The Uniqueness of Signature Problem in the Non-Markov Setting

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Abstract

We establish a general framework for a class of multidimensional stochastic processes over $[0, 1]$ under which with probability one, the signature (the collection of iterated path integrals in the sense of rough paths) is well-defined and determines the sample paths of the process up to reparametrization. In particular, by using the Malliavin calculus we show that our method applies to a class of Gaussian processes including fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge.

1 Introduction

The set of continuous paths forms a semigroup with involution, with the group operation and involution given by the concatenation and reversal of paths. In as early as 1954 K.T. Chen [5] observed that the map sending a bounded variation path $x : [0, 1] \to \mathbb{R}^d$ to the formal series

$$1 + \int_0^1 dx_s^i X_i + \int_0^1 \int_0^{s_2} dx_{s_1}^i dx_{s_2}^j X_i X_j + \dots,$$

where X_i $(i = 1, \cdots, d)$ are indeterminates and x^i denote the i-th coordinate component of x, is a homomorphism from the semigroup of paths to the algebra of non-commutative formal power series. Unfortunately, this map is not injective. The homomorphism property of the map implies that any path concatenated with its reversal will be mapped to the trivial formal series. It seems however that the map is essentially injective if we restrict our attention to paths that "do not track back along itself". Indeed, Chen himself [6] proved that the map is injective on the space of regular, irreducible paths. In [16], B. Hambly and T. Lyons extended Chen's result to the space of paths with bounded variation and introduced the notion of *tree-like paths* to describe paths that track

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back along itself. In particular, they proved that the formal series corresponding to a path, which they called the signature of a path, is trivial if and only if the path is tree-like.

Aside from its interesting algebraic properties, the map also gains attention through the fundamental role it plays in rough path theory. In [26], L.C. Young defined the Stieltjes type integral $\int_0^1 y_t dx_t$ in terms of a Riemann sum when $x.$ and $y.$ have finite p and q-variation respectively, where $\frac{1}{p} + \frac{1}{q} > 1$. In particular, it allows us to define, for a Lipschitz one form ϕ, the integral $\int_0^1 \phi(x_t) dx_t$ when x is a multidimensional path with finite p-variation for $p < 2$. In the same paper, Young gave an example where the integral $\int_0^1 \phi(x_t) dx_t$ defined using Riemann sum would diverge when x has only finite 2-variation. In other words, the Stieltjes integration map $x \to \int_0^1 \phi(x_t) dx_t$ does not have a closable graph with respect to p-variation if $p \geqslant 2$. The seemingly insurmountable $p = 2$ barrier, at least in the deterministic setting, is to remain for another sixty years. In [19], T. Lyons showed that the Stieltjes integration map would have a closable graph with respect to the p-variation metric if the path x takes value in a step-$\lfloor p \rfloor$ nilpotent Lie group. He called these paths *weakly geometric p-rough paths*. The first step in the construction of such integral is to define the signature for weakly geometric p-rough paths. The integration of one forms against such paths is then defined via polynomial approximations. There has been excellent progress in extending the rough path theory to even more general paths in, for example, the work of M. Gubinelli [14], M. Hairer and D. Kelly [15], etc.

From a theoretical standpoint, once Lyons defined the signature for weakly geometric rough paths, it is natural to ask if the signature of a weakly geometric rough path determines the path uniquely up to tree-like equivalence as in the bounded variation case. From a practical point of view, there has also been works done in, for example, D. Levin, T. Lyons and H. Ni [18] on analyzing time series data using the signature map. The justification of their method implicitly used the fact that the map from a path to its signature is injective in some sense. The solution of this long standing open problem in rough path theory is contained in the very recent work by H. Boedihardjo, X. Geng, T. Lyons and D. Yang [3].

There has also been exciting progress of the problem in the probabilistic setting. In [17], Y. Le Jan and Z. Qian proved that with probability one, the Stratonovich signatures of Brownian motion determine the Brownian sample paths. Their strategy, in particular the approximation scheme constructed in the proof, was originated from the study of cyclic cohomology in algebraic topology. The proof relies on the strong Markov property and the potential theory for the Laplace operator in an essential way. This result was then extended to hypoelliptic diffusions by X. Geng and Z. Qian [13]. Similar results were also established for Chordal SLE_κ curves with $\kappa \leqslant 4$ by H. Boedihardjo, H. Ni and Z. Qian [4].

