C^\infty(\mathbb{R}^2 \backslash \{0\})$, then we have

$$\int_0^{2\pi} \frac{d}{d\theta} f(\cos\theta, \sin\theta)d\theta = f(1, 0) - f(1, 0) = 0. \tag{3.2}$$

But

$$\begin{aligned}
\frac{d}{d\theta} f(\cos\theta, \sin\theta) &= -\frac{\partial f}{\partial x}(\cos\theta, \sin\theta)\sin\theta + \frac{\partial f}{\partial y}(\cos\theta, \sin\theta)\cos\theta \\
&= -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta \\
&= 1,
\end{aligned}$$

which certainly yields a contradiction to (3.2). In other words, such f does not exist.

Let's summarize these two situations in the language of differential forms. The pair (f_1, f_2) is equivalent to a 1-form

$$\omega = f_1 dx + f_2 dy,$$

and (3.1) is equivalent to saying that $d\omega = 0$. The existence of f such that (f_1, f_2) is the gradient of f is equivalent to saying that $df = \omega$. If such f exists, since $d^2 = 0$, it certainly implies that $d\omega = 0$ and hence (3.1) holds. This is true on $\mathbb{R}^2$ or $\mathbb{R}^2 \backslash \{0\}$ (in fact on any open subsets of $\mathbb{R}^2$). However, the converse problem behaves very differently in the previous two situations. The fundamental reason is the difference between the topology of $\mathbb{R}^2$ and $\mathbb{R}^2\backslash\{0\}$: one does not have holes while the other have one hole.

Once we've reformulated the problem by using differential forms, of course we can ask the same question on $\mathbb{R}^3$ and for differential forms of each degree. For example, one can ask: if a smooth vector field Y on $\mathbb{R}^3$ is divergence free, is it the curl of some smooth vector field X (this correspondence to the question for 2-forms on $\mathbb{R}^3$)? In general, this leads us to the following definition.

Definition 3.1. Let M be a manifold. An r-form ω on M is called *closed* if $d\omega = 0$; it is called *exact* if $d\tau = \omega$ for some $(r-1)$-form τ. The space of closed (exact, respectively) r-forms on M is denoted by $Z^r(M)$ ($B^r(M)$, respectively).

Since there is no (-1)-forms, each 0-form is exact. Moreover, it is easy to see that $Z^r(M)$ and $B^r(M)$ are (usually infinite dimensional) real vector spaces. Since $d^2 = 0$, it is immediate that $B^r(M)$ is a subspace of $Z^r(M)$ for all $r \geqslant 0$.

The two examples we gave previously are essentially about studying how far is a closed r-form from being exact. More precisely, it is about the study of the quotient space $Z^r(M)/B^r(M)$.

Definition 3.2. The sequence $\Omega^*(M) = (\Omega^r(M))_{r\geqslant 0}$ of differential forms together with the exterior derivative d is called the *de Rham complex* over M. The quotient space

$$\frac{Z^r(M)}{B^r(M)} = \frac{\mathrm{Ker}(d:\ \Omega^r(M) \to \Omega^{r+1}(M))}{\mathrm{Im}(d:\ \Omega^{r-1} \to \Omega^r(M))}$$

as a real vector space is called the *r-th de Rham cohomology group* of M, and it is denoted by $H^r(M)$. Elements in $H^r(M)$ are called *cohomology classes* of degree r.

Remark 3.1. Although we call $H^r(M)$ a group, it is just a real vector space (usually of finite dimension). In the context of algebraic topology, we will encounter more general cohomology groups and homology groups, which are free abelian groups with coefficients not just $\mathbb{R}$.

By definition, under the quotient map a closed r-form ω defines a cohomology class $[\omega]$ of degree r. If $[\omega] = [\omega']$, then there exists some $(r-1)$-form τ such that $\omega' - \omega = d\tau$.

$H^0(M)$ is just the space of C^∞ functions f with $df = 0$, which is equivalent to the fact that f is locally constant. Therefore, if M is connected, $H^0(M)$ is naturally isomorphic to $\mathbb{R}$. In general, $H^0(M)$ is isomorphic to $\mathbb{R}^N$, where N is the cardinality of the set of connected components of M (at most countable since the topology of M is second countable). In particular, if M is compact (so there will be only finitely many connected components of M since the set of connected components form an open cover of M), then $H^0(M)$ is finite dimensional. Later on we will prove that for a compact manifold M, $H^r(M)$ is finite dimensional for all $r \geqslant 0$.

If $r > n = \mathrm{dim} M$, by definition we have $H^r(M) = 0$.

By using the wedge product, we are able to multiply cohomology classes of different degrees. Let $a \in H^r(M)$ and $b \in H^s(M)$. We define

$$a \cdot b = [\omega \wedge \tau],$$

where ω (τ, respectively) is a representative in the cohomology class a (b, respectively). This is well-defined since $\omega \wedge \tau$ is closed and if

$$\omega' - \omega = d\alpha,\ \tau' - \tau = d\beta$$

for some α, β, then

$$\begin{aligned} \omega' \wedge \tau' &= (\omega + d\alpha) \wedge (\tau + d\beta) \\ &= \omega \wedge \tau + d((-1)^r \omega \wedge \beta + \alpha \wedge \tau + (-1)^r d\alpha \wedge \beta), \end{aligned}$$

where r is the degree of ω. Therefore, "$\cdot$" defines a bilinear product on

$$H(M) = \oplus_{r \geqslant 0} H^r(M)$$

which makes $H(M)$ a graded algebra over $\mathbb{R}^1$. $H(M)$ is usually called the *de Rham cohomology*. Note that due to the antisymmetry of the wedge product, we have

$$a \cdot b = (-1)^{rs} b \cdot a,\ \forall a \in H^r(M), b \in H^s(M).$$

A crucial point on the de Rham cohomology is that a C^∞ map $F: M \to N$ between manifolds induces a pullback on cohomology classes by acting on representatives, which is indeed an algebra homomorphism. More precisely, defined $F^*: H^r(N) \to H^r(M)$ by

$$F^*[\omega] = [F^*\omega].$$

This is well-defined since F^* commutes with d (see Proposition 2.7). Moreover, if $[\omega] \in H^r(N), [\tau] \in H^s(N)$,then

$$\begin{aligned} F^*([\omega] \cdot [\tau]) &= F^*[\omega \wedge \tau] = [F^*(\omega \wedge \tau)] = [F^*\omega \wedge F^*\tau] \\ &= [F^*\omega] \cdot [F^*\tau] = F^*[\omega] \cdot F^*[\tau]. \end{aligned}$$

Therefore, $F^*: H(N) \to H(M)$ is an algebra homomorphism.

From (2.21) we can see immediately that if $F: M \to N$ and $G: N \to P$ are C^∞ maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^*: H(P) \to H(M). \tag{3.3}$$

As we shall see later on, the de Rham cohomology reveals the global topology of the manifold to some extend. Therefore, the computation of the de Rham cohomology groups is a significant problem. At the moment we only have very limited tools in computing these groups of a general manifold. But for some special examples, it doesn't take too much efforts.

First consider the simplest case: $M = \mathbb{R}^1$. Since $\mathbb{R}^1$ is connected, we know that $H^0(\mathbb{R}^1) = \mathbb{R}$. Now we only need to compute $H^1(\mathbb{R}^1)$. In this case every one form $\omega = fdx$ is closed. If we let

$$F(x) = \int_0^x f(t)dt \in C^\infty(\mathbb{R}^1),$$

then it is immediate that $dF = \omega$. In other words, every closed form on $\mathbb{R}^1$ is exact, and hence $H^1(\mathbb{R}^1) = 0$. Therefore, we have

$$H^r(\mathbb{R}^1) = \begin{cases} \mathbb{R}, & r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Now we want to consider higher dimensional Euclidean spaces. Let U be some open subset of $\mathbb{R}^n$, and use

$$s_0, s_1 : \ U \to U \times \mathbb{R}^1$$

to denote the inclusions (the 0-section and the 1-section)

$$s_0(x) = (x, 0), \ s_1(x) = (x, 1), \ x \in U.$$

The key to the computation is the following result.

Proposition 3.1. *For each $r \geqslant 0$, there exists a linear operator $\widehat{S}_r : \ \Omega^r(U \times \mathbb{R}^1) \to \Omega^{r-1}(U)$ ($\Omega^{-1}(U) := 0$), such that*

$$d \circ \widehat{S}_r + \widehat{S}_{r+1} \circ d = s_1^* - s_0^* : \ \Omega^r(U \times \mathbb{R}^1) \to \Omega^r(U), \tag{3.4}$$

where s_0^*, s_1^* are the induced pullbacks of differential forms by s_0, s_1, respectively. In particular, when acting on the de Rham cohomology groups,

$$s_0^* = s_1^* : \ H^r(U \times \mathbb{R}^1) \to H^r(U). \tag{3.5}$$

Proof. Let's first see how (3.4) implies (3.5). In fact, when $r \geqslant 1$, for any closed r-form ω on $U \times \mathbb{R}^1$, (3.4) implies that

$$s_1^*\omega - s_0^*\omega = d\widehat{S}_r(\omega),$$

and hence $s_0^*\omega$ and $s_1^*\omega$ defines the same cohomology class. When $r = 0$, if f is a C^∞ function on $U \times \mathbb{R}^1$ with $df = 0$, then (3.4) implies that $s_1^* f = s_0^* f$.

Now we show the existence of the linear operator $\widehat{S}_r$ such that (3.4) holds. Intuitively, $\widehat{S}_r$ is just the "integration along each fiber".

Let

$$\begin{aligned} \omega \ = \ & \sum_{i_1 < \cdots < i_r} a_{i_1 \cdots i_r}(x,t) dx^{i_1 \cdots i_r} \\ & + \sum_{j_1 < \cdots < j_{r-1}} b_{j_1 \cdots j_{r-1}}(x,t) dt \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{r-1}} \in \Omega^r(U \times \mathbb{R}^1), \end{aligned}$$

and define

$$\widehat{S}_r(\omega) = \sum_{j_1<\cdots<j_{r-1}} (\int_0^1 b_{j_1\cdots j_{r-1}}(x,t)dt)dx^{j_1}\wedge\cdots\wedge dx^{j_{r-1}}.$$

If $r=0$ simply define $\widehat{S}_r=0$. It follows that

$$d\widehat{S}_r(\omega) = \sum_{j_1<\cdots<j_{r-1}} (\int_0^1 \frac{\partial b_{j_1\cdots j_{r-1}}(x,t)}{\partial x^k}dt)dx^k\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_{r-1}}.$$

On the other hand,

$$\begin{aligned} d\omega &= \sum_{i_1<\cdots<i_r} \frac{\partial a_{i_1\cdots i_r}(x,t)}{\partial x^k}dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_r} \\ &\quad + \sum_{i_1<\cdots<i_r} \frac{\partial a_{i_1\cdots i_r}(x,t)}{\partial t}dt\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_r} \\ &\quad + \sum_{j_1<\cdots<j_{r-1}} \frac{\partial b_{j_1\cdots j_{r-1}}(x,t)}{\partial x^k}dx^k\wedge dt\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_{r-1}}. \end{aligned}$$

By the definition of $\widehat{S}_r$, we have

$$\begin{aligned} \widehat{S}_{r+1}(d\omega) &= \sum_{i_1<\cdots<i_r} a_{i_1\cdots i_r}(x,1)dx^{i_1}\wedge\cdots\wedge dx^{i_r} \\ &\quad - \sum_{i_1<\cdots<i_r} a_{i_1\cdots i_r}(x,0)dx^{i_1}\wedge\cdots\wedge dx^{i_r} \\ &\quad - \sum_{j_1<\cdots<j_{r-1}} (\int_0^1 \frac{\partial b_{j_1\cdots j_{r-1}}(x,t)}{\partial x^k}dt)dx^k\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_{r-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} d\widehat{S}_r(\omega)+\widehat{S}_{r+1}d(\omega) &= \sum_{i_1<\cdots<i_r} a_{i_1\cdots i_r}(x,1)dx^{i_1}\wedge\cdots\wedge dx^{i_r} \\ &\quad - \sum_{i_1<\cdots<i_r} a_{i_1\cdots i_r}(x,0)dx^{i_1}\wedge\cdots\wedge dx^{i_r}. \end{aligned}$$

But it is easy to see that the R.H.S. of the above identity is just $s_1^*\omega - s_0^*\omega$, hence (3.4) holds. Note that (3.4) also holds for $r=0$. □

By using coordinate charts and showing invariance under change of coordinates, one can prove the following result in the same way. We leave the proof as an exercise.

Proposition 3.2. *Proposition 3.1 holds when U is replaced by a manifold M.*

Proposition 3.2 yields immediately the homotopy invariance for the de Rham cohomology. This is perhaps the most important property for the de Rham cohomology.

Definition 3.3. Let $f, g: M \to N$ be two C^∞ maps between manifolds M and N. A *smooth homotopy* from f to g is a C^∞ map

$$F: M \times \mathbb{R}^1 \to N$$

such that

$$F(p,0) = f(p),\ F(p,1) = g(p),\ \forall p \in M.$$

f and g are called *smoothly homotopic* (denoted by $f \simeq g$) if there exists a smooth homotopy from f to g.

Now we have the following result.

Theorem 3.1. *Let $f, g: M \to N$ be two C^∞ maps between manifolds M and N. If f and g are smoothly homotopic, then $f^* = g^*: H^r(N) \to H^r(M)$ for all $r \geqslant 0$.*

Proof. Let F be a smooth homotopy from f to g, and use $s_0, s_1: M \to M \times \mathbb{R}^1$ to denote the inclusions as before. Then we have

$$f = F \circ s_0,\ g = F \circ s_1,$$

and hence

$$f^* = s_0^* \circ F^*,\ g^* = s_1^* \circ F^*.$$

By Proposition 3.2, we have $f^* = g^*$ when acting on the de Rham cohomology groups. $\square$

As a corollary, we obtain the so-called *Poincáre lemma.*

Proposition 3.3. *Let U be a star-shaped open subset of $\mathbb{R}^n$ (i.e., there exists some $x_0 \in U$ such that for any $x \in U$, the line segment $\{tx_0 + (1-t)x : t \in [0,1]\}$ is contained in U). Then*

$$H^r(U) = \begin{cases} \mathbb{R}, & r = 0; \\ 0, & \textit{otherwise}. \end{cases} \tag{3.6}$$

In particular, (3.6) holds when $U = \mathbb{R}^n$.

Proof. Obviously U is connected and hence $H^0(U) = \mathbb{R}$.

Now consider $H^r(U)$ with $r \geqslant 1$. Let $h(t) \in C^\infty(\mathbb{R}^1)$ be such that

$$h(t) = \begin{cases} 1, & t \geqslant 1; \\ 0, & t \leqslant 0, \end{cases}$$

and $0 \leqslant h(t) \leqslant 1$ for $t \in [0,1]$. It follows that

$$F(x,t) = (1 - h(t))x_0 + h(t)x,\ (x,t) \in U \times \mathbb{R}^1$$

defines a smooth homotopy from the constant map

$$\mathrm{e}_{x_0}(x) = x_0,\ x \in U,$$

to the identity map id_U. Therefore, by Theorem 3.1 we have $\mathrm{e}_{x_0}^* = \mathrm{id}_U^*$. But it is easy to see that for any $a \in H^r(U)$,

$$\mathrm{e}_{x_0}^* a = 0,\ \mathrm{id}_U^* a = a.$$

It follows that $a = 0$ and hence $H^r(U) = 0$. □

By taking $U = \mathbb{R}^n$ in Proposition 3.3, we obtain immediately that

$$H^r(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & r = 0; \\ 0, & \text{otherwise}. \end{cases}$$

In particular, the problem proposed at the very beginning for $\mathbb{R}^2$ is solved completely. However, the case of $\mathbb{R}^2 \backslash \{0\}$ is still unclear, and we will come to this point later on.

Definition 3.4. Two manifolds M and N are said to be *homotopy equivalent* (*in the smooth sense*) if there exists C^∞ maps $f:\ M \to N$ and $g:\ N \to M$ such that

$$g \circ f \simeq \mathrm{id}_M,\ f \circ g \simeq \mathrm{id}_N.$$

Another important consequence of Theorem 3.1 is the following.

Theorem 3.2. *If manifolds M and N are homotopy equivalent, then $H^r(M) \cong H^r(N)$ for all $r \geqslant 0$.*

Proof. Let f, g be C^∞ maps given in Definition 3.4, then by Theorem 3.1 we have

$$f^* \circ g^* = \mathrm{id}:\ H^r(M) \to H^r(M),$$

and

$$g^* \circ f^* = \mathrm{id}:\ H^r(N) \to H^r(N).$$

Therefore, f^* is an isomorphism from $H^r(N)$ to $H^r(M)$ with inverse g^*. □

Example 3.1. Let M be a manifold. It is easy to see that $M \times \mathbb{R}^n$ and M are homotopy equivalent, under the maps

$$\pi:\ M \times \mathbb{R}^n \to M,\ \pi(p, x) = p,$$

and

$$i:\ M \to M \times \mathbb{R}^n,\ i(p) = (p, 0).$$

More generally, one can show that for any real vector bundle, the total space E and the base space B are homotopy equivalent, and hence they have the same de Rham cohomology (up to isomorphism).

Example 3.2. $\mathbb{R}^{n+1}\backslash\{0\}$ and S^n are homotopy equivalent under the maps

$$r:\ \mathbb{R}^{n+1}\backslash\{0\} \to S^n,\ r(x) = \frac{x}{\|x\|},$$

and the inclusion $i:\ S^n \to \mathbb{R}^{n+1}\backslash\{0\}$. Therefore, they have the same de Rham cohomology.

Example 3.3. A manifold M is called *contractible* if it is homotopy equivalent to the single point space $\{x_0\}$ which is regarded as a 0-dimensional manifold (equivalently, if the identity map is smoothly homotopic to a constant map on M). If M is contractible, then

$$H^r(M) = \begin{cases} \mathbb{R}, & r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, a contractible manifold is always connected.

An interesting geometric application of the homotopy invariance of de Rham cohomology is the so-called *Hairy Ball Theorem*.

Theorem 3.3. *There exists a smooth non-vanishing vector field on the n-sphere S^n if and only if n is odd.*

Proof. Assume that X is a smooth non-vanishing vector field on S^n. Note that for each $x \in S^n$, we can think of X_x as a vector in $\mathbb{R}^{n+1}$ canonically. In this manner, we can regard X as a smooth non-vanishing map $v:\ S^n \to \mathbb{R}^{n+1}$, and we may further assume that v is normalized so that

$$\|v(x)\| = 1,\ \forall x \in S^n, \tag{3.7}$$

where $\|\cdot\|$ is the Euclidean norm.

Define a C^∞ map $F:\ S^n \times \mathbb{R}^1 \to \mathbb{R}^{n+1}$ by

$$F(x,t) = (\cos \pi t)x + (\sin \pi t)v(x),\ (x,t) \in S^n \times \mathbb{R}^1.$$

Since $\langle x, v(x)\rangle = 0$ for all $x \in S^n$ (here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product), by (3.7) we can easily see that F takes value in S^n. Moreover,

$$F(x,0) = x,\ F(x,1) = -x,\ \forall x \in S^n.$$

Therefore, F defines a smooth homotopy from the identity map to the antipodal map

$$a(x) = -x,\ x \in S^n,$$

on S^n. It follows from Theorem 3.1 that

$$\mathrm{id}_{S^n}^* = a^*:\ H^n(S^n) \to H^n(S^n). \tag{3.8}$$

On the other hand, if we take the standard orientation form ω on S^n, which can be defined by the pullback of the form

$$\sum_{i=0}^{n}(-1)^i x_i dx^1 \wedge \cdots\cdots \wedge \widehat{dx^i} \wedge \cdots\cdots dx^n$$

on $\mathbb{R}^{n+1}$ by the inclusion map $i: S^n \to \mathbb{R}^{n+1}$, then it is easy to see that (one may probably encounter this in the Problem Sheet 2)

$$a^*\omega = (-1)^{n+1}\omega.$$

In particular, if n is even, then $a^*\omega = -\omega$. By (3.8) we conclude that $[\omega] = 0 \in H^n(S^n)$, which means $\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^n)$. It then follows from Stokes' theorem that

$$\int_{S^n} \omega = \int_{S^n} d\alpha = 0.$$

But this is a contradiction since the integration of any orientation form on S^n is strictly positive by definition (in fact, for our choice of ω here, $\int_{S^n} \omega$ gives the volume of S^n).

Therefore, if n is even, any smooth vector field on S^n must vanish at some point.

If n is odd, write $n = 2k-1$, and for any $x = (a^1, b^1, \cdots, a^k, b^k) \in S^n \subset \mathbb{R}^{2k}$, define

$$v(x) = (-b^1, a^1, \cdots, -b^k, a^k) \in S^n.$$

It is easy to see that $\langle x, v(x)\rangle = 0$ for all $x \in S^n$ and hence v defines a smooth non-vanishing vector field on S^n. $\square$

3.2 The top de Rham cohomology group and the degree of a C^∞ map

The integration theory developed in Section 2.5 is a very useful tool in studying the de Rham cohomology of a manifold and related topological problems. In particular, for a compact, connected and oriented manifold M of dimension n, the integral operator induces an isomorphism from the top de Rham cohomology group $H^n(M)$ to $\mathbb{R}$, so that $H^n(M)$ is always one dimensional.

Since integration is defined on the space of compactly supported top forms, there is no point to assume compactness of the manifold in the first place. Therefore, we start with the more general situation.

Let M be an n-dimensional connected and oriented manifold.

For $r \geqslant 0$, let $\Omega_c^r(M)$ be the real vector space of compactly supported r-forms. By the locality of d, it is easy to see that $d(\Omega_c^{r-1}(M)) \subset \Omega_c^r(M)$. Define the quotient space

$$H_c^n(M) = \Omega_c^n(M)/d(\Omega_c^{n-1}(M)).$$

This is called the *n-th* (or *top*) *de Rham cohomology group* of M *with compact supports*. In general (except the case when M is compact), $H_c^n(M)$ and $H^n(M)$ are not isomorphic.

Now we are going to show that, the integral operator

$$\int_M : \Omega_c^n(M) \to \mathbb{R},$$

which is well-defined in Section 2.5 with respect to the given orientation on M, induces a linear isomorphism

$$\int_M : H_c^n(M) \to \mathbb{R}$$

defined by

$$\int_M [\omega] = \int_M \omega, \ [\omega] \in H_c^n(M). \tag{3.9}$$

Note that $\int_M$ is well-defined by Stokes' theorem.

By using a bump function in an oriented coordinate chart, we can always construct a compactly supported n-form ω such that

$$\int_M \omega = 1.$$

Therefore $\int_M$ given by (3.9) is surjective. Now it suffices to show that: if $\omega \in \Omega_c^n(M)$ with

$$\int_M \omega = 0,$$

then $\omega = d\alpha$ for some $\alpha \in \Omega_c^{n-1}(M)$.

We first look that the simplest example: $M = \mathbb{R}^n$. Essentially this is equivalent to the following result.

Proposition 3.4. *If $f \in C_c^\infty(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} f(x) dx^1 \cdots dx^n = 0,$$

then there exists $f_1, \cdots, f_n \in C_c^\infty(\mathbb{R}^n)$ such that

$$f = \sum_{i=1}^n \frac{\partial f_i}{\partial x^i}.$$

Proof. We prove the result by induction on the dimension n.

If $n = 1$, $f \in C_c^\infty(\mathbb{R}^1)$ with

$$\int_{\mathbb{R}^1} f(x) dx = 0, \tag{3.10}$$

let

$$f_1(x) = \int_{-\infty}^x f(t) dt, \ x \in \mathbb{R}^1.$$

From (3.10) it is easy to see that $f_1 \in C_c^\infty(\mathbb{R}^1)$, and obviously we have

$$f = \frac{df_1}{dx}.$$

Now assume that the result is true for dimension n. Let $f \in C_c^\infty(\mathbb{R}^{n+1})$ with

$$\int_{\mathbb{R}^{n+1}} f(x)dx^1 \cdots dx^{n+1} = 0.$$

Define

$$g(x^1, \cdots, x^n) = \int_{-\infty}^{\infty} f(x^1, \cdots, x^n, x^{n+1})dx^{n+1}, \ (x^1, \cdots, x^n) \in \mathbb{R}^n.$$

It follows that $g \in C_c^\infty(\mathbb{R}^n)$ and by Fubini's theorem we have

$$\int_{\mathbb{R}^n} g(x)dx^1 \cdots dx^n = 0.$$

By induction hypothesis, there exists $g_1, \cdots, g_n \in C_c^\infty(\mathbb{R}^n)$, such that

$$g = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x^i}.$$

Choose a bump function $\rho \in C_c^\infty(\mathbb{R}^1)$ with

$$\int_{-\infty}^{\infty} \rho(x)dx = 1.$$

For $1 \leqslant i \leqslant n$, define

$$f_i(x^1, \cdots, x^n, x^{n+1}) = g_i(x^1, \cdots, x^n)\rho(x^{n+1}) \in C_c^\infty(\mathbb{R}^{n+1}),$$

and let

$$f_{n+1}(x^1, \cdots, x^{n+1}) = \int_{-\infty}^{x^{n+1}} (f(x^1, \cdots, x^n, t) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x^i}(x^1, \cdots, x^n, t))dt.$$

One may easily verify that $f_{n+1} \in C_c^\infty(\mathbb{R}^{n+1})$. Therefore, we have

$$f = \sum_{i=1}^{n+1} \frac{\partial f_i}{\partial x^i}.$$

□

So far we have proved that $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ via the integral operator (note that this already shows $H_c^n(\mathbb{R}^n)$ is not isomorphic to $H^n(\mathbb{R}^n) = 0$). The idea of proving the isomorphism for a general connected and oriented manifold M is to reduce to coordinate charts which are diffeomorphic to $\mathbb{R}^n$ and use connectedness to go from one coordinate chart to another.

First we need to following lemma.

Lemma 3.1. *Let $\{U_i\}_{i\in I}$ be an atlas on M. Then for any $p, q \in M$, there exists a finite sequence of indices $i_0, \cdots, i_k \in A$, such that*

$$p \in U_{i_0},\ q \in U_{i_k},$$

and

$$U_{i_{l-1}} \cap U_{i_l} \neq \emptyset,\ \forall l = 1, \cdots, k. \tag{3.11}$$

Proof. Fix $p \in M$ and some U_{i_0} containing p. Define V to be the set of $q \in M$ such that there exists a finite sequence $i_1, \cdots, i_k \in A$ with $q \in U_{i_k}$ and (3.11) holds. Obviously $V \neq \emptyset$ as $U_{i_0} \subset V$. If $q \in V$ with $i_1, \cdots, i_k$ being the associated finite sequence, then $U_{i_k} \subset V$, and hence V is open. Moreover, let $q \notin V$ and assume that $q \in U_i$ for some $i \in I$. It follows from the definition of V that $U_i \subset V^c$. Therefore, V is closed. By the connectedness of M, we know that $V = M$. □

Now we are able to prove the following result.

Theorem 3.4. *The integral operator $\int_M : H_c^n(M) \to \mathbb{R}$ is a linear isomorphism.*

Proof. As we've discussed before, it suffices to show that: if $\omega \in \Omega_c^n(M)$ with zero integral, then $\omega = d\alpha$ for some $\alpha \in \Omega_c^{n-1}(M)$.

Choose an oriented atlas $\{U_i\}_{i\in I}$ on M such that each U_i is diffeomorphic to $\mathbb{R}^n$ under the coordinate map, and take a partition of unity $\{\varphi_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$ with the same index. It follows that

$$\omega = \sum_{i\in I} \varphi_i \omega, \tag{3.12}$$

which is in fact a finite sum by local finiteness. Fix $i_0 \in I$ and use a bump function to define an n-form ω_0 supported in U_{i_0} with

$$\int_M \omega_0 = 1.$$

For each fixed $i \in I$ in the (finite) sum (3.12), we know that $\mathrm{supp}\varphi_i\omega \subset U_i$. From Lemma 3.11, it is not hard to see that there exists a finite sequence of indices $i_1, \cdots, i_{k-1} \in I$, such that

$$U_{i_{l-1}} \cap U_{i_l} \neq \emptyset,\ l = 1, \cdots, k,$$

where $i_k := i$. By using bump functions, for each $1 \leqslant l \leqslant k$, we can construct an n-form τ_l supported in $U_{i_{l-1}} \cap U_{i_l}$ such that

$$\int_M \tau_l = 1.$$

It follows from the choice of the atlas and Proposition 3.4 that for each $1 \leqslant l \leqslant k$, there exists an n-form β_l supported in $U_{i_{l-1}}$, such that

$$\begin{aligned} \omega_0 - \tau_1 &= d\beta_1, \\ \tau_{l-1} - \tau_l &= d\beta_l, \ \forall 2 \leqslant l \leqslant k. \end{aligned}$$

Therefore,

$$\omega_0 - \tau_k = d(\sum_{l=1}^{k} \beta_l). \tag{3.13}$$

Now let

$$c_i = \int_M \varphi_i \omega,$$

it follows that

$$\int_M (\varphi_i \omega - c_i \tau_k) = 0,$$

and $\varphi_i \omega - c_i \tau_k$ is supported in $U_{i_k} = U_i$. Again by Proposition 3.4, there exists an n-form γ supported in U_i, such that

$$\varphi_i \omega - c_i \tau_k = d\gamma.$$

Combining with (3.13), we have

$$\varphi_i \omega = c_i \omega_0 + d(\gamma - c_i \sum_{l=1}^{k} \beta_l).$$

We use α_i to denote $\gamma - c_i \sum_{l=1}^{k} \beta_l$.

Finally, let $\alpha = \sum_i \alpha_i \in \Omega_c^{n-1}(M)$ (note that this is a finite sum). Since $\int_M \omega = 0$, we have

$$\sum_i c_i = 0.$$

Therefore,

$$\omega = \sum_i \varphi_i \omega = d\alpha,$$

and the proof is complete. □

One immediate corollary of Theorem 3.4 is the following.

Corollary 3.1. *If M is a compact, connected and oriented manifold of dimension n, then the integral operator $\int_M$ induces a linear isomorphism from $H^n(M)$ to $\mathbb{R}$. In particular,*

$$dim H^n(M) = 1,$$

and any n-form ω on M with nonzero integral induces a generator $[\omega]$ of $H^n(M)$.

Remark 3.2. By Corollary 3.1 and using a bump function, we can always choose a generator of $H^n(M)$ represented by some n-form ω supported in any given open subset $U \subset M$ whose integral is nonzero.

An interesting application of Corollary 3.1 is the so-called *Brouwer Fixed Point Theorem*. Let D^n be the closed n-dimensional unit ball, so $\partial D^n = S^{n-1}$.

Theorem 3.5. *Let $F: D^n \to D^n$ be a C^∞ map. Then there exists some $x \in D^n$ such that $F(x) = x$.*

Proof. The case of $n = 1$ can be proved easily by using properties of continuous functions on a closed interval. Let's consider $n \geqslant 2$.