It should be pointed out that in the probabilistic setting, the result of Le Jan and Qian is stronger than the general deterministic result in [3] from a theoretical point of view, as it not only gives the injectivity but also yields an

explicit way of how a sample path can be reconstructed from its signature outside a null set in the path space. In the deterministic setting, such reconstruction was studied by T. Lyons and W. Xu [22] for C^1-paths via symmetrization, and by H. Boedihardjo and X. Geng [2] for planar Jordan curves with finite p-variation for $1 \leqslant p < 2$ via Fourier transform. A general inversion scheme for the signature of a weakly geometric rough path remains a significant open problem in rough path theory.

The main purpose of the present paper is to strengthen the technique of Le Jan and Qian to include a class of non-Markov processes. In particular, we shall establish the almost-sure uniqueness of signature (up to reparametrization) for a class of Gaussian processes including fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge. More importantly, our technique also yields an explicit inversion scheme for the signature of sample paths. The fundamental difficulty in exploiting the idea of Le Jan and Qian lies in the unavailability of those probabilistic and analytic tools arising from the strong Markov property and the potential theory which were used in their proof and also in [13]. The key of getting around this difficulty is to develop methods which enable us to investigate the problem essentially from a pathwise point of view.

The well-definedness of signature when the sample paths of the process have finite p-variation for $p \geqslant 1$ are well studied in probability literatures. For instance, it was shown by L. Coutin and Z. Qian [7] that with probability one, the sample paths of fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$ can be lifted canonically as geometric rough paths. Moreover, it is believed that no such canonical lift exists for $H \leqslant \frac{1}{4}$. There are similar results for lots of interesting stochastic processes, such as martingales, Markov processes, Gaussian processes, solutions to Gaussian rough differential equations, SLE_κ curves with $\kappa \leqslant 4$ etc., under certain regularity conditions. See for example [12].

In establishing our main result, we shall state explicitly under what conditions on the law of the process would the almost-sure uniqueness of the signature hold. We hope that this provides a general framework for solving the almost-sure uniqueness of signature problem for other interesting processes. Note that our result is *not* a direct corollary of the result in [3], since it is highly nontrivial to prove the existence of a null set outside which no two paths can be tree-like deformation of each other.

2   Preliminaries on Rough Path Theory

We first recall some basic notions from rough path theory, which we will use throughout the rest of the present paper.

Let $T\left(\left(\mathbb{R}^d\right)\right)$ denote the infinite dimensional tensor algebra over $\mathbb{R}^d$. Let π_k denote the projection map from $T\left(\left(\mathbb{R}^d\right)\right)$ to $\left(\mathbb{R}^d\right)^{\otimes k}$ and $\pi^{(k)}$ denote the projection map from $T\left(\left(\mathbb{R}^d\right)\right)$ to the truncated k-th tensor algebra

$$T^k\left(\mathbb{R}^d\right) := \oplus_{j=0}^k \left(\mathbb{R}^d\right)^{\otimes j}.$$

Here we shall equip $\left(\mathbb{R}^d\right)^{\otimes k}$ with the Euclidean norm by identifying it with $\mathbb{R}^{d^k}$. Let $\triangle := \{(s,t) : 0 \leqslant s \leqslant t \leqslant 1\}$ be the standard 2-simplex.

Definition 2.1. A *multiplicative functional of degree* $N \in \mathbb{N}$ is a map $\mathbf{X} : \triangle \to T^N\left(\mathbb{R}^d\right)$ satisfying the following so-called Chen's identity:

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}, \ \forall 0 \leqslant s \leqslant u \leqslant t \leqslant 1.$$

Let $\mathbf{X}, \mathbf{Y}$ be two multiplicative functionals of degree N. For $p \geqslant 1$, define

$$d_p\left(\mathbf{X}, \mathbf{Y}\right) = \max_{1 \leqslant i \leqslant N} \sup_{\mathcal{P}} \left(\sum_l \left| \pi_i\left(\mathbf{X}_{t_{l-1},t_l} - \mathbf{Y}_{t_{l-1},t_l}\right) \right|^{\frac{p}{i}} \right)^{\frac{i}{p}},$$

where the supremum is taken over all possible finite partitions of $[0,1]$. d_p is called the *p-variation metric*. If $d_p\left(\mathbf{X}, \mathbf{1}\right) < \infty$ where $\mathbf{1} = (1, 0, \cdots, 0)$, we then say that $\mathbf{X}$ has *finite p-variati*on. A multiplicative functional of degree $\lfloor p \rfloor$ with finite p-variation is called a *p-rough path*.

The following so-called Lyons' extension theorem (see [19]) says that the signature of a p-rough path is well defined.