Assume on the contrary that F does not have a fixed point. For any $x \in D^n$, extend the straight line segment $\overline{F(x)x}$ until it meets the boundary $\partial D^n = S^{n-1}$ at the point $g(x)$. This defines a C^∞ map

$$g: D^n \to S^{n-1}$$

such that $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$.

Let $h \in C_c^\infty(\mathbb{R}^1)$ be defined as in the proof of Proposition 3.3. It follows that

$$g(h(t)x), \ (x,t) \in S^{n-1} \times \mathbb{R}^1,$$

defines a smooth homotopy from the identity map on S^{n-1} to a constant map which maps the whole sphere S^{n-1} to the point $g(0)$. The same reason as in the proof of Proposition 3.3 shows that the r-th de Rham cohomology group of S^{n-1} is trivial for $r \geqslant 1$. But this contradicts Corollary 3.1 since we know that $H^{n-1}(S^{n-1}) \cong \mathbb{R}$.

Therefore, F has a fixed point on D^n. □

Corollary 3.1 allows us to introduce the concept of degree of a C^∞ map between two compact, connected and oriented manifolds M, N of the same dimension n.

More precisely, let $F: M \to N$ be a C^∞ map. Since the integral operator induces a natural isomorphism from the top de Rham cohomology group to $\mathbb{R}$, there exists a unique real number k such that the following diagram commutes:

$$\begin{array}{ccc} H^n(N) & \xrightarrow{F^*} & H^n(M) \\ \downarrow {\scriptstyle \int_N} & & \downarrow {\scriptstyle \int_M} \\ \mathbb{R} & \xrightarrow{k\cdot} & \mathbb{R}. \end{array}$$

Here the bottom row means multiplication by k.

Definition 3.5. This real number k is called the *degree of* F, and it is denoted by $\deg F$.

By definition, for any $\omega \in \Omega^n(N)$,

$$\int_M F^*\omega = \deg F \cdot \int_N \omega. \tag{3.14}$$

Since there exists unique cohomology classes $[\omega_M]$ on M and $[\omega_N]$ on N whose integrals are both equal to 1, we can equivalently define $\deg F$ to be the number such that

$$F^*[\omega_N] = \deg F \cdot [\omega_M].$$

It follows from Theorem 3.1 that if F and G are smoothly homotopic, then $\deg F = \deg G$. Moreover, if

$$F: \ M \to N, \ G: \ N \to P,$$

are C^∞ maps between compact, connected and oriented manifolds of the same dimension, then by (3.3) we have

$$\deg(G \circ F) = \deg(F) \cdot \deg(G).$$

We now prove a surprising and crucial fact: the degree of a C^∞ map is always an integer. Let $F: \ M \to N$ be as before.

Definition 3.6. A point $q \in N$ is called a *regular value of* F if either $F^{-1}(q) = \emptyset$, or for each $p \in F^{-1}(q)$, the differential $(dF)_p$ is surjective.

If q is a regular value of F, then for any $p \in F^{-1}(q)$ (if not empty), by definition we know that $(dF)_p$ is a linear isomorphism from T_pM to T_qN.

Let $T: \ V \to W$ be a linear isomorphism between two real oriented vector spaces of the same dimension. We define the sign of T to be

$$\mathrm{sgn} T = \begin{cases} 1, & \text{if } T \text{ preserves orientation;} \\ -1, & \text{otherwise.} \end{cases}$$

Now we have the following result.

Theorem 3.6. *Let q be a regular value of F. Then*

$$deg F = \sum_{p \in F^{-1}(q)} sgn(dF)_p.$$

Proof. If $F^{-1}(q) = \emptyset$, since $F(M)$ is compact in N, there exists some open neighborhood V of q, such that $V \cap F(M) = \emptyset$. Use a bump function to choose some $\omega \in \Omega^n(N)$ supported in V with

$$\int_N \omega = 1.$$

It follows that $F^*\omega = 0$ and by (3.14) we have

$$\deg F = 0.$$

If $F^{-1}(q) \neq \emptyset$, for any $p \in F^{-1}(q)$, by the inverse function theorem there exists some open neighborhoods U_p of p and V_p of q such that $F|_{U_p}: \ U_p \to V_p$ is a diffeomorphism. In particular,

$$U_p \cap F^{-1}(q) = \{p\}.$$

By a compactness argument and shrinking those open neighborhoods if necessary, we can see that

$$F^{-1}(q) = \{p_1, \cdots, p_k\}$$

is a finite set and there exists disjoint open neighborhoods U_i of p_i and an open neighborhood V of q, such that

$$F^{-1}(V) = \cup_{i=1}^{k} U_i$$

and

$$F|_{U_i}: \ U_i \to V$$

is a diffeomorphism. We may further assume that U_i, V are connected oriented coordinate charts ($i = 1, \cdots, k$).

Now choose $\omega \in \Omega^n(N)$ supported in V with

$$\int_N \omega = 1.$$

It follows that

$$\int_M F^*\omega = \sum_{i=1}^{k} \int_{U_i} F|_{U_i}^*\omega.$$

Write

$$\omega = f(y)dy^1 \wedge \cdots \wedge dy^n$$

under V. For each $1 \leqslant i \leqslant k$, under U_i we have

$$F|_{U_i}^*\omega = f(F(x))\det(\frac{\partial y^\alpha}{\partial x^\beta})_{1\leqslant \alpha,\beta \leqslant n} dx^1 \wedge \cdots \wedge dx^n,$$

where

$$y^\alpha = F^\alpha(x^1, \cdots, x^n), \ \alpha = 1, \cdots, n.$$

Therefore,

$$\int_{U_i} F|_{U_i}^*\omega = \int_{U_i} f(F(x))\det(\frac{\partial y^\alpha}{\partial x^\beta})_{1\leqslant \alpha,\beta \leqslant n} dx^1 \cdots dx^n.$$

On the other hand, we can regard $F|_{U_i}$ as a change of variables for Lebesgue integrals from V to U_i, therefore we have

$$\begin{aligned} \int_N \omega &= \int_V f(y)dy^1 \cdots dy^n \\ &= \int_{U_i} f(F(x))|\det(\frac{\partial y^\alpha}{\partial x^\beta})_{1\leqslant \alpha,\beta \leqslant n}| dx^1 \cdots dx^n \\ &= 1. \end{aligned}$$

By the connectedness of U_i, the sign of $\det(\frac{\partial y^\alpha}{\partial x^\beta})_{1\leqslant \alpha,\beta \leqslant n}$ is constant on U_i, and it equals the sign of $(dF)_{p_i}$. It follows that

$$\int_{U_i} F|_{U_i}^*\omega = \text{sgn}(dF)_{p_i}.$$

Therefore,

$$\int_M F^*\omega = \sum_{i=1}^{k} \mathrm{sgn}(dF)_{p_i}.$$

Now the proof is complete. □

An immediate corollary of Theorem 3.6 is the following.

Corollary 3.2. *If F is not surjective, then deg$F = 0$.*

One may wonder whether F always has a regular value. In fact, a much stronger result holds in a very general setting, known as *Sard's theorem*. Here we state a special case of Sard's theorem without proof, which asserts that there are lots of points as regular values of F. A subset S of a manifold M is called a *null set* in M if there exists a countable atlas $\{(U_n, \varphi_n)\}_{n\geqslant 1}$ on M such that for each $n \geqslant 1$, $\varphi_n(S \cap U_n)$ is a Lebesgue null set.

Theorem 3.7. *Let $F:\ M \to N$ be a C^∞ map between manifolds M and N of the same dimension. Let S be the set of $p \in M$ such that $(dF)_p$ is not surjective. Then $F(S)$ is a null set in N.*

Example 3.4. Consider the antipodal map a on the n-sphere S^n. Obviously every point on S^n is a regular value of a. Fix $q \in S^n$, there is exactly one pre-image p of q under a, namely, $p = -q$. If we visualize the orientation on S^n in terms of a normal vector field, it is not hard to see $\{v_1, \cdots, v_n\}$ on T_pS^n, $\{(da)_p(v_1), \cdots, (da)_p(v_n)\}$ is a positively oriented basis on T_qS^n if and only if n is odd (we leave it as an exercise). Therefore, deg$a = (-1)^{n+1}$.

In particular, this gives another proof of the Hairy Ball Theorem. In fact, if there exists a smooth non-vanishing vector field X on S^n, from the proof of Theorem 3.3 we know that the identity map is smoothly homotopic to the antipodal map. Therefore,

$$1 = \mathrm{degid}_{S^n} = \mathrm{deg}a = (-1)^{n+1},$$

which implies that n is odd.

By using the degree of a C^∞ map, one can give a geometric proof of the fundamental theorem of algebra.

Theorem 3.8. *Let $p(z) = z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ be a polynomial with complex coefficients of degree $n \geqslant 1$, then there exists some $z_0 \in \mathbb{C}$, such that $p(z_0) = 0$.*

Proof. Let $M = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, which is diffeomorphic to S^2 (see Problem 1 in Problem Sheet 1). We can think of p as a C^∞ map from $\mathbb{C}$ to itself, which can be extended to a C^∞ map (still denoted by p) from M to itself by defining $p(\infty) = \infty$. Moreover,

$$F(z,t) = \begin{cases} z^n + t(a_1z^{n-1} + \cdots + a_{n-1}z + a_n), & z \in \mathbb{C}; \\ \infty, & z = \infty, \end{cases}$$

defines a smooth homotopy from C^∞ map

$$p_0(z) = \begin{cases} z^n, & z \in \mathbb{C}; \\ \infty, & z = \infty, \end{cases}$$

to the map p. Therefore, the degree of p is the same as the degree of p_0.

Now consider the oriented chart $\mathbb{C} \subset M$ with coordinates (x, y). Choose a bump function f with compact support in $\mathbb{C}$ which depends only on $\|z\|$ and does not contain the origin, and consider the 2-form

$$\omega = f(x, y)dx \wedge dy.$$

By using polar coordinates (r, θ), we may write

$$\omega = f(r)rdr \wedge d\theta$$

and

$$p_0^*\omega = f(r^n)r^n d(r^n) \wedge d(n\theta).$$

It follows from (3.14) that

$$\text{deg}p_0 \cdot \int_{\mathbb{R}^2} f(r)rdrd\theta = n \cdot \int_{\mathbb{R}^2} f(r^n)r^n d(r^n)d\theta = n \cdot \int_{\mathbb{R}^2} f(r)rdrd\theta.$$

Therefore, $\text{deg}p = \text{deg}p_0 = n \geqslant 1$. By Corollary 3.2, we know that p is surjective. In particular, $p^{-1}(0) \neq \emptyset$ and hence there exists some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. □

3.3 The cohomology of cochain complexes

We need to develop more tools to study the de Rham cohomology of a manifold. One natural idea, which is quite fundamental in algebraic topology, is to decompose the manifold into open subsets and to study the relation between the de Rham cohomology of these subsets and of the original manifold. This technique is formally known as the Mayer-Vietoris argument.

Before going into the geometric setting, we shall first develop some algebraic tools.

Definition 3.7. A *cochain complex* $A^* = (A^\cdot, d^\cdot)$ is a sequence $\{A^r\}$ of real vector spaces (not necessarily finite dimensional) connected by a sequence $\{d^r : A^r \to A^{r+1}\}$ of linear maps, called *coboundary maps*, such that $d^{r+1} \circ d^r = 0$ for all r.

We can use the diagram

$$\cdots \longrightarrow A^{r-1} \xrightarrow{d^{r-1}} A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} A^{r+2} \longrightarrow \cdots$$

to describe a cochain complex A^*.

For each r, elements in A^r are called *r-cochains*, elements in $\mathrm{Ker}d^r$ are called *r-cocycles*, and elements in $\mathrm{Im}d^{r-1}$ are called *r-coboundaries*. Since $d^{r-1} \circ d^r = 0$, similar to the de Rham cohomology, we can also form the quotient space

$$H^r(A^*) = \frac{\mathrm{Ker}d^r}{\mathrm{Im}d^{r-1}}.$$

Definition 3.8. $H^r(A^*)$ is called the *r-th cohomology group of the cochain complex* A^*.

Definition 3.9. A cochain complex A^* is called *exact* if $\mathrm{Im}d^{r-1} = \mathrm{Ker}d^r$ (equivalently, $H^r(A^*) = 0$) for all r.

We may talk about maps between cochain complexes.

Definition 3.10. A *cochain map* $f: A^* \to B^*$ between two cochain complexes consists of a sequence $\{f^r: A^r \to B^r\}$ of linear maps, such that $f^{r+1} \circ d_A^r = d_B^r \circ f^r$ for all r. In other words, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{r-1} & \xrightarrow{d_A^{r-1}} & A^r & \xrightarrow{d_A^r} & A^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{r-1}} & & \downarrow{\scriptstyle f^r} & & \downarrow{\scriptstyle f^{r+1}} & & \\
\cdots & \longrightarrow & B^{r-1} & \xrightarrow{d_B^{r-1}} & B^r & \xrightarrow{d_B^r} & B^{r+1} & \longrightarrow & \cdots .
\end{array}$$

By the commutativity of the previous diagram, it is easy to see that a cochain map induces a linear map

$$f^*: H^r(A^*) \to H^r(B^*)$$

for each r, by acting on representatives.

Example 3.5. The de Rham complex together with the exterior derivative, and the pullback of differential forms by a C^∞ map between manifolds is a standard example of these algebraic concepts.

Now we introduce a particularly important concept: a short exact sequence.

Definition 3.11. A *short exact sequence* is an exact cochain complex of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \tag{3.15}$$

A *short exact sequence of cochain complexes*

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0 \tag{3.16}$$

consists of chain maps f and g such that

$$0 \longrightarrow A^r \xrightarrow{f^r} B^r \xrightarrow{g^r} C^r \longrightarrow 0.$$

is a short exact sequence for all r.

It is easy to see that a cochain complex of the form (3.15) is a short exact sequence if and only if f is injective, g is surjective, and $\mathrm{Ker}g = \mathrm{Im}f$.

The main result we are going to present here is that a short exact sequence of cochain complexes induces a long exact sequence on cohomology, and the construction is canonical.

Assume that we have a short exact sequence of cochain complexes of the form (3.16). To simplify the notation, we will always use the same d to denote coboundary maps even in different cochain complexes and use the same f and g to denote cochain maps acting on each degree, although the domain of definition for d, f, g may depend on the context.

Now we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{r-1} & \xrightarrow{d} & A^{r} & \xrightarrow{d} & A^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow f & & \downarrow f & & \downarrow f & & \\
\cdots & \longrightarrow & B^{r-1} & \xrightarrow{d} & B^{r} & \xrightarrow{d} & B^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow g & & \downarrow g & & \downarrow g & & \\
\cdots & \longrightarrow & C^{r-1} & \xrightarrow{d} & C^{r} & \xrightarrow{d} & C^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}
\tag{3.17}
$$

where each row represents a cochain complex and each column is a short exact sequence.

We are going to construct a linear map $\partial^* : H^r(C^*) \to H^{r+1}(A^*)$ which connects the whole diagram (3.17) up and down on the level of cohomology.

Let $[c] \in H^r(C^*)$ be represented by an r-cocycle $c \in C^r$. By short exactness, we know that g is surjective, hence there exists $b \in B^r$ such that $c = g(b)$. From the commutativity of the diagram, we have

$$g(d(b)) = d(g(b)) = d(c) = 0,$$

which means that $d(b) \in \mathrm{Ker}g$. Again by short exactness, we have $d(b) \in \mathrm{Im}f$, and hence there exists some $a \in A^{r+1}$ with $d(b) = f(a)$. From commutativity again, we have

$$f(d(a)) = d(f(a)) = d^2(b) = 0,$$

which implies that $d(a) = 0$ since f is injective. Therefore, a is an $(r+1)$-cocycle which determines a cohomology class $[a] \in H^{r+1}(A^*)$. We define $\partial^*([c]) = [a]$. This procedure of defining ∂^* is usually known as *diagram chasing*.

Of course we need to prove the following fact.

Lemma 3.2. $\partial^* : H^r(C^*) \to H^{r+1}(A^*)$ *is a well-defined linear map.*

Proof. Assume that we start with another r-cocycle $c' \in C^r$ in the cohomology class $[c]$, and define

$$c' \mapsto b' \mapsto d(b') \mapsto a'$$

by the previous diagram chasing procedure. We need to show that $a' - a$ is an $(r+1)$-coboundary, namely, there exists some $a_0 \in A^r$ such that $d(a_0) = a' - a$.

First notice that c', c represent the same cohomology class, and hence there exists $c_0 \in C^{r-1}$ with $dc_0 = c' - c$. By the surjectivity of g, we can find some $b_0 \in B^{r-1}$ with $g(b_0) = c_0$. It follows from commutativity that

$$\begin{aligned} g(b' - b - d(b_0)) &= c' - c - g(d(b_0)) \\ &= c' - c - d(g(b_0)) \\ &= 0. \end{aligned}$$

Therefore, by short exactness we have $b' - b - d(b_0) \in \mathrm{Im} f$, which implies that there exists some $a_0 \in A^r$ such that

$$b' - b - d(b_0) = f(a_0).$$

Now again by commutativity we have

$$\begin{aligned} f(d(a_0)) &= d(f(a_0)) \\ &= d(b' - b - d(b_0)) \\ &= f(a') - f(a). \end{aligned}$$

It follows from the injectivity of f that $d(a_0) = a' - a$, which concludes the proof. $\square$

Now we have the following important result.

Theorem 3.9. *A short exact sequence of cochain complexes of the form (3.16) induces the following long exact sequence:*

$$\cdots \xrightarrow{\partial^*} H^r(A^*) \xrightarrow{f^*} H^r(B^*) \xrightarrow{g^*} H^r(C^*) \xrightarrow{\partial^*} H^{r+1}(A^*) \xrightarrow{f^*} \cdots, \tag{3.18}$$

where ∂^* is the linear map defined previously.

Proof. It is straight forward from definition that the sequence (3.18) satisfies

$$g^* \circ f^* = 0,\ \partial^* \circ g^* = 0,\ f^* \circ \partial^* = 0,$$

and hence it is a cochain complex. Now we show exactness at each part.

(1) $\mathrm{Ker} g^* = \mathrm{Im} f^*$.

Let $[b] \in \mathrm{Ker} g^*$, then $g(b)$ is a coboundary, so $g(b) = d(c_1)$ for some cochain c_1. By the surjectivity of g, there exists some cochain b_1 such that $g(b_1) = c_1$. It follows that

$$g(d(b_1)) = d(g(b_1)) = d(c_1) = g(b),$$

and thus $b - d(b_1) \in \mathrm{Im} f$, which implies

$$b - d(b_1) = f(a)$$

for some cochain a. Since b is a cocycle, it is easy to see that a is also a cocycle, which implies that $[b] = f^*[a] \in \mathrm{Im} f^*$.

(2) $\mathrm{Ker}\partial^* = \mathrm{Im} g^*$.

Let $[c] \in \mathrm{Ker}\partial^*$, and take a, b as in the construction of ∂^*. It follows that there exists some cochain a_1 such that $d(a_1) = a$. Let $b_1 = f(a_1)$, then we have

$$g(b - b_1) = c - g(f(a_1)) = c.$$

On the other hand,

$$d(b - b_1) = f(a) - df(a_1) = f(d(a_1)) - d(f(a_1)) = 0,$$

which means that $b - b_1$ is a cocycle. It follows from the definition of g^* that $[c] = g^*([b - b_1]) \in \mathrm{Im} g^*$.

(3) $\mathrm{Ker} f^* = \mathrm{Im}\partial^*$.

Let $[a] \in \mathrm{Ker} f^*$, then $f(a) = d(b)$ for some cochain b. Let $c = g(b)$. It follows that

$$d(c) = d(g(b)) = g(d(b)) = g(f(a)) = 0,$$

which implies that c is a cocycle. By the definition of ∂^*, it is obvious that $[a] = \partial^*([c]) \in \mathrm{Im}\partial^*$.

Now the proof is complete. □

Another very useful algebraic result about exact sequences is the following so-called the *Five Lemma*, which will be used later on.

Theorem 3.10. *Let f be a cochain map between two exact cochain complexes A^* and B^* of the form*

$$\begin{array}{ccccccccc} A^1 & \xrightarrow{d_A^1} & A^2 & \xrightarrow{d_A^2} & A^3 & \xrightarrow{d_A^3} & A^4 & \xrightarrow{d_A^4} & A^5 \\ \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & \downarrow{\scriptstyle f^4} & & \downarrow{\scriptstyle f^5} \\ B^1 & \xrightarrow{d_B^1} & B^2 & \xrightarrow{d_B^2} & B^3 & \xrightarrow{d_B^3} & B^4 & \xrightarrow{d_B^4} & B^5. \end{array}$$

Assume further that f^1, f^2, f^4, f^5 are isomorphisms. Then f^3 is also an isomorphism.

Proof. Exercise. □

Finally, by using the notions of cochain complexes and cochain maps, we may recapture what we've essentially done when proving the homotopy invariance for de Rham cohomology from an algebraic point of view.

Definition 3.12. Two chain maps $f, g: A^* \to B^*$ between cochain complexes A^* and B^* are said to be *cochain-homotopic*, if for each r, there exists a linear map $s^r: A^r \to B^{r-1}$ such that

$$d_B^{r-1} \circ s^r + s^{r+1} \circ d_A^r = g^r - f^r: A^r - B^r.$$

This sequence $s = (s^\cdot)$ of linear maps is called a *homotopy operator* from f to g.

It is immediate from definition that if f and g are cochain-homotopic, then $f^* = g^*: H^r(A^*) \to H^r(B^*)$ for all r.

If we look back into the proofs Proposition 3.1 and Proposition 3.2, actually what we are doing is to construct a homotopy operator $\widehat{S}$ from the cochain maps s_0^* to s_1^* on de Rham complexes. This implies immediately that $s_0^* = s_1^*$ on the de Rham cohomology, which is the key to proving the homotopy invariance for the de Rham cohomology. More directly, from the proof of Theorem 3.1, it is easy to see that $\widehat{S} \circ F^*$ defines a homotopy operator from f^* to g^*, which concludes that $f^* = g^*$ on the de Rham cohomology.

3.4 The Mayer-Vietoris sequence

Perhaps the most important application of Theorem 3.9 in the context of de Rham cohomology is the so-called Mayer-Vietoris sequence, which provides us with a fundamental tool of computing the de Rham cohomology groups in general. The basic idea is that the de Rham cohomology of the union $U_1 \cup U_2$ of open subsets of a manifold can be computed via $H^*(U_1), H^*(U_2)$, and $H^*(U_1 \cap U_2)$. By induction, this approach generalizes to the case where we have finitely many open subsets.

Let M be a manifold, and let U_1, U_2 be non-empty open subsets of M. For $\alpha = 1, 2$, we use $i_\alpha: U_\alpha \to U_1 \cup U_2$ and $j_\alpha: U_1 \cap U_2 \to U_\alpha$ to denote the inclusions.

Now we have the following result.

Proposition 3.5. *For each $r \geqslant 0$, define $I^*: \Omega^r(U_1 \cup U_2) \to \Omega^r(U_1) \oplus \Omega^r(U_2)$ by*

$$I^*(\omega) = (i_1^*\omega, i_2^*\omega),$$

and define $J^*: \Omega^r(U_1) \oplus \Omega^r(U_2) \to \Omega^r(U_1 \cap U_2)$ by

$$J^*(\omega, \tau) = j_1^*\omega - j_2^*\tau.$$

Then

$$0 \longrightarrow \Omega^*(U_1 \cup U_2) \xrightarrow{I^*} \Omega^*(U_1) \oplus \Omega^*(U_2) \xrightarrow{J^*} \Omega^*(U_1 \cap U_2) \longrightarrow 0 \tag{3.19}$$

is a short exact sequence of cochain complexes (de Rham complexes).

Remark 3.3. When $U_1 \cap U_2 = \emptyset$, we define J^* to be the zero map, and obviously I^* is an isomorphism.

Proof. Let $r \geqslant 0$.

(1) $I^*\ \Omega^r(U_1 \cup U_2) \to \Omega^r(U_1) \oplus \Omega^r(U_2)$ is injective.

This is trivial by using local coordinate charts.

(2) $J:\ \Omega^r(U_1) \oplus \Omega^r(U_2) \to \Omega^r(U_1 \cap U_2)$ is surjective.

Let $\alpha \in \Omega^r(U_1 \cap U_2)$, and choose a partition of unity $\{\varphi_1, \varphi_2\}$ subordinate to the open cover $\{U_1, U_2\}$ of $U_1 \cup U_2$ with the same index. Define

$$\omega(p) = \begin{cases} \varphi_2(p) \cdot \alpha(p), & p \in \text{supp}\varphi_2 \cap U_1; \\ 0, & p \in (\text{supp}\varphi_2)^c \cap U_1, \end{cases} \tag{3.20}$$

and

$$\tau(p) = \begin{cases} -\varphi_1(p) \cdot \alpha(p), & p \in \text{supp}\varphi_1 \cap U_2; \\ 0, & p \in (\text{supp}\varphi_1)^c \cap U_2. \end{cases} \tag{3.21}$$

It follows from the properties of $\{\varphi_1, \varphi_2\}$ that $\omega \in \Omega^r(U_1), \tau \in \Omega^r(U_2)$ and

$$J^*(\omega, \tau) = j_1^*(\omega) - j_2^*(\tau) = \alpha.$$

(3) $\text{Ker}J^* = \text{Im}I^*$.

It is trivial to see that $\text{Im}I^* \subset \text{Ker}J^*$. Conversely, if $(\omega, \tau) \in \text{Ker}J^*$, we can simply define

$$\xi(p) = \begin{cases} \omega(p), & p \in U_1; \\ \tau(p), & p \in U_2. \end{cases}$$

This is well-defined and it is an r-form on M such that $I^*(\xi) = (\omega, \tau)$.

Now the proof is complete. □

Note that for two cochain complexes A^* and B^*, we have a canonical isomorphism between $H^r(A^* \oplus B^*)$ and $H^r(A^*) \oplus H^r(B^*)$ defined by $[(a, b)] \mapsto ([a], [b])$ (see Problem 2 of Problem Sheet 3). By Theorem 3.9, we have the following important result. This is known as the *Mayer-Vietoris sequence*.

Theorem 3.11. *The short exact sequence (3.19) of cochain complexes induces a long exact sequence*

$$\cdots \xrightarrow{\partial^*} H^r(U_1 \cup U_2) \xrightarrow{I^*} H^r(U_1) \oplus H^r(U_2) \xrightarrow{J^*} H^r(U_1 \cap U_2)$$

$$\xrightarrow{\partial^*} H^{r+1}(U_1 \cup U_2) \xrightarrow{I^*} H^{r+1}(U_1) \oplus H^{r+1}(U_2) \xrightarrow{J^*} \cdots \tag{3.22}$$

on the level of cohomology, where the coboundary map ∂^* is defined explicitly via diagram chasing as before.

It is better to describe ∂^* in our context here. According to its construction and the proof of Proposition 3.5, for any $[\alpha] \in H^r(U_1 \cap U_2)$, we can find $(\omega, \tau) \in \Omega^r(U_1) \oplus \Omega^r(U_2)$ (defined by (3.20) and (3.21)) such that

$$j_1^*\omega - j_2^*\tau = \alpha.$$

Since $d\alpha = 0$, we know that $d\omega$ coincides with $d\tau$ on $U_1 \cap U_2$. It follows that $d\omega$ and $d\tau$ glue to a closed $(r+1)$-form ξ on $U_1 \cup U_2$, and we have $[\xi] = \partial^*([\alpha])$.

An important application of the Mayer-Vietoris sequence is the computation of the de Rham cohomology of the n-sphere S^n.

First of all, we have seen in Section 3.2 that

$$H^r(S^1) = \begin{cases} \mathbb{R}, & r = 0, 1; \\ 0, & \text{otherwise}. \end{cases} \tag{3.23}$$

Now suppose $n \geqslant 2$. Let

$$\begin{aligned} U_1 &= \{(x^0, x^1, \cdots, x^n) \in S^n : \ x^n > -\frac{1}{2}\}, \\ U_2 &= \{(x^0, x^1, \cdots, x^n) \in S^n : \ x^n < \frac{1}{2}\}. \end{aligned}$$

It follows that U_1 and U_2 are both contractible. Moreover, $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is diffeomorphic to $S^{n-1} \times (-\frac{1}{2}, \frac{1}{2})$.

For $r \geqslant 1$, from the Mayer-Vietoris sequence we know that the sequence

$$0 = H^r(U_1) \oplus H^r(U_2) \xrightarrow{J^*} H^r(U_1 \cap U_2)$$

$$\xrightarrow{\partial^*} H^{r+1}(U_1 \cup U_2) \xrightarrow{I^*} H^{r+1}(U_1) \oplus H^{r+1}(U_2) = 0$$

is exact. This implies that ∂^* is an isomorphism and hence we have

$$H^r(S^{n-1} \times (-\frac{1}{2}, \frac{1}{2})) \cong H^r(U_1 \cap U_2) \cong H^{r+1}(U_1 \cup U_2) = H^{r+1}(S^n).$$

On the other hand, since $S^{n-1} \times (-\frac{1}{2}, \frac{1}{2})$ and S^{n-1} are homotopy equivalent, it follows that

$$H^r(S^{n-1}) \cong H^{r+1}(S^n). \tag{3.24}$$

For $r = 0$, we consider the exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{I^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{J^*} \mathbb{R} \xrightarrow{\partial^*} H^1(S^n) \longrightarrow 0, \tag{3.25}$$

where we identify each de Rham cohomology group of degree 0 with $\mathbb{R}$ by connectedness. Then we have

$$H^1(S^n) = \mathrm{Im}(\partial^*), \ \mathrm{Ker}(\partial^*) = \mathrm{Im}(J^*), \ \mathrm{Ker}(J^*) = \mathrm{Im}(I^*).$$

Since I^* is injective, it follows that

$$\dim\mathrm{Im}(I^*) = \dim\mathrm{Ker}(J^*) = 1,$$

and hence

$$\dim\mathrm{Ker}(\partial^*) = \dim\mathrm{Im}(J^*) = 2 - \dim\mathrm{Ker}(J^*) = 1.$$

This implies that ∂^* is a zero map and therefore

$$H^1(S^n) = 0. \tag{3.26}$$

Combining with (3.23), (3.24) and (3.26), by induction we obtain the following result.

Proposition 3.6. *For $n \geqslant 1$,*

$$H^r(S^n) = \begin{cases} \mathbb{R}, & r = 0, n; \\ 0, & \textit{otherwise}. \end{cases} \tag{3.27}$$

Remark 3.4. If we examine the proof more carefully, the only place we've used the result on the top cohomology of a compact, connected and oriented manifold (namely, Corollary 3.1) is for the circle S^1. The use of the Mayer-Vietoris sequence and the induction argument are purely algebraic.