Theorem 2.1. *For $p \geqslant 1$, let $\mathbf{X}$ be a p-rough path. Then there exists a unique multiplicative functional $S\left(\mathbf{X}\right) : \triangle \to T\left(\left(\mathbb{R}^d\right)\right)$ such that $\pi^{(N)}\left(S\left(\mathbf{X}\right)\right)$ has finite p-variation for each $N \in \mathbb{N}$ and*

$$\pi^{(\lfloor p \rfloor)}\left(S\left(\mathbf{X}\right)\right) = \mathbf{X}.$$

Definition 2.2. $S\left(\mathbf{X}\right)_{0,1} \in T\left(\left(\mathbb{R}^d\right)\right)$ is called the *signature* of the p-rough path $\mathbf{X}$.

If $x : [0,1] \to \mathbb{R}^d$ is a path with finite p-variation for some $1 \leqslant p < 2$, then as a p-rough path no higher levels of x are needed and we can express the signature of x explicitly as

$$S\left(x\right)_{0,1} = \left(1, \int_{0 < s_1 < 1} dx_{s_1}, \ldots, \int_{0 < s_1 < \cdot < s_n < 1} dx_{s_1} \otimes \ldots \otimes dx_{s_n}, \ldots \right),$$

where the iterated integrals are defined in the sense of Young.

A fundamental result in rough path theory, proved by Lyons [19], is the continuity of rough path integrals and the solution map for rough differential equations with respect to the driving path under the p-variation metric.

There is a special class of rough paths called geometric rough paths. They are the simplest and very natural examples of rough paths with which we can define path integrals against one forms.

Definition 2.3. Given $p \geqslant 1$. Let $G\Omega_p\left(\mathbb{R}^d\right)$ denote the completion of the set

$$\left\{ S_{\lfloor p \rfloor}(x) := \pi^{(\lfloor p \rfloor)}\left(S(x)\right) : \ x \text{ has bounded total variation} \right\}$$

with respect to the p-variation metric d_p. $G\Omega_p(\mathbb{R}^d)$ is called the space of *geometric p-rough paths*.

In [7], Coutin and Qian showed that under certain conditions on the decorrelation of the increment of a Gaussian process, with probability one the lifting of the dyadic piecewise linear interpolation of the Gaussian sample paths in $G\Omega_p\left(\mathbb{R}^d\right)$ is a Cauchy sequence under the p-variation metric. In [12], P. Friz and N. Victoir extended this result to a larger class of Gaussian processes under certain regularity condition on the covariance function. Moreover, they showed that the lifting of any sequence of piecewise linear interpolation of the Gaussian sample paths in $G\Omega_p$ converges to the same limit. From here onwards, this limit will be known as the *canonical lifting* of the Gaussian process in $G\Omega_p\left(\mathbb{R}^d\right)$. A fundamental example of these results is fractional Brownian motion with Hurst parameter $H > 1/4$. It follows from Theorem 2.1 that the signature $S(x)_{0,1} \in T(\mathbb{R}^d)$ of fractional Brownian motion with $H > 1/4$ is well-defined for almost surely through the canonical lifting.

A detailed study on the geometric rough path nature of many interesting and important stochastic processes can be found in [12].

3   Main Results

In this section we are going to state main results of the present paper and illustrate the idea of proofs.

Let $X = \{X_t : t \in [0, 1]\}$ be a d-dimensional continuous stochastic process starting at the origin, where $d \geqslant 2$. We will always assume that X is realized on the path space $(W, \overline{\mathcal{B}(W)}, \mathbb{P})$, where W is the space of $\mathbb{R}^d$-valued continuous paths over $[0, 1]$ starting at the origin, $\overline{\mathcal{B}(W)}$ is the completion of the Borel σ-algebra over W with respect to the law $\mathbb{P}$ of X.

In the rest of the present paper, we will make the following assumptions on the law $\mathbb{P}$.

Assumption (A): There exists a $\mathbb{P}$-null set $\mathcal{N}_0$ and a map $S : W \backslash \mathcal{N}_0 \to C\left(\triangle; T\left(\left(\mathbb{R}^d\right)\right)\right)$, such that for each $x \in W \backslash \mathcal{N}_0$ and $(s, t) \in \Delta$, $\pi_1\left(S(x)_{s,t}\right) = x_t - x_s$ and $S(x)$ is the multiplicative extension of some geometric rough path $\mathbf{X}$ (see Theorem 2.1). We will call such a map S a $\mathbb{P}$-*almost sure lifting*. The integrals with respect to x will then be defined as integrating against the geometric rough path $\mathbf{X}$.

Assumption (B): For any $0 < t < 1$, the law of x_t is absolutely continuous with respect to the Lebesgue measure.

Assumption (C): For any open cube $H \subset \mathbb{R}^d$, there exists a differential one form $\phi = \sum_{i=1}^{d} \phi_i dx^i$ supported on the closure of H, such that for any $0 \leqslant s < t \leqslant 1$, if we let

$$A_{s,t}^H = \{x \in W : \text{ there exists some } u \in (s, t) \text{ such that } x_u \in H\}, \qquad (3.1)$$

then

$$\mathbb{P}\left(\left\{x \in W : \int_s^t \phi(dx_u) = 0\right\} \cap A_{s,t}^H\right) = 0.$$

Here $\int_s^t \phi(dx_u) = \sum_{i=1}^d \int_s^t \phi_i(x_u)dx_u^i$ is defined in the sense of rough paths according to Assumption (A).

Remark 3.1. As we have mentioned before, Assumption (A) is quite natural for a large class of stochastic processes. Assumption (B) is also verified for most of these processes, e.g., hypoelliptic diffusions, Gaussian processes, solutions to hypoelliptic rough differential equations driven by Gaussian processes, etc. These examples are well studied in [12]. Assumption (C) suggests certain kind of nondegeneracy for sample paths of the process, which is essential for the recovery of a path from its signature in our setting. By a closer look at Assumption (C), it actually excludes the possibility of the sample paths being tree-like. Therefore, with probability one the sample paths are already "reduced" paths in the tree-like equivalence classes and it is natural to expect an inversion scheme for the signature in our setting (see [3], [16] for the notion of tree-like paths). This is the main goal of the present paper.

In the last section, as a fundamental example we will show that these assumptions are all verified for a class of Gaussian processes including fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge.

Since we aim at recovering a path up to reparametrization from its signature, we first give the definition of reparametrization.

Definition 3.1. A *reparametrization* is a continuous, strictly increasing map $\sigma : [0,1] \to [0,1]$ with $\sigma(0) = 0$ and $\sigma(1) = 1$. The group of reparametrizations is denoted by $\mathcal{R}$.

Now we are in a position to state our main results.

Theorem 3.1. *Assume that the law $\mathbb{P}$ of the stochastic process satisfies Assumption (A), (B) and (C). Let S be the $\mathbb{P}$-almost sure lifting as in Assumption (A). Then there exists a $\mathbb{P}$-null set $\mathcal{N}$, such that for any $x, x' \in \mathcal{N}^c$, if $S(x)_{0,1} = S(x')_{0,1}$, then there exists some $\sigma \in \mathcal{R}$, such that*

$$x_t = x'_{\sigma(t)}, \ \forall t \in [0,1].$$

As a fundamental example, we will prove the following result for a class of Gaussian processes satisfying conditions to be specified later on in the final section.

Theorem 3.2. *Let $\mathbb{P}$ be the law of a Gaussian process satisfying conditions specified in Section 6. Then $\mathbb{P}$ satisfies Assumption (A), (B), (C). In particular, the result holds for fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge.*

Before going into the mathematical proofs, we first describe the strategy informally. The approximation scheme we are going to use is an adaptation from the work of Le Jan and Qian [17]. However, the main difficulties are in the development of each step in the non-Markov setting, which will become clear in the detailed proofs.

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Step One. Prove that if two paths have the same signature, then the iterated integrals of the paths along any finite sequence of smooth one forms are the same. Following [17], these iterated integrals along one forms will be called *extended signatures.*

Step Two. Decompose the Euclidean space $\mathbb{R}^d$ into disjoint identical open cubes with small tunnels between them. For each such cube, we define a differential one form supported on the closure of the cube according to Assumption (C).

Step Three. Show that, for each path x outside a $\mathbb{P}$-null set, the ordered sequence of cubes visited by x corresponds to the unique maximal sequence of differential one forms along which the extended signature of x is nonzero. This together with step one allows us to recover the ordered sequence of cubes visited by x from its signature.

Step Four. Construct a polygonal approximation of x by joining the centers of cubes visited by x in order. This polygonal path will be parametrized so that it is at the center of the cube at the time when the cube is first visited by x. Show that with probability one, as the size of cubes tends to zero, the polygonal path converges to the original path x under the uniform topology.