One might ask if we can compute the de Rham cohomology of S^1 directly from the Mayer-Vietoris sequence without using Corollary 3.1. This is of course possible. In fact, we can use the same exact sequence as (3.25), but here

$$H^0(U_1 \cap U_2) \cong \mathbb{R} \oplus \mathbb{R}$$

since $U_1 \cap U_2$ has two connected components. The same argument as before yields that

$$\mathrm{dim} H^1(S^1) = \mathrm{dim}\mathrm{Im}\partial^* = 1,$$

and therefore $H^1(S^1) = 1$.

From Remark 3.2 we know how to construct a generator of the top de Rham cohomology group $H^n(S^n)$ via integration. Alternatively, we can use the Mayer-Vietoris seqeunce to do this. Firstly, for the circle S^1, let

$$\alpha(p) = \begin{cases} 1, & p \in U_1 \cap U_2 \cap \{(x^0, x^1) \in S^1 : \ x^0 > 0\}; \\ 0, & p \in U_1 \cap U_2 \cap \{(x^0, x^1) \in S^1 : \ x^0 < 0\}. \end{cases}$$

This is a closed 0-form on $U_1 \cap U_2$. From the construction of the coboundary map ∂^*, we can define a closed 1-form ξ_1 on S^1 via diagram chasing, which represents the de Rham cohomology class $\partial^*([\alpha])$. We leave it as an exercise to show that ξ_1 is not exact so $[\xi_1]$ is a generator of $H^1(S^1)$. Moreover, by using partition of unity (see (3.20) and (3.21)) we can choose ξ_1 to be supported in (in fact, any open subset of) $U_1 \cap U_2$. For $n \geqslant 2$, notice that the pullback of differential forms by the projection $\pi : \ S^{n-1} \times (-\frac{1}{2}, \frac{1}{2}) \to S^{n-1}$ induces an isomorphism

$$\pi^* : \ H^{n-1}(S^{n-1}) \to H^{n-1}(S^{n-1} \times (-\frac{1}{2}, \frac{1}{2})).$$

Therefore, by using induction on the exact sequence

$$0 \xrightarrow{J^*} H^{n-1}(S^{n-1} \times (-\tfrac{1}{2}, \tfrac{1}{2})) \xrightarrow{\partial^*} H^n(S^n) \xrightarrow{I^*} 0,$$

we can construct a generator $[\xi_n]$ of $H^n(S^n)$ which comes from a generator of $H^{n-1}(U_1 \cap U_2) \cong H^{n-1}(S^{n-1})$ via the isomorphism ∂^*, and by the same reason as before ξ_n can be chosen to be supported in $U_1 \cap U_2$. If we look at the construction more consistently, we can see that all these ξ_n's are obtained by

$$\alpha \to \xi_1 \to \pi^*\xi_1 \to \xi_2 \to \pi^*\xi_2 \to \xi_3 \to \pi^*\xi_3 \to \cdots$$

by propagating along the coboundary operator ∂^* in the Mayer-Vietoris sequence for each n.

Now we use (3.27) to compute the de Rham cohomology of the real projective space $\mathbb{R}P^n$.

Let $q: S^n \to \mathbb{R}P^n$ be the projection map and $a: S^n \to S^n$ be the antipodal map. We leave it as an exercise to show that for each $r \geqslant 0$, $q^*: \mathbb{R}P^n \to S^n$ is injective and

$$q^*(\Omega^r(\mathbb{R}P^n)) = \{\omega \in \Omega^r(S^n) : a^*\omega = \omega\}.$$

On the other hand, since $a^2 = \mathrm{id}_{S^n}$, from standard linear algebra we have the eigen-space decomposition

$$\Omega^r(S^n) = \Omega^r_+(S^n) \oplus \Omega^r_-(S^n),$$

where

$$\Omega^r_\pm(S^n) = \{\omega \in \Omega^r(S^n) : a^*\omega = \pm\omega\}.$$

From Problem 2 (1) in Problem Sheet 3, we know that

$$H^*(S^n) \cong H^*_+(S^n) \oplus H^*_-(S^n),$$

where $H^*_\pm(S^n)$ is the cohomology of the cochain complex $(\Omega^*_\pm(S^n), d)$.

Now we see that

$$H^*(\mathbb{R}P^n) \cong H^*_+(S^n),$$

which already implies by (3.27) that

$$H^r(\mathbb{R}P^n) = 0, \ \forall 1 \leqslant r \leqslant n-1.$$

Moreover, if we take the standard orientation form ω on S^n, then $[\omega]$ is a generator of $H^n(S^n) \cong \mathbb{R}$ and we've seen before that

$$a^*\omega = (-1)^{n+1}\omega.$$

Therefore, if n is even, then

$$H^n(\mathbb{R}P^n) \cong H^n_+(S^n) = 0$$

and if n is odd,

$$H^n(\mathbb{R}P^n) \cong H^n_+(S^n) \cong H^n(S^n) = \mathbb{R}.$$

Finally, combining with the fact that $\mathbb{R}P^n$ is connected, we obtain the following result.

Proposition 3.7. *For* $n \geqslant 1$*,*

$$H^r(\mathbb{R}P^n) = \begin{cases} \mathbb{R}, & r = 0 \text{ or } r = n \text{ if } n \text{ is odd}; \\ 0, & \text{otherwise.} \end{cases}$$

This method does not work for computing the de Rham cohomology of the complex projective space $\mathbb{C}P^n$. Instead, let's try to use the Mayer-Vietoris sequence directly to do this.

Let $U = \{[(z^0, \cdots, z^n)] \in \mathbb{C}P^n : \ z^n \neq 0\}$, and $V = \{[(0, \cdots, 0, 1)]\}^c$. It follows that $U \cup V = \mathbb{C}P^n$. Moreover, it is easy to see that U is diffeomorphic to $\mathbb{C}^n$, and $U \cap V$ is diffeomorphic to $\mathbb{C}^n \backslash \{0\}$. Now we are going to show that V is homotopy equivalent to $\mathbb{C}P^{n-1}$. To see this, first notice that $\mathbb{C}P^{n-1}$ is canonically embedded into $\mathbb{C}P^n$ by identifying $\mathbb{C}P^{n-1}$ with the space $\{[(z^0, \cdots, z^n)] \in \mathbb{C}P^n : \ z^n = 0\} \subset V$. Under such identification, if we let $i: \ \mathbb{C}P^{n-1} \to V$ be the inclusion and $r: \ V \to \mathbb{C}P^{n-1}$ be the projection map defined by

$$r([(z^0, \cdots, z^n)]) = [(z^0, \cdots, z^{n-1}, 0)],$$

then $r \circ i = \mathrm{id}_{\mathbb{C}P^{n-1}}$. Furthermore, the C^∞ map

$$\begin{aligned} F: \ V \times [0,1] &\to V, \\ ([(z^0, \cdots, z^n)], t) &\mapsto [(z^0, \cdots, z^{n-1}, tz^n)], \end{aligned}$$

defines a smooth homotopy from $i \circ r$ to id_V. Therefore, V and $\mathbb{C}P^{n-1}$ are homotopy equivalent, and hence they have isomorphic de Rham cohomology.

Now we have the Mayer-Vietoris sequence for U, V on $\mathbb{C}P^n$. To compute $H^*(\mathbb{C}P^n)$ inductively based on this sequence, we need to know $H^*(\mathbb{C}P^1)$. However, from the differential structure of $\mathbb{C}P^1$, it is not hard to see that $\mathbb{C}P^1$ is diffeomorphic to S^2 (one may use the stereographic projection to see this), and hence $H^*(\mathbb{C}P^1) \cong H^*(S^2)$.

Now we have enough data to compute $H^*(\mathbb{C}P^n)$. We state the result in the following and leave the algebraic computation as an exercise.

Proposition 3.8. *For* $n \geqslant 1$*;*

$$H^r(\mathbb{C}P^n) = \begin{cases} \mathbb{R}, & r \text{ is even and } 0 \leqslant r \leqslant 2n; \\ 0, & \text{otherwise.} \end{cases}$$

The power of Mayer-Vietoris sequence is much more than just computing the de Rham cohomology of some special manifolds. In the rest of the notes, we will use the Mayer-Vietoris sequence to prove three important results on de Rham cohomology: the Poincaré duality, the Künneth formula, and the Thom isomorphism. They are all very useful in understanding the de Rham cohomology quantitatively.

3.5 The Poicaré duality

Firstly, we prove an important result on the relation between the de Rham cohomology and the one with compact supports for an orientable manifold: the Poincaré duality. Here we will again see the power of integration.

First we introduce the concept of de Rham cohomology with compact supports (we have seen this for the top degree before).

Let M be a manifold, and let $(\Omega_c^*(M), d)$ be the cochain complex of compactly supported differential forms on M, and consider the associated cohomology

$$H_c^r(M) = \frac{\mathrm{Ker}(d:\ \Omega_c^r(M) \to \Omega_c^{r+1}(M))}{\mathrm{Im}(d:\ \Omega_c^{r-1}(M) \to \Omega_c^r(M))},\ r \geqslant 0.$$

This is called the *de Rham cohomology of M with compact supports*, and it is denoted by $H_c^*(M)$.

To get some feeling about this cohomology, let's compute $H_c^*(\mathbb{R}^n)$.

It is obvious that $H_c^0(\mathbb{R}^n) = 0$. Furthermore, by Theorem 3.4 we know that $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$. The key of computing $H_c^*(\mathbb{R}^n)$ is induction based on the following fact.

Proposition 3.9. *Let $\pi_*:\ \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1) \to \Omega_c^{r-1}(\mathbb{R}^n)$ be the linear map which sends $\omega \in \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ of the form*

$$(I):\ a(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

to zero and sends $\omega \in \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ of the form

$$(II):\ b(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dt$$

to $(\int_{\mathbb{R}^1} b(x,t)dt)dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \in \Omega_c^{r-1}(\mathbb{R}^n)$, where $a(x,t)$ and $b(x,t)$ are C^∞ functions on $\mathbb{R}^n \times \mathbb{R}^1$ with compact supports. Then π_* commutes with d, and it induces a linear isomorphism on cohomology.

Proof. The commutativity of π_* and d follows from straight forward calculation if we express $\omega \in \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ in terms of the natural basis $\{dx^1, \cdots, dx^n, dt\}$. Therefore, π_* induces a linear homomorphism on cohomology.

To show that $\pi_*:\ H_c^r(\mathbb{R}^n \times \mathbb{R}^1) \to H_c^{r-1}(\mathbb{R}^n)$ is an isomorphism, we construct the inverse of π_* explicitly. Let $e(t) \in C_c^\infty(\mathbb{R}^1)$ be such that $\int_{\mathbb{R}^1} e(t)dt = 1$. Define the linear map $e_*:\ \Omega_c^{r-1}(\mathbb{R}^n) \to \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ by

$$e_*(\omega) = \pi^*\omega \wedge e(t)dt,$$

where $\pi:\ \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ is the natural projection. It is easy to see that e_* commutes with d and hence it induces a linear homomorphism on cohomology. Moreover, it is trivial that $\pi_* \circ e_* = \mathrm{id}_{\Omega_c^{r-1}(\mathbb{R}^n)}$. Therefore, e_* is the right inverse of π_* on cohomology.

In order to show that e_* is also the left inverse of π_* on cohomology, we need to construct a homotopy operator $K^r:\ \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1) \to \Omega_c^{r-1}(\mathbb{R}^n \times \mathbb{R}^1)$ from the cochain maps $e_* \circ \pi_*$ to id. It is constructed explicitly in the following way.

For $\omega \in \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ of type (I), we define $K(\omega) = 0$. For $\omega \in \Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)$ of type (II), we define

$$\begin{aligned} K(\omega) &= (-1)^{r+1}((\int_{-\infty}^{t} b(x,s)ds) - (\int_{-\infty}^{\infty} b(x,s)ds) \cdot (\int_{-\infty}^{t} e(s)ds)) \\ &\quad \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}}. \end{aligned}$$

If ω is of type (I), then

$$(\mathrm{id}_{\Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)} - e_* \circ \pi_*)(\omega) = \omega,$$

and

$$\begin{aligned} (dK + Kd)(\omega) &= Kd\omega \\ &= K(\frac{\partial a}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \\ &\quad +(-1)^r \frac{\partial a}{\partial t} dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dt) \\ &= a(x,t) dx^{i_1} \wedge \cdots \wedge dx^{i_r} \\ &= \omega. \end{aligned}$$

If ω is of type (II), then

$$\begin{aligned} &(\mathrm{id}_{\Omega_c^r(\mathbb{R}^n \times \mathbb{R}^1)} - e_* \circ \pi_*)(\omega) \\ =&(b(x,t) - (\int_{-\infty}^{\infty} b(x,s)ds) \cdot e(t)) dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} K(\omega) &= (-1)^{r+1}((\int_{-\infty}^{t} b(x,s)ds) - (\int_{-\infty}^{\infty} b(x,s)ds) \cdot (\int_{-\infty}^{t} e(s)ds)) \\ &\quad \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}}, \\ d\omega &= \frac{\partial b}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dt, \end{aligned}$$

and

$$\begin{aligned} dK(\omega) &= (b(x,t) - (\int_{-\infty}^{\infty} b(x,s)ds) \cdot e(t)) dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dt \\ &\quad +(-1)^{r+1}(\int_{-\infty}^{t} \frac{\partial b}{\partial x^k} ds - (\int_{-\infty}^{\infty} \frac{\partial b}{\partial x^k} ds) \cdot \int_{-\infty}^{t} e(s)ds) \\ &\quad \cdot dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}}, \\ Kd(\omega) &= (-1)^r(\int_{-\infty}^{t} \frac{\partial b}{\partial x^k} ds - (\int_{-\infty}^{\infty} \frac{\partial b}{\partial x^k} ds) \cdot \int_{-\infty}^{t} e(s)ds) \\ &\quad \cdot dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}}. \end{aligned}$$

It follows that

$$(\mathrm{id}_{\Omega_c^r(\mathbb{R}^n\times\mathbb{R}^1)} - e_*\circ\pi_*)(\omega) = (dK + Kd)(\omega).$$

Therefore, K is a homotopy operator from the cochain maps $e_*\circ\pi_*$ to id, and the proof is complete. □

By induction we can easily obtain the following result.

Proposition 3.10. *For $n \geqslant 1$,*

$$H_c^r(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & r = n; \\ 0, & \textit{otherwise}. \end{cases}$$

From this we can see that the de Rham cohomology with compact supports is not homotopy invariant. However, it is obvious that it is invariant under diffeomorphism.

For the usual de Rham complex, we have the short exact sequence (3.19) which induces the long exact sequence (the Mayer-Vietoris sequence) on the de Rham cohomology. Similarly, we are going to show that, for the de Rham complex with compact supports, we also have a short exact sequence which of course induces a long exact sequence on cohomology. But here the direction is reversed: instead of pulling back forms by the inclusions, we use push-forwards.

Let U_1, U_2 be open subsets of M and define the inclusions i_1, i_2, j_1, j_2 as before. A r-form ω with compact support in U_1 can be regarded as a differential form $i_{1*}\omega$ with compact support in $U_1\cup U_2$ by trivial extension. Therefore, the push-forward $i_{1*}:\ \Omega_c^*(U_1)\to\Omega_c^*(U_1\cup U_2)$, and similarly i_{2*}, j_{1*}, j_{2*} are all well-defined. Moreover, they all commutes with the exterior derivative and hence they are cochain maps.

Now we have the following result.