Step Five. Since the signature is invariant under the reparametrization of the path, it is not possible to recover the exact visit times of the cubes. If two paths have the same signature, then the corresponding polygonal paths constructed in (3) are only equal up to a reparametrization. Therefore, we need to introduce a variant of the FrÃ©chet distance on W measuring the distance of two paths modulo parametrization. We should also prove that outside a $\mathbb{P}$-null set this is indeed a metric. It will then follow from step four that if two paths x and x' have the same signature, their corresponding approximation paths converge to the same limit under this metric, which will imply that x and x' are equal up to a reparametrization.

For the Gaussian case, Assumption (A) is verified from [12] and Assumption (B) is trivial by definition. By using the Malliavin calculus, for each open cube H we will explicitly construct a differential one form ϕ supported on $\overline{H}$ such that the functional $x \to \int_s^t \phi(dx_u)$ has a density conditioned on the set $A_{s,t}^H$. This certainly verifies Assumption (C).

4   Signature Determines Extended Signatures

Starting from this section, we are going to develop the detailed proofs of our main results.

As the first step, here we will prove that if two sample paths as geometric rough paths have the same signatures, then they have the same extended signatures. Note that the signatures and extended signatures are well-defined for $\mathbb{P}$-almost surely according to Assumption (A). For the general theory of integration along one forms against rough paths, see for example [12], [21].

From now on, for a geometric rough path $\mathbf{X}$ and a finite sequence $(\phi^1, \cdots, \phi^n)$ of differential one forms $\phi^1, \ldots, \phi^n$, we will use $\left[\phi^1, \cdots, \phi^n\right]_{0,t}(x)$ to denote the

first level of the iterated rough path integral $\int_0^t \cdots \int_0^{s_2} \phi^1 (d\mathbf{X}_{s_1}) \cdots \phi^n (d\mathbf{X}_{s_n})$, where $x. := \pi_1(S(\mathbf{X})_{0,.})$ is the first level path of $\mathbf{X}$. A simple way of understanding this integral is through

$$\int_0^t \cdots \int_0^{s_2} \phi^1 (d\mathbf{X}_{s_1}) \ldots \phi^n (d\mathbf{X}_{s_n}) = \lim_{k \to \infty} \int_0^t \cdots \int_0^{s_2} \phi^1 (dx_{s_1}^{(k)}) \cdots \phi^n (dx_{s_n}^{(k)}),$$

where by the definition of geometric rough paths $x^{(k)}$ is a sequence of paths with bounded total variation whose lifting converges to $\mathbf{X}$ under the p-variation metric. An alternative and equivalent way of defining this integral is through the following rough differential equation:

$$\begin{cases} dx_t^i = dx_t^i, & 1 \leqslant i \leqslant d, \\ dy_t^j = y_t^{j-1} \sum_{i=1}^d \phi_i^j(x_t) dx_t^i, & 1 \leqslant j \leqslant n, \\ x_0 = 0, y_0 = 0, \end{cases} \qquad (4.1)$$

where $y_t^{-1} := 1$.

Sometimes we will also use the notation $\int_0^t \cdots \int_0^{s_2} \phi^1 (dx_{s_1}) \ldots \phi^n (dx_{s_n})$ to denote the path integral. Note that the ordering of $(\phi^1, \cdots, \phi^n)$ is noncommutative in this notation.

Now we have the following result.

Proposition 4.1. *Given $p \geqslant 1$, let $\mathbf{X}, \mathbf{X}' \in G\Omega_p (\mathbb{R}^d)$ be two geometric p-rough paths. Suppose that $\phi^1, \ldots, \phi^n$ are n compactly supported C^∞-one forms. If $S(\mathbf{X})_{0,1} = S(\mathbf{X}')_{0,1}$, then*

$$[\phi^1, \ldots, \phi^n]_{0,1} (x) = [\phi^1, \ldots, \phi^n]_{0,1} (x'),$$

where x and x' are the first level paths of $\mathbf{X}$ and $\mathbf{X}'$ respectively.

To prove Proposition 4.1, first notice that the case of polynomial one forms follows immediately from integration by parts and the shuffle product formula for the signature (see [17], p.4 and [20], Theorem 2.15).

Lemma 4.1. *Let $\phi_1, \ldots, \phi_n$ be n polynomial one forms. Let x be a continuous path with bounded total variation with $x_0 = 0$. Then there exists a linear functional f on $T(\mathbb{R}^d)$ such that*

$$[\phi^1, \ldots, \phi^n]_{0,1} (x) = f\left(S(x)_{0,1} \right).$$

Proposition 4.1 then follows from polynomial approximations.