Proposition 3.11. *For each $r\geqslant 0$, define $J_*:\ \Omega_c^r(U_1\cap U_2)\to\Omega_c^r(U_1)\oplus\Omega_c^r(U_2)$ by*

$$J_*(\alpha) = (j_{1*}\alpha, -j_{2*}\alpha),$$

and define $I_*:\ \Omega_c^r(U_1)\oplus\Omega_c^r(U_2)\to\Omega_c^r(U_1\cup U_2)$ by

$$I_*(\omega,\tau) = i_{1*}\omega + i_{2*}\tau.$$

Then

$$0 \longrightarrow \Omega_c^*(U_1\cap U_2) \xrightarrow{J_*} \Omega_c^*(U_1)\oplus\Omega_c^*(U_2) \xrightarrow{I_*} \Omega_c^*(U_1\cup U_2) \longrightarrow 0 \tag{3.28}$$

is a short exact sequence of cochain complexes.

Proof. Obviously this is a cochain complex. Let $r\geqslant 0$. It is trivial to see that J_* is injective. Moreover, let $\xi\in\Omega_c^r(U_1\cup U_2)$ and choose a partition of unity

$\{\varphi_1, \varphi_2\}$ subordinate to $\{U_1, U_2\}$ on $U_1 \cup U_2$ with the same index. Then $\varphi_i \xi$ has compact support in U_i $(i = 1, 2)$, and

$$I_*((\varphi_1 \xi)|_{U_1}, (\varphi_2 \xi)|_{U_2}) = \xi.$$

Therefore, I_* is surjective. Finally, let $(\omega, \tau) \in \mathrm{Ker} I_*$. This implies that ω has to be compactly supported in $U_1 \cap U_2$, and certainly we have

$$J_*(\omega|_{U_1 \cap U_2}) = (\omega, \tau).$$

Therefore, $\mathrm{Ker} I_* = \mathrm{Im} J_*$. □

By Theorem 3.9, the short exact sequence (3.11) induces a long exact sequence

$$\cdots \xrightarrow{\partial_*} H^r_c(U_1 \cap U_2) \xrightarrow{J_*} H^r_c(U_1) \oplus H^r_c(U_2) \xrightarrow{I_*} H^r_c(U_1 \cup U_2)$$

$$\xrightarrow{\partial_*} H^{r+1}_c(U_1 \cap U_2) \xrightarrow{J^*} H^{r+1}_c(U_1) \oplus H^{r+1}_c(U_2) \xrightarrow{I_*} \cdots \tag{3.29}$$

on cohomology, where ∂_* is the coboundary map defined via diagram chasing. This is known as the *Mayer-Vietoris sequence with compact supports*.

In our context here, we can describe the coboundary map ∂_* explicitly. Take a closed r-form ξ on $U_1 \cup U_2$ with compact support, and define a pre-image

$$(\omega, \tau) = ((\varphi_1 \xi)|_{U_1}, (\varphi_2 \xi)|_{U_2})$$

of ξ under I_* by using a partition of unity as in the proof of Proposition 3.11. It follows that $d\omega$ is compactly supported in $U_1 \cap U_2$ and

$$J_*((d\omega)|_{U_1 \cap U_2}) = (d\omega, d\tau).$$

The cohomology class determined by $(d\omega)|_{U_1 \cap U_2}$ is the image of $[\xi]$ under ∂_*.

The Poincaré duality is about the relation between $H^*(M)$ and $H^*_c(M)$ on an orientable manifold M.

Assume that M is an oriented manifold of dimension n. Since we are able to integrate n-forms with compact supports on M, integration defines a bilinear functional

$$\int_M : \ \Omega^r(M) \times \Omega^{n-r}_c(M) \to \mathbb{R}$$

by

$$\int_M (\omega, \tau) = \int_M \omega \wedge \tau.$$

By Stokes' theorem, it induces a homomorphism

$$\int_M : \ H^r(M) \to H^{n-r}_c(M)^*$$

by

$$\langle \int_M [\omega], [\tau] \rangle = \int_M \omega \wedge \tau, \ [\tau] \in H_c^{n-r}(M),$$

where $H_c^{n-r}(M)^*$ is the dual space of $H_c^{n-r}(M)$.

The *Poincaré duality* says that $\int_M$ is an isomorphism for all $0 \leqslant r \leqslant n$.

For simplicity, we only prove this result for the case when M has a finite good cover. It holds for the general case by more careful analysis on the topology of the manifold (see [3] for the details).

Definition 3.13. An open cover $\{U_\alpha\}_{\alpha \in A}$ of M is called a *good cover* if any non-empty finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is diffeomorphic to $\mathbb{R}^n$. It is called a *finite good cover* if A is a finite set.

It can be proved that each manifold has a good cover, and each compact manifold has a finite good cover. The proof is to introduce a Riemannian metric on the manifold and use geodesically convex neighborhoods as a good cover. We are not going to present the details here.

If a manifold has a finite good cover, then its de Rham cohomology is relatively simple.

Proposition 3.12. *Let M be a manifold with a finite good cover. Then $H^r(M)$ and $H_c^r(M)$ are finite dimensional for all r.*

Proof. We prove by induction. If M is diffeomorphic to $\mathbb{R}^n$, then the result is trivial. Assume that the de Rham cohomology groups for any manifold with a finite good cover containing at most k open subsets are all finite dimensional. Let M be a manifold with a finite good cover $\{U_1, \cdots, U_{k+1}\}$, and let $N = U_1 \cup \cdots \cup U_k$. It follows from induction hypothesis that $H^r(N)$, $H^r(U_{k+1})$ and

$$H^r(N \cap U_{k+1}) = H^r((U_1 \cap U_{k+1}) \cup \cdots \cup (U_k \cap U_{k+1}))$$

are all finite dimensional for all r. The finite dimensionality of $H^r(M)$ then follows easily from the Mayer-Vietoris sequence for N and U_{k+1}.

Similarly, by using the Mayer-Vietoris sequence with compact supports, we can show by induction that $H_c^r(M)$ is finite dimensional for all r. $\square$

From now on, we always assume that M has a *finite* good cover. As we've seen in the proof of Proposition 3.12, this is for the reason of applying induction arguments on the cardinality of a finite good cover.

To initiate the induction for proving the Poincaré duality, let's first look at two open subsets U_1, U_2 of M such that $U_1, U_2, U_1 \cap U_2$ (if non-empty) are all diffeomorphic to $\mathbb{R}^n$. The idea of proving the Poincaré duality for $U_1 \cup U_2$ (i.e.,

$$\int_{U_1 \cup U_2} : \ H^r(U_1 \cup U_2) \to H_c^{n-r}(U_1 \cup U_2)^*$$

is an isomorphism for all r) is to apply the Five Lemma (see Problem 2 in Problem Sheet 3) to two long exact sequences: the classical Mayer-Vietoris sequence and the *dual* of the Mayer-Vietoris sequence with compact supports, which are connected by the homomorphisms defined via integration. But we need to prove several preliminary results to make the Five Lemma work.

The first one is about the exactness of the dual of (3.29).

Lemma 3.3. *Let*

$$V \xrightarrow{f} W \xrightarrow{g} Z$$

be an exact cochain complex of finite dimensional real vector spaces, then its dual

$$Z^* \xrightarrow{g^*} W^* \xrightarrow{f^*} V^*$$

is also an exact cochain complex, where $\cdot^*$ *means taking dual in the sense of real vector spaces and linear maps.*

Proof. It is obvious that $\mathrm{Im} g^* \subset \mathrm{Ker} f^*$. Conversely, if $w^* \in \mathrm{Ker} f^*$, we have

$$\langle f^* w^*, v\rangle = w^*(f(v)) = 0, \ \forall v \in V.$$

Since (3.3) is exact, we know that w^* annihilates $\mathrm{Ker} g$. It follows that

$$z_0^*(g(w)) = w^*(w), \ w \in W,$$

is a well-defined linear functional on $\mathrm{Im} g \subset Z$. Now take any linear extension z^* of z_0^* to the whole space Z, it follows that

$$\langle g^* z^*, w\rangle = z^*(g(w)) = z_0^*(g(w)) = w^*(w), \ \forall w \in W.$$

Therefore, $g^* z^* = w^* \in \mathrm{Im} g^*$. □

Remark 3.5. Lemma 3.3 holds for infinite dimensional real vector spaces. The proof then relies on Zorn's lemma which guarantees a linear extension of a linear map defined on a subspace to the whole space.

The second one is about the commutativity of the diagram.

Lemma 3.4. *The following three diagram commutes:*

(1)

$$\begin{array}{ccc} H^r(U_1 \cup U_2) & \xrightarrow{I^*} & H^r(U_1) \oplus H^r(U_2) \\ \Big\downarrow \int_{U_1 \cup U_2} & & \Big\downarrow \int_{U_1} \oplus \int_{U_2} \\ H_c^{n-r}(U_1 \cup U_2)^* & \xrightarrow{I'_*} & H_c^{n-r}(U_1)^* \oplus H_c^{n-r}(U_2)^*; \end{array}$$

(2)

$$\begin{array}{ccc} H^r(U_1) \oplus H^r(U_2) & \xrightarrow{J^*} & H^r(U_1 \cap U_2) \\ \Big\downarrow \int_{U_1} \oplus \int_{U_2} & & \Big\downarrow \int_{U_1 \cap U_2} \\ H_c^{n-r}(U_1)^* \oplus H_c^{n-r}(U_2)^* & \xrightarrow{J'_*} & H_c^{n-r}(U_1 \cap U_2)^*; \end{array}$$

(3)

$$\begin{array}{ccc} H^r(U_1\cap U_2) & \xrightarrow{\partial^*} & H^{r+1}(U_1\cup U_2) \\ \big\downarrow {\scriptstyle \int_{U_1\cap U_2}} & & \big\downarrow {\scriptstyle \int_{U_1\cup U_2}} \\ H_c^{n-r}(U_1\cap U_2)^* & \xrightarrow{(-1)^{r+1}\cdot\partial'_*} & H_c^{n-r-1}(U_1\cup U_2)^*. \end{array}$$

Here $\cdot'$ denotes the dual maps, and $\int_{U_1}\oplus\int_{U_2}$ is defined by

$$\langle\int_{U_1}\oplus\int_{U_2}([\omega],[\tau]),([\alpha],[\beta])\rangle=\int_{U_1}\omega\wedge\alpha+\int_{U_2}\tau\wedge\beta,$$

where $[\omega]\in H^r(U_1)$, $[\tau]\in H^r(U_2)$, $[\alpha]\in H_c^{n-r}(U_1)$ and $[\beta]\in H_c^{n-r}(U_2)$.

Remark 3.6. In the third diagram, there is a sign $(-1)^{r+1}$ on ∂_*', but this won't affect the exactness of the whole sequence.

Proof. (1) Let $[\xi]\in H^r(U_1\cup U_2)$ and $[\omega]\in H_c^{n-r}(U_1)$, $[\tau]\in H_c^{n-r}(U_2)$. It follows that

$$\langle\int_{U_1}\oplus\int_{U_2}(I^*[\xi]),([\omega],[\tau])\rangle=\int_{U_1}i_1^*\xi\wedge\omega+\int_{U_2}i_2^*\xi\wedge\tau.$$

On the other hand, we have

$$\begin{aligned}\langle I_*'\int_{U_1\cup U_2}([\xi]),([\omega],[\tau])\rangle &= \langle\int_{U_1\cup U_2}([\xi]),I_*([\omega],[\tau])\rangle\\ &= \int_{U_1\cup U_2}\xi\wedge(i_{1*}\omega+i_{2*}\tau)\\ &= \int_{U_1}i_1^*\xi\wedge\omega+\int_{U_2}i_2^*\xi\wedge\tau,\end{aligned}$$

since ω is compactly supported in U_1 and τ is compactly supported in U_2. Therefore, the first diagram commutes.

(2) The commutativity of the second diagram follows from a similar argument.

(3) Let $[\alpha]\in H^r(U_1\cap U_2)$ and $[\xi]\in H_c^{n-r-1}(U_1\cup U_2)$. It follows that

$$\langle\int_{U_1\cup U_2}(\partial^*[\alpha]),[\xi]\rangle=\int_{U_1\cup U_2}\partial^*\alpha\wedge\xi. \tag{3.30}$$

where by diagram chasing, $\partial^*\alpha$ is defined by

$$\partial^*\alpha=\begin{cases}d(\varphi_2\alpha), & \text{on } U_1;\\ -d(\varphi_1\alpha), & \text{on } U_2,\end{cases}$$

in which $\{\varphi_1, \varphi_2\}$ is a partition of unity subordinate to $\{U_1, U_2\}$ with the same index on $U_1 \cup U_2$. Therefore, (3.30) is equal to

$$\begin{aligned}
&\int_{U_1 \cup U_2} \partial^* \alpha \wedge (\varphi_1 \xi + \varphi_2 \xi) \\
&= \int_{U_1} \partial^* \alpha \wedge (\varphi_1 \xi) + \int_{U_2} \partial^* \alpha \wedge (\varphi_2 \xi) \\
&= \int_{U_1} d(\varphi_2 \alpha) \wedge (\varphi_1 \xi) - \int_{U_2} d(\varphi_1 \alpha) \wedge (\varphi_2 \xi), \qquad (3.31)
\end{aligned}$$

since $\varphi_i \xi$ is compactly supported in U_i $(i = 1, 2)$. It follows from Stokes' theorem that

$$\int_{U_1} d(\varphi_2 \alpha) \wedge (\varphi_1 \xi) = (-1)^{r+1} \int_{U_1} (\varphi_2 \alpha) \wedge d(\varphi_1 \xi),$$

and

$$\int_{U_2} d(\varphi_1 \alpha) \wedge (\varphi_2 \xi) = (-1)^{r+1} \int_{U_2} (\varphi_1 \alpha) \wedge d(\varphi_2 \xi).$$

Therefore, (3.31) is equal to

$$(-1)^{r+1} (\int_{U_1} (\varphi_2 \alpha) \wedge d(\varphi_1 \xi) - \int_{U_2} (\varphi_1 \alpha) \wedge d(\varphi_2 \xi)).$$

On the other hand, by the description of ∂_* we know that $d(\varphi_1 \xi) = -d(\varphi_2 \xi)$ is compactly supported in $U_1 \cap U_2$. It follows that (3.31) is equal to

$$(-1)^{r+1} \int_{U_1 \cap U_2} \alpha \wedge d(\varphi_1 \xi),$$

which is exactly the same as

$$\langle (-1)^{r+1} \partial'_* \int_{U_1 \cap U_2} ([\alpha]), [\xi] \rangle = (-1)^{r+1} \int_{U_1 \cap U_2} \alpha \wedge \partial_* \xi.$$

Consequently, the third diagram commutes. □

The third one is the about isomorphisms on the vertical direction.

Lemma 3.5. *The Poincaré duality holds for U_1, U_2 and $U_1 \cap U_2$.*

Proof. We first compute the de Rham cohomology of $\mathbb{R}^n$ with compact supports.

We know from the Poincaré lemma and Problem 1 in Problem Sheet 3 that

$$H^r(\mathbb{R}^n) \cong H_c^{n-r}(\mathbb{R}^n)^* = \begin{cases} \mathbb{R}, & r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since U_1, U_2 and $U_1 \cap U_2$ are all diffeomorphic to $\mathbb{R}^n$, it suffices to show that $\int_{U_1}, \int_{U_2}$ and $\int_{U_1 \cap U_2}$ are all non-trivial homomorphisms when acting on the 0-th de Rham cohomology groups. But this is obvious since by using a bump function we can always construct an n-form with compact support inside any given open set whose integral is nonzero. □

As we've pointed out in Remark 3.6, a sign change on the map ∂'_* does not affect the exactness of the dual of the Mayer-Vietoris sequence with compact supports. Therefore, the previous three lemmas enable us to apply the Five Lemma to conclude that:

Proposition 3.13. *If U_1, U_2 and $U_1 \cap U_2$ are all diffeomorphic to $\mathbb{R}^n$, then the Poincaré duality holds for $U_1 \cup U_2$.*

Of course the previous argument is just a special case of a general induction which is used to prove the Poincaré duality for an oriented manifold with a finite good cover.