Proof of Proposition 4.1. We write ϕ^i as $\phi^i = \sum_{j=1}^d \phi_j^i(x) dx^j$. Let K be a compact neighborhood of $x([0,1]) \cup x'([0,1])$. According to [1], Theorem 1, for each $\alpha > 0$ and each j, there exists a polynomial sequence $\phi_j^{i(m)}$ such that

$$\sup_K \left| D^\alpha \left(\phi_j^i - \phi_j^{i(m)} \right) \right| \to 0$$

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as $m \to \infty$. Let $\phi^{i(m)}(x) = \sum_{j=1}^{d} \phi_j^{i(m)}(x) \, dx^j$. As $\mathbf{X} \in G\Omega_p(\mathbb{R}^d)$, by definition there exists a sequence $x^{(k)}$ of paths with bounded total variation, such that $d_p(S_{\lfloor p \rfloor}(x^{(k)}), \mathbf{X}) \to 0$ as $k \to \infty$. Since the integration map $(\phi, \mathbf{X}) \to \int_0^1 \phi(d\mathbf{X}_t)$ is jointly continuous under the $\mathrm{Lip}(\alpha)$ and p-variation norms whenever $\alpha > p + 1$ (see [12], Theorem 10.47), we have

$$\left[\phi^{1(m)}, \ldots, \phi^{n(m)}\right]_{0,1}\left(x^{(k)}\right) \to \left[\phi^1, \ldots, \phi^n\right]_{0,1}(x),$$

as $m, k \to \infty$. Now the result follows from Lemma 4.1. $\qquad\square$

To end this section, we present a lemma on extended signatures which is used in the next section. The intuition behind this result is easy and clear.

Lemma 4.2. *Let $\mathbf{X}$ be a geometric p-rough path, and let $(\phi^1, \cdots, \phi^n)$ be a sequence of compactly supported C^∞-one forms. Assume that for some $1 \leqslant j_0 \leqslant n$, the first level path x of $\mathbf{X}$ does not visit the interior of the support of ϕ^{j_0}. Then*

$$[\phi^1, \cdots, \phi^n]_{0,t}(x) = 0, \ \forall t \in [0, 1].$$

Proof. For $1 \leqslant j \leqslant n$, define

$$y_t^j = [\phi^1, \cdots, \phi^j]_{0,t}(x), \ t \in [0, 1].$$

It follows that the path $z_t := (x_t, y_t^j)_{1 \leqslant j \leqslant n} \in \mathbb{R}^d \oplus \mathbb{R}^n$ solves the rough differential equation (4.1).

According to the Euler estimates for rough differential equations projected onto the j_0-th component (see [12], Corollary 10.15), we have

$$\left| y_{s,t}^{j_0} - \mathcal{E}_{(V)}^{j_0}\left(z_s, S_{\lfloor \gamma \rfloor}(\mathbf{X})_{s,t}\right) \right| \leqslant C \|\mathbf{X}\|_{p-\mathrm{var};[s,t]}^\gamma, \ \forall 0 \leqslant s \leqslant t \leqslant 1,$$

where $\mathcal{E}_{(V)}^{j_0}\left(z_s, S_{\lfloor \gamma \rfloor}(\mathbf{X})_{s,t}\right)$ is the j_0-th component of the Euler scheme (see [12], Definition 10.1) for the generating vector fields V of (4.1), $\gamma > p$ is some fixed constant, and C is a constant depending only on p, γ and the one forms. By writing out the vector fields V explicitly, it is not hard to see that the j_0-th component of the Euler scheme is a sum in which every term contains ϕ^{j_0} or its higher order derivatives evaluated at x_s. Since the path x does not visit the interior of the support of ϕ^{j_0}, we conclude that

$$\mathcal{E}_{(V)}^{j_0}\left(z_s, S_{\lfloor \gamma \rfloor}(\mathbf{X})_{s,t}\right) = 0, \ \forall 0 \leqslant s \leqslant t \leqslant 1. \tag{4.2}$$

Therefore, for any partition $\{t_k\}$ of $[s, t]$, we have

$$|y_{t_{k-1}, t_k}^{j_0}| \leqslant C \|\mathbf{X}\|_{p-\mathrm{var};[t_{k-1}, t_k]}^\gamma.$$

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It follows that

$$
\begin{aligned}
|y_{s,t}^{j_0}| &\leqslant \sum_k |y_{t_{k-1},t_k}^{j_0}| \\
&\leqslant C \left(\max_k \|\mathbf{X}\|_{p-\text{var};[t_{k-1},t_k]}^{\gamma-p} \right) \sum_k \|\mathbf{X}\|_{p-\text{var};[t_{k-1},t_k]}^{p} \\
&\leqslant C \left(\max_k \|\mathbf{X}\|_{p-\text{var};[t_{k-1},t_k]}^{\gamma-p} \right) \|\mathbf{X}\|_{p-\text{var};[s,t]}^{p}.
\end{aligned}
$$

By taking $\max_k (t_k - t_{k-1}) \to 0$ we obtain that $y_{s,t}^{j_0} = 0$. Since $y_0^{j_0} = 0$, it follows that $y_t^{j_0} = 0$ for all $t \in [0,1]$.

Now the result follows from inductively projecting the Euler estimate ([12], Corollary 10.15) onto components $j_0 \leqslant j \leqslant n$. In this case by induction we conclude that (4.2) holds for each component $j_0 \leqslant j \leqslant n$, and thus the same argument as before yields that

$$
y_t^n = [\phi^1, \cdots, \phi^n]_{0,t}(x) = 0
$$

for all $t \in [0,1]$. Alternatively we can use the fact

$$
[\phi^1, \cdots, \phi^j]_{0,t}(x) = \sum_{i=1}^{d} \int_0^t [\phi^1, \cdots, \phi^{j-1}]_{0,s}(x) \phi_i^j(x_s) dx_s^i, \ \ j_0 \leqslant j \leqslant n,
$$

to reach the same conclusion.

$\square$

5 The Strengthened Le Jan-Qian Approximation Scheme and the Uniqueness of Signature

Now fix $\varepsilon, \delta > 0$ with $\delta << \varepsilon$.

For any integer point $z = (z^1, z^2, \ldots, z^d) \in \mathbb{Z}^d$, let $H_z^{\varepsilon,\delta}$ be the open cube centered at εz with edges of length $\varepsilon - \delta$. In other words,

$$
H_z^{\varepsilon,\delta} = \left\{ x \in \mathbb{R}^d : \ |x^i - \varepsilon z^i| < \frac{\varepsilon - \delta}{2}, \ \forall i = 1, \cdots, d \right\}.
$$

Geometrically, the space $\mathbb{R}^d$ is divided into disjoint identical open cubes and small closed tunnels.

For any $x \in W$ and $k \geqslant 1$, define recursively

$$
\tau_k^{\varepsilon,\delta} = \inf \left\{ t \in \left[\tau_{k-1}^{\varepsilon,\delta}, 1 \right] : \ x_t \in \bigcup_{z \neq \boldsymbol{m}_{k-1}^{\varepsilon,\delta}} H_z^{\varepsilon,\delta} \right\},
$$

and $\boldsymbol{m}_k^{\varepsilon,\delta}$ to be the integer point $z \in \mathbb{Z}^d$ such that

$$x_{\tau_k^{\varepsilon,\delta}} \in H_z^{\varepsilon,\delta},$$

where $\tau_0^{\varepsilon,\delta} = 0$, $\boldsymbol{m}_0^{\varepsilon,\delta} = 0 \in \mathbb{Z}^d$. Let

$$N^{\varepsilon,\delta} = \sup \left\{ k \geqslant 1 : \ \tau_k^{\varepsilon,\delta} < 1 \right\},$$

where $\sup \emptyset := 0$. The sequence $\left\{ \tau_k^{\varepsilon,\delta} \right\}$ records the successive visit times of the open cubes by the path, the sequence $\left\{ \boldsymbol{m}_k^{\varepsilon,\delta} \right\}$ records the cubes visited in order, and $N^{\varepsilon,\delta}$ records the total number of cubes visited. Note that revisit of the same cube after visiting some other cubes counts, but revisit before visiting any other cube does not count. By continuity and compactness, it is easy to see that for any $x \in W$, $0 \leqslant N^{\varepsilon,\delta} < \infty$.

Here and thereafter, for notation simplicity we drop the dependence on x for these random variables on W.

Remark 5.1. It is important to use the open cubes instead of the closed ones, as we are only interested in the case when a path x travels through the interior of a cube. Hence these $\tau_k^{\varepsilon,\delta}$ are not stopping times with respect to the natural filtration.

For each cube $H_z^{\varepsilon,\delta}$, let $\phi_z^{\varepsilon,\delta}$ be the differential one form given in Assumption (C). In particular, $\phi_z^{\varepsilon,\delta}$ is supported on the closure of $H_z^{\varepsilon,\delta}$, and $\phi_z^{\varepsilon,\delta} = 0$ on ∂H.