Theorem 3.12. *If M is an oriented manifold of dimension n with a finite good cover, then Poincaré duality holds for M.*

Proof. We prove by induction on the number of open subsets in a finite good cover of M.

If M is diffeomorphic to $\mathbb{R}^n$, then the result follows from Lemma 3.5.

Assume that the Poincaré duality holds for any oriented manifold of dimension n with a finite good cover containing at most k open subsets. Let $\{U_1, \cdots, U_{k+1}\}$ be a finite good cover of M containing $k+1$ open subsets. Then by induction hypothesis, the Poincaré duality holds for $U = U_1 \cup \cdots \cup U_k$, $V = U_{k+1}$, and

$$U \cap V = (U_1 \cap U_{k+1}) \cup \cdots \cup (U_k \cap U_{k+1}).$$

By applying the Five Lemma as before for the two long exact sequences for U, V, we conclude that the Poincaré duality holds for M.

Now the proof is complete. □

In particular, when M is compact, then the Poincaré duality takes the form

$$\textstyle\int_M : \; H^r(M) \xrightarrow{\cong} H^{n-r}(M)^* \; , \; \forall 0 \leqslant r \leqslant n.$$

Remark 3.7. The Poincaré duality implies Theorem 3.4 and Corollary 3.1. In fact, if M is connected, then $H^0(M) \cong \mathbb{R}$. By the Poincaré duality, $H^n_c(M)^* \cong \mathbb{R}$ and hence $H^n_c(M)$ is 1-dimensional. Moreover, $\int_M(1)$ is a non-trivial element in $H^n_c(M)^*$ where "1" is the constant function with value 1. But $\int_M(1)$ is exactly the linear map given by (3.9). Therefore,

$$\int_M : \; H^n_c(M) \to \mathbb{R}$$

is a non-trivial linear map and by comparing dimensions it has to be an isomorphism.

3.6 The Künneth formula

Secondly, we are going to establish a way of computing the de Rham cohomology of product manifolds. This is known as the Künneth formula.

Let M, F be two manifolds. Here we also assume that M has a finite good cover. This is necessary in applying the induction argument based on the Mayer-Vietoris sequence as before.

Consider the natural projections $\pi: M \times F \to M$ and $\rho: M \times F \to F$. For differential forms $\omega \in \Omega(M),\ \theta \in \Omega(F)$, define

$$\psi(\omega \otimes \theta) = \pi^*(\omega) \wedge \rho^*(\omega) \in \Omega(M \times F).$$

It is easy to see that ψ induces a linear map

$$\psi: \oplus_{r=0}^{n} H^r(M) \otimes H^{n-r}(F) \to H^n(M \times F)$$

for all $n \geqslant 0$.

The *Künneth* formula is the following.

Theorem 3.13. *For manifolds M and F, where M has a finite good cover, ψ is a linear isomorphism for all $n \geqslant 0$.*

Proof. Similar to the proof of the Poincaré duality, we try to use induction based on the Mayer-Vietoris sequence.

If M is diffeomorphic to $\mathbb{R}^n$, it follows directly from Theorem 3.2 and Example 3.1 that ψ is an isomorphism.

Now assume that the theorem holds for all manifolds with a finite good cover consisting of at most k open subsets, and let M be a manifold with a finite good cover $\{U_1, \cdots, U_k, U_{k+1}\}$.

Let

$$U = \cup_{i=1}^{k} U_i,\ V = U_{k+1}.$$

Based on the Mayer-Vietoris sequence (3.22) for U, V, we have the following cochain complex

$$\cdots \xrightarrow{\widetilde{\partial}^*} \oplus_{r=0}^{n} H^r(U \cup V) \otimes H^{n-r}(F) \tag{3.32}$$

$$\xrightarrow{\widetilde{I}^*} \oplus_{r=0}^{n} (H^r(U) \otimes H^{n-r}(F) \oplus H^r(V) \otimes H^{n-r}(F))$$

$$\xrightarrow{\widetilde{J}^*} \oplus_{r=0}^{n} H^r(U \cap V) \otimes H^{n-r}(F)$$

$$\xrightarrow{\widetilde{\partial}^*} \oplus_{r=0}^{n+1} H^r(U \cup V) \otimes H^{n+1-r}(F) \xrightarrow{\widetilde{I}^*} \cdots.$$

Here the coboundary maps $\widetilde{I}^*, \widetilde{J}^*$ and $\widetilde{\partial}^*$ are defined canonically by tensoring I^*, J^* and ∂^* in (3.22) with the identity map on $H^{n-r}(F)$ and then taking direct sum. But we should be careful that when defining $\widetilde{\partial}^*$, $\oplus_{r=0}^{n+1} H^r(U \cup V) \otimes H^{n-r}(F)$ should be viewed as

$$H^0(U \cup V) \otimes H^{n+1}(F) \oplus (\oplus_{r=0}^{n} H^{r+1}(U \cup V) \otimes H^{n-r}(F))$$

and $\oplus_{r=0}^{n} \partial^* \otimes \mathrm{id}_{H^{n-r}(F)}$ takes value in the second component. It is not hard to see that (3.32) is a long exact sequence.

The linear map ψ, induces a chain of linear maps between the long exact sequence (3.32) and the Mayer-Vietoris sequence for $U \times F, V \times F$ on $M \times F$. Since we can write

$$U \cap V = (U_1 \cap U_{k+1}) \cup \cdots \cup (U_k \cap U_{k+1}),$$

by the induction hypothesis, we know that

$$\psi : \otimes_{r=0}^{n} H^r(U \cap V) \otimes H^{n-r}(F) \to H^n((U \cap V) \times F)$$

and

$$\psi : \oplus_{r=0}^{n} (H^r(U) \otimes H^{n-r}(F) \oplus H^r(V) \otimes H^{n-r}(F)) \to H^n(U \times F) \oplus H^n(V \times F)$$

are isomorphisms. The induction step will be completed by the Five lemma once we show the commutativity of the whole diagram.

The only non-trivial part is the commutativity of the diagram:

$$\begin{array}{ccc} \oplus_{r=0}^{n} H^r(U \cap V) \otimes H^{n-r}(F) & \xrightarrow{\widetilde{\partial}^*} & \oplus_{r=0}^{n} H^r(U \cap V) \otimes H^{n-r}(F) \\ \downarrow \psi & & \downarrow \psi \\ H^n((U \cap V) \times F) & \xrightarrow{\partial^*} & H^n((U \cup V) \times F). \end{array} \qquad (3.33)$$

In fact, let $\{\varphi_1, \varphi_2\}$ be a partition of unity subordinate to $\{U, V\}$ on $U \cup V$. For $[\alpha] \in H^r(U \cap V)$ and $[\theta] \in H^{n-r}(F)$, by the description of ∂^*, we know that $\widetilde{\partial}^*([\alpha] \otimes [\theta])$ is the cohomology class on $(U \cup V) \times F$ determined by $\pi^* \partial^* \xi \wedge \rho^* \theta$, where

$$\xi = \begin{cases} d(\rho_V \alpha), & \text{on } U; \\ -d(\rho_U \alpha), & \text{on V}, \end{cases}$$

which is well-defined on $U \cup V$. On the other hand, we have

$$\begin{aligned} \partial^* \psi(\alpha \otimes \theta) &= \partial^*(\pi^* \alpha \wedge \rho^* \theta) \\ &= \begin{cases} \pi^* d(\rho_V \alpha) \wedge \rho^* \theta, & \text{on } U \times F; \\ -\pi^* d(\rho_U \alpha) \wedge \rho^* \theta, & \text{on } V \times F. \end{cases} \\ &= \pi^* \xi \wedge \rho^* \theta. \end{aligned}$$

Note that here we've used the fact that $\{\pi^* \varphi_U, \pi^* \varphi_V\}$ is a partition of unity subordinate to $\{U \times F, V \times F\}$ on $(U \cup V) \times F$, and θ is closed. Therefore, the diagram (3.33) commutes, which concludes the proof of the theorem. □

Remark 3.8. The Künneth formula may not hold for arbitrary manifolds, and we do need some kind of finiteness assumption. In fact, it holds under the assumption that the de Rham cohomology groups of M are finite dimensional.

By using similar Mayer-Vietoris arguments, it is possible to establish the Künneth formula for de Rham cohomology with compact supports. More precisely, if M, N are manifolds with finite good cover, then

$$\psi: \oplus_{r=0}^{n} H_c^r(M) \otimes H_c^{n-r}(N) \to H_c^n(M \times N)$$

is a linear isomorphism for all $n \geqslant 0$. We left the proof as an exercise.

Remark 3.9. It should be pointed out that, unlike the case of the de Rham cohomology, the Künneth formula for de Rham cohomology with compact supports is true for all manifolds. We need to strengthen the induction arguments to prove this, as we should for proving the Poincaré duality without assuming that the manifold has a finite good cover.

We can think of $H^*(M) \otimes H^*(F)$ as a graded real vector space by setting

$$(H^*(M) \otimes H^*(F))^n = \oplus_{r=0}^{n} H^r(M) \otimes H^{n-r}(F), \ n \geqslant 0,$$

and introduce an algebra structure on $H^*(M) \otimes H^*(F)$ by setting

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{rs}(a \cdot c) \otimes (b \cdot d)$$

for $b \in H^r(F)$ and $c \in H^s(M)$. In this way, $H^*(M) \otimes H^*(F)$ becomes a graded algebra and

$$\psi: \ H^*(M) \otimes H^*(F) \to H^*(M \times F)$$

becomes an algebra isomorphism. The same conclusion applies to the de Rham cohomology with compact supports.

Example 3.6. Let M be the n-torus $T^n = S^1 \times \cdots \times S^1$. The Künneth formula shows that

$$\dim H^r(M) = \begin{pmatrix} n \\ r \end{pmatrix}, \ \forall 0 \leqslant r \leqslant n.$$

3.7 The Thom isomorphism

The last part of the present notes will be devoted to the study of the cohomology of real vector bundles. As in the Poincaré duality and the Künneth formula, it is also based on the Mayer-Vietoris argument.

We first introduce some basic notions about real vector bundles.

Definition 3.14. Let $\pi: \ E \to M$ be a surjective C^∞ map between manifolds such that $\pi^{-1}(p)$ is a real vector space for all $p \in M$. We call (E, M, π) a *real vector bundle of rank n* if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of M together with diffeomorphisms $\{\phi_\alpha: \ \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n\}_{\alpha \in A}$ such that each ϕ_α restricts to a linear isomorphism

$$\phi_\alpha|_{\pi^{-1}(p)}: \ \pi^{-1}(p) \to \{(p,t): \ t \in \mathbb{R}^n\}$$

for each $p \in U_\alpha$. $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is called a *local trivialization of* (E, M, π).

For a real vector bundle (E, M, π), E is called the *total space*, M is called the *base space*. For each $p \in M$, the real vector space $\pi^{-1}(p)$ is called the *fiber at* p. A *section of* (E, M, π) is a C^∞ map $s: M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

Example 3.7. The simplest example of real vector bundles is the product manifold $M \times \mathbb{R}^n$. For each non-zero vector in $\mathbb{R}^n$,

$$s_v(p) = (p, v), \ p \in M,$$

defines a non-vanishing section of the bundle.

Example 3.8. A non-trivial example we've met before is the tangent bundle TM. We can take an atlas as the open cover and define ϕ_α in a natural way under the natural basis in a coordinate chart U_α. A section of TM is just a smooth vector field on M. As we've seen in the Hairy Ball Theorem, the tangent bundle over the n-sphere has a non-vanishing section if and only if n is odd.

More generally, tensor bundles and exterior bundles are common examples of real vector bundles.

A fundamental concept about real vector bundles is transition functions.

Definition 3.15. Let (E, M, π) be a real vector bundle of rank n equipped with a local trivialization $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$. Then for any $\alpha, \beta \in A$, the map

$$\phi_\beta \circ \phi_\alpha^{-1}: \ (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

restricts to a linear automorphism on $\{(p, t): \ t \in \mathbb{R}^n\} \cong \mathbb{R}^n$ for each $p \in U_\alpha \cap U_\beta$ (if not empty). It gives rise to a C^∞ map

$$g_{\alpha\beta}: \ U_\alpha \cap U_\beta \to GL(n; \mathbb{R})$$

if we identify the linear automorphism group on $\mathbb{R}^n$ as the general linear group $GL(n; \mathbb{R})$ of order n. The family $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ of C^∞ maps is called *transition functions of* (E, M, π).

Example 3.9. For the tangent bundle TM, when we fix a local trivialization $\{(U_\alpha, \phi_\alpha)\}$ using an atlas, the transition functions are given by the Jacobians of change of coordinates.

The transition functions $\{g_{\alpha,\beta}\}_{\alpha,\beta \in A}$ of a real vector bundle (E, M, π) of rank n satisfy the following compatibility conditions:

(1) $g_{\alpha\alpha}(p) = I$ for all $p \in U_\alpha$;

(2) $g_{\alpha\beta}(p) \cdot g_{\beta\gamma}(p) = g_{\alpha\gamma}(p)$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$ (if not empty).

It is a fundamental result that if we are given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M together with a family $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ of C^∞ functions satisfying (1) and (2), then we can always construct a real vector bundle (E, M, π) with transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$. We are not going to prove this result here.

Lemma 3.6. *Let $\{(U_\alpha, \phi'_\alpha)\}_{\alpha \in A}$ be another local trivialization of (E, M, π) with the same open cover, and $\{g'_{\alpha\beta}\}_{\alpha,\beta \in A}$ be the associated transition functions. Then for each $\alpha \in A$, there exists some C^∞ map $\lambda_\alpha : U_\alpha \to GL(n;\mathbb{R})$, such that*

$$g'_{\alpha\beta} = \lambda_\beta \cdot g_{\alpha\beta} \cdot \lambda_\alpha^{-1}, \text{ on } U_\alpha \cap U_\beta.$$

Proof. For each $\alpha \in A$, let $\lambda_\alpha : U_\alpha \to GL(n;\mathbb{R})$ defined by $\phi'_\alpha \circ \phi_\alpha^{-1}$. It follows that

$$\begin{aligned} g'_{\alpha\beta} &= \phi'_\beta \circ (\phi'_\alpha)^{-1} \\ &= \lambda_\beta \cdot \phi_\beta \circ \phi_\alpha^{-1} \circ \lambda_\alpha^{-1} \\ &= \lambda_\beta \cdot g_{\alpha\beta} \cdot \lambda_\alpha^{-1}. \end{aligned}$$

□

Definition 3.16. Two family of transition functions related in the way of Lemma 3.6 are said to be *equivalent*.

Definition 3.17. Let (E, M, π) be a vector bundle of rank n with transition functions $\{g_{\alpha\beta}\}$. If we can find an equivalent family of transition functions $\{g'_{\alpha\beta}\}$ which take values in a subgroup H of $GL(n;\mathbb{R})$, then we say that *the structure group of (E, M, π) may be reduced to H*. A vector bundle is *orientable* if its structure group can be reduced to $GL^+(n;\mathbb{R})$, the group of invertible matrices with positive determinant.

By using the partition of unity, it is not hard to show that (left as exercise) the structure group any vector bundle can always be reduced to the orthogonal group. Therefore, a vector bundle is orientable if and only if its structure group can be reduced to the special orthogonal group, the group of orthogonal matrices with positive determinant.

The notion of orientability as a vector bundle is different from the notion of orientability as a manifold. However, we have the following result.

Proposition 3.14. *Let (E, M, π) be an orientable vector bundle of rank n over an m-dimensional orientable manifold. Then the total space E is orientable as a manifold.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a local trivialization with transition functions $\{g_{\alpha\beta}\}$ valued in $GL^+(n;\mathbb{R})$ such that $\{U_\alpha\}$ is an orientation compatible atlas on M with associated chart maps $\{\varphi_\alpha\}$. It follows that the collection of

$$\begin{aligned} \Phi_\alpha : \ \varphi_\alpha(U_\alpha) \times \mathbb{R}^n &\to \pi^{-1}(U_\alpha), \\ (x, t) &\mapsto \phi_\alpha^{-1}(\varphi_\alpha^{-1}(x), t), \end{aligned}$$

defines an atlas on E. Moreover, the change of coordinates for this atlas is given by

$$(x, s) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(x), g_{\alpha\beta}(\varphi_\alpha^{-1}(x)) \cdot s),$$

which has Jacobian

$$\begin{pmatrix} \frac{\partial \varphi_\beta \circ \varphi_\alpha^{-1}}{\partial x} & * \\ 0 & g_{\alpha\beta}(\varphi_\alpha^{-1}(x)) \end{pmatrix}.$$

By our choice we know that the determinant of this matrix is positive. Therefore, $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ is an orientation compatible atlas on E, and thus E is orientable as a manifold. □

Example 3.10. From the computation in the proof of Proposition 3.14, it is not hard to see that the tangent bundle TM over any manifold M is always orientable as a manifold. However, TM is orientable as a vector bundle if and only if M is orientable.

If we have specified the transition functions of an orientable vector bundle taking values in $GL^+(n;\mathbb{R})$, we say that the vector bundle is *oriented*.

We can talk about maps between real vector bundles.

Definition 3.18. A *homomorphism* between two vector bundles (E_1, M_1, π_1) and (E_2, M_2, π_2) is a pair (F, f) of C^∞ maps

$$F : \ E_1 \to E_2, \ f : \ M_1 \to M_2$$

such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2, \end{array}$$

and F restricts to a linear map between fibers $\pi_1^{-1}(p)$ and $\pi_2^{-1}(f(p))$ for all $p \in M_1$. A vector bundle homomorphism is an *isomorphism* if it has an inverse also being a vector bundle homomorphism.

Note that for a vector bundle homomorphism (F, f), by the surjectivity of the covering map, f is uniquely determined by F.

More frequently we shall consider vector bundle homomorphisms over a fixed base space.

Definition 3.19. A *homomorphism* between two vector bundles (E_1, M, π_1) and (E_2, M, π_2) *over the base space* M is a vector bundle homomorphism (F, id_M). It is an *isomorphism* if it has an inverse also being a vector bundle homomorphism over the base space M.

Definition 3.20. A vector bundle (E, M, π) is said to be *trivial* if it is isomorphic to the trivial bundle $(M \times \mathbb{R}^n, M, \pi')$ over the base space M.

It is easy to see that a vector bundle is trivial if and only if it has transition functions taking values only at the identity matrix.

In general it is hard to see whether a vector bundle is trivial or not.

The following characterization of triviality is very useful. The proof of this result is beyond the scope of the notes.

Proposition 3.15. *If M is contractible, then any vector bundle over M is trivial.*

On the other hand, to show that a vector bundle is non-trivial, it suffices to show that it does not have a non-vanishing section, since a trivial vector bundle always does.

For example, the tangent bundle of the n-sphere is non-trivial when n is even.

Another example is called the *canonical line bundle over* $\mathbb{R}P^n$. For each $[x] \in \mathbb{R}P^n$, let $L_{[x]}$ be the one dimensional vector space in $\mathbb{R}^{n+1}$ determined by the straight line $[x]$. Then the set of all $L_{[x]}$ forms a vector bundle over $\mathbb{R}P^n$ of rank 1 in a natural way (one may use the projection map $q: S^n \to \mathbb{R}P^n$, which is a local diffeomorphism, to construct a local trivialization). We use γ_n^1 to denote this vector bundle.

Assume that $s: \mathbb{R}P^n \to \gamma_n^1$ is a section of γ_n^1 and consider

$$s \circ q: S^n \to \gamma_n^1.$$

By definition, for each $x \in S^n$, there exists a unique real number $t(x)$ such that $s([x]) = t(x) \cdot x$. It is easy to see that t defines a continuous function on S^n such that

$$t(-x) = -t(x), \ \forall x \in S^n.$$

By the connectedness of S^n, t vanishes at some point. In other words, γ_n^1 does not have a non-vanishing section. Therefore, γ_n^1 is non-trivial.

Now we come to the study of cohomology of a vector bundle. Let (E, M, π) be a vector bundle of rank n.

From homotopy invariance it is trivial to see that the de Rham cohomology of the total space E is isomorphic to the de Rham cohomology of the base space M.

For the de Rham cohomology with compact supports, we have the following simple result.

Proposition 3.16. *If E and M are orientable manifolds with finite good cover, then*

$$H_c^r(E) \cong H_c^{r-n}(M)$$

for all r.

Proof. It follows from Proposition 3.12 and the Poincaré duality that

$$\begin{aligned} H_c^r(E) &\cong (H^{m+n-r}(E))^* \\ &\cong (H^{m+n-r}(M))^* \\ &\cong H_c^{r-n}(M), \end{aligned}$$

for all r, where $m = \dim M$. □

Remark 3.10. Proposition 3.16 may fail in general. One can think about the example of the open Möbius band, which is a non-orientable vector bundle over S^1 of rank 1.

Now we are going to introduce a third type of cohomology which is more natural than the previous two in the study of vector bundles.

Let $(\Omega^*_{cv}(E), d)$ be the cochain complex of differential forms ω on E such that for any compact set K on M, $\pi^{-1}(K) \cap \mathrm{supp}\omega$ is compact in E. The cohomology of $(\Omega^*_{cv}(E), d)$ is called the *de Rham cohomology with compact supports in the vertical direction*, and it is denoted by $H^*_{cv}(E)$. Obviously $H^*_{cv}(E)$ is invariant under vector bundle isomorphism.

From now on, assume further that (E, M, π) is orientable.

Similar to the case of de Rham cohomology with compact supports, we define a linear operator π_* called *integration along fiber* in the following way.

Take a local trivialization $\{(U_\alpha, \phi_\alpha)\}$ with transition functions $\{g_{\alpha\beta}\}$ valued in $GL^+(n; \mathbb{R})$, such that $\{U_\alpha\}$ is an atlas on M. Let ω be a differential form with compact support in the vertical direction. On $U_\alpha \times \mathbb{R}^n$ it can be written as

$$\omega = \sum_{I,|J|<n} a_{I,J}(x,s) dx^I \wedge ds^J + \sum_I a_{I,J=(1,\cdots,n)}(x,s) dx^I \wedge ds^1 \wedge \cdots \wedge ds^n,$$

where the summation is over all multi-indices $I = (i_1 < \cdots < i_p)$ and $J = (j_1 < \cdots < j_q)$, $|J|$ is the cardinality of the multi-index J, and

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \ ds^J = ds^{j_1} \wedge \cdots \wedge ds^{j_q}.$$

We then define

$$\pi^\alpha_*(\omega) = \sum_I \left(\int_{\mathbb{R}^n} a_{I,J=(1,\cdots,n)}(x,s) ds^1 \cdots ds^n\right) dx^I, \tag{3.34}$$

which is a differential form on $U_\alpha \subset M$ with degree decreased by n. On another chart $U_\beta \times \mathbb{R}^n$, write

$$\omega = \sum_{I,|J|<n} b_{I,J}(y,t) dy^I \wedge dt^J + \sum_I b_{I,J=(1,\cdots,n)}(y,t) dy^I \wedge dt^1 \wedge \cdots \wedge dt^n,$$

then by definition

$$\pi^\beta_*(\omega) = \sum_I \left(\int_{\mathbb{R}^n} b_{I,J=(1,\cdots,n)}(y,t) dt^1 \cdots dt^n\right) dy^I.$$

If we let $y = y(x)$ and $t = g_{\alpha\beta}(x) \cdot s$ (change of coordinates from $\pi^{-1}(U_\alpha)$ to $\pi^{-1}(U_\beta)$), since $g_{\alpha\beta}(x) \in GL^+(n; \mathbb{R})$, the change of variables formula for Lebesgue integrals implies that

$$\pi^\beta_*(\omega) = \sum_I \left(\int_{\mathbb{R}^n} b_{I,J=(1,\cdots,n)}(y(x), g_{\alpha\beta}(x) \cdot s) \cdot \det g_{\alpha\beta}(x) \cdot ds^1 \cdots ds^n\right) dy^I(x).$$

On the other hand, it is not hard to see that

$$\sum_I a_{I,J=(1,\cdots,n)}(x,s) dx^I = \sum_I b_{I,J=(1,\cdots,n)}(y(x), g_{\alpha\beta}(x) \cdot s) \cdot \det g_{\alpha\beta}(x) \cdot dy^I(x).$$

Therefore,

$$\pi_*^\alpha(\omega) = \pi_*^\beta(\omega)$$

on $U_\alpha \cap U_\beta$. In other words, (3.34) defines a global linear operator $\pi_* : \Omega_{cv}^r(E) \to \Omega^{r-n}(M)$ for each r.

Remark 3.11. π_* depends on how the vector bundle is oriented (i.e., the choice of local trivialization whose transition functions take values in $GL^+(n;\mathbb{R})$). Here we always fix an orientation.

The linear operator π_* satisfies the following important property called the *projection formula*.

Proposition 3.17. *Let τ be a differential form on M and ω be a differential form on E with compact supports in the vertical direction. Then*

$$\pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega. \tag{3.35}$$

Proof. It suffices to prove the result locally. Take $\{(U_\alpha, \phi_\alpha)\}$ as before. Without loss of generality, we can assume

$$\tau = a(x)dx^I,\ \omega = b(x,t)dx^J \wedge dt^K,$$

where I, J, K are multi-indices and b is compactly supported in the vertical direction.

If $|K| < n$, then

$$\pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega = 0.$$

If $K = (1, \cdots, n)$, then

$$\begin{aligned} \pi_*(\pi^*\tau \wedge \omega) &= (\int_{\mathbb{R}^n} a(x)b(x,t)dt^1 \cdots dt^n)dx^I \wedge dx^J \\ &= a(x)dx^I \wedge (\int_{\mathbb{R}^n} b(x,t)dt^1 \cdots dt^n)dx^J \\ &= \tau \wedge \pi_*\omega. \end{aligned}$$

Therefore, (3.35) holds. □

Moreover, by definition and straight forward calculation it is easy to see that π_* commutes with d. Therefore, π_* induces a linear map on cohomology. The *Thom isomorphism* asserts the following.

Theorem 3.14. *Let (E, M, π) be an oriented vector bundle of rank n and M has a finite good cover. Then the induced map $\pi_* : H_{cv}^r(E) \to H^{r-n}(M)$ on cohomology is a linear isomorphism for all r.*

Proof. As before, we prove by induction on the cardinality of the finite good cover.

First assume that M is diffeomorphic to $\mathbb{R}^m$. By Proposition 3.15, the vector bundle is trivial, and hence we may assume that $E = \mathbb{R}^m \times \mathbb{R}^n$. In this case, it follows from the same proof as Proposition 3.9 that π_* can be written as the composition of isomorphisms

$$H^r_{cv}(\mathbb{R}^m \times \mathbb{R}^n) \to H^{r-1}_{cv}(\mathbb{R}^m \times \mathbb{R}^{n-1}) \to \cdots \to H^{r-n+1}_{cv}(\mathbb{R}^m \times \mathbb{R}^1) \to H^{r-n}(\mathbb{R}^m).$$

Therefore π_* is an isomorphism.

Now assume that the theorem holds for all manifolds with a finite good cover containing at most k open sets, and let M be a manifold with a finite good cover $\{U_1, \cdots, U_k, U_{k+1}\}$. Let

$$U = U_1 \cup \cdots \cup U_k, \ V = U_{k+1}.$$

Then we have the Mayer-Vietoris sequence for U, V on M. On the other hand, since the restriction of a differential form on E with compact support in the vertical direction on any open subset $\pi^{-1}(U)$ also has compact support in the vertical direction, it follows from the same argument as in the case of the de Rham complex that we have the short exact sequence

$$0 \longrightarrow \Omega^*_{cv}(\pi^{-1}(U \cup V)) \xrightarrow{I^*} \Omega^*_{cv}(\pi^{-1}(U)) \oplus \Omega^*_{cv}(\pi^{-1}(V))$$

$$\xrightarrow{J^*} \Omega^*_{cv}(\pi^{-1}(U \cap V) \longrightarrow 0$$

of cochain complexes, which induces a long exact sequence on the de Rham cohomology with compact supports in the vertical direction.

The linear map π_*, induces a chain of linear maps between this long exact sequence and the Mayer-Vietoris sequence for U, V on M. To conclude the induction by using the Five Lemma, it suffices to show the commutativity of the whole diagram.

The only non-trivial part is the commutativity of

$$\begin{array}{ccc} H^r_{cv}(\pi^{-1}(U \cap V)) & \xrightarrow{\partial^*} & H^{r+1}_{cv}(\pi^{-1}(U \cup V)) \\ \downarrow \pi_* & & \downarrow \pi_* \\ H^{r-n}(U \cap V) & \xrightarrow{\partial^*} & H^{r+1-n}(U \cup V). \end{array} \tag{3.36}$$

Let $\{\varphi_1, \varphi_2\}$ be a partition of unity subordinate to $\{U, V\}$ on $U \cup V$. It follows that $\{\pi^*\varphi_1, \pi^*\varphi_2\}$ is a partition of unity subordinate to $\{\pi^{-1}(U), \pi^{-1}(V)\}$ on $\pi^{-1}(U \cup V)$. Let $[\alpha] \in H^r_{cv}(\pi^{-1}(U \cap V))$, then on U we have

$$\begin{aligned} \pi_*\partial^*\alpha &= \pi_* d(\pi^*\varphi_2 \cdot \alpha) \\ &= \pi_*(\pi^*(d\varphi_2) \wedge \alpha). \end{aligned}$$

By the projection formula (3.35), on U we have

$$\begin{aligned} \pi_*\partial^*\alpha &= (d\varphi_2) \wedge \pi_*\alpha \\ &= d(\varphi_2 \cdot \pi_*\alpha) \\ &= \partial^*\pi_*\alpha. \end{aligned}$$

Similar result holds on V. Therefore, the diagram 3.36 commutes, which completes the proof of the theorem. □

Remark 3.12. The Thom isomorphism holds without the assumption that the base manifold has a finite good cover.

Let $\mathcal{T}: H^*(M) \to H^{*+n}_{cv}(E)$ be the inverse of π_*. Since constant functions are closed forms, $\Phi = \mathcal{T}([1])$ is a cohomology class of degree n in $H^*_{cv}(E)$. This is called the *Thom class*. Once we've known the Thom class, the Thom isomorphism $\mathcal{T}$ is given explicitly by

$$\mathcal{T}([\omega]) = [\pi^*\omega] \cdot \Phi. \tag{3.37}$$

In fact, if we take a representative $\phi \in \Phi$, then by the projection formula (3.35), we have

$$\pi_*(\pi^*\omega \wedge \phi) = \omega \wedge \pi_*\phi = \omega,$$

and (3.37) follows.

The computation of the Thom class is a fundamental problem in the theory of vector bundles, and we refer the readers to [1] for more stories and related topics.

References

[1] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology.

[2] S.S. Chern, W.H. Chen, K.S. Lam, Lectures on Differential Geometry.

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[4] I.M. Singer, J.A. Thorpe, Lecture Notes on Elementary Topology and Geometry.

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