5.1 Recovery of Cubes Visited in Order by Using the Extended Signature

Let $\mathcal{W}_m$ $(m \geqslant 0)$ be the set of words $(z_0 = 0, z_1, \cdots, z_m)$ with $z_i \neq z_{i+1}$, $z_i \in \mathbb{Z}^d$, and let $\mathcal{W} = \bigcup_{m \geqslant 0} \mathcal{W}_m$. Elements of $\mathcal{W}$ are called *admissible words*.

For $w = (z_0, z_1, \cdots, z_m) \in \mathcal{W}$, define

$$E_w^{\varepsilon,\delta} = \left\{ x \in W : \ N^{\varepsilon,\delta} = m, \ \boldsymbol{m}_k^{\varepsilon,\delta} = z_k, \ k = 0, \cdots, m \right\}.$$

It follows that W can be written as the disjoint union $W = \bigcup_{w \in \mathcal{W}} E_w^{\varepsilon,\delta}$.

Now we have the following result.

Lemma 5.1. *For any $m \geqslant 0$, if $w = (z_0 = 0, \cdots, z_m) \in \mathcal{W}_m$ and $x \in E_w^{\varepsilon,\delta}$, then*

(1)

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta} \right]_{0,1} (x) = \prod_{i=1}^{m+1} \int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \phi_{z_{i-1}}^{\varepsilon,\delta} (dx_t), \tag{5.1}$$

where $\tau_{m+1}^{\varepsilon,\delta} = 1$ by definition since $x \in E_w^{\varepsilon,\delta}$.

(2) For any $w' = (z_0, z_1', \cdots, z_n') \in \mathcal{W}_n$ with $n > m$,

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_n'}^{\varepsilon,\delta} \right]_{0,1} (x) = 0.$$

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(3) For any $w' = (z_0, z_1', \cdots, z_m')$ with $w' \neq w$,

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,1}(x) = 0.$$

Proof. We prove this result by induction on m.

If $m = 0$, assume that $x \in E_{(z_0)}^{\varepsilon,\delta}$. Then (1) and (3) are trivial. To see (2), let $w' = (z_0, z_1', \cdots, z_n') \in \mathcal{W}_n$ with $n > 0$. Since w' is an admissible word, the path x does not visit the open cube $H_{z_1'}^{\varepsilon,\delta}$. According to Lemma 4.2, we know that the corresponding extended signature is zero. If $m = 1$, assume that $w = (z_0, z_1) \in \mathcal{W}_1$ and $x \in E_w^{\varepsilon,\delta}$. Then (3) follows by the same argument as before. To see (1), first we have

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \phi_{z_1}^{\varepsilon,\delta}\right]_{0,1}(x) = \int_0^1 \left[\phi_{z_0}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_1}^{\varepsilon,\delta}(dx_t)$$

$$= \int_{\tau_1^{\varepsilon,\delta}}^1 \left[\phi_{z_0}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_1}^{\varepsilon,\delta}(dx_t),$$

since $\phi_{z_1}^{\varepsilon,\delta}$ is supported in $H_{z_1}^{\varepsilon,\delta}$. Moreover, if $\tau_1^{\varepsilon,\delta} \leqslant t \leqslant 1$, then

$$\left[\phi_{z_0}^{\varepsilon,\delta}\right]_{0,t}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}\right]_{0,\tau_1^{\varepsilon,\delta}}(x),$$

since $\phi_{z_0}^{\varepsilon,\delta}$ is supported in $H_{z_0}^{\varepsilon,\delta}$. Therefore,

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \phi_{z_1}^{\varepsilon,\delta}\right]_{0,1}(x) = \left(\int_0^{\tau_1^{\varepsilon,\delta}} \phi_{z_0}^{\varepsilon,\delta}(dx_t)\right)\left(\int_{\tau_1^{\varepsilon,\delta}}^1 \phi_{z_1}^{\varepsilon,\delta}(dx_t)\right)$$

and (1) follows. If $w' = (z_0, z_1', \cdots, z_n') \in \mathcal{W}_n$ with $n > 1$, there are two cases. The first case is that there is some $0 < k \leqslant n$ such that z_k' is different from z_0 and z_1. In this case (2) follows again from Lemma 4.2. The second case is

$$w' = (z_0, z_1, z_0, z_1, \cdots, z_n'),$$

where $n > 1$ and z_n' is either z_0 or z_1. If $z_n' = z_0$, then

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_n'}^{\varepsilon,\delta}\right]_{0,1}(x) = \int_0^{\tau_1^{\varepsilon,\delta}} \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'=z_1}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_0}(dx_t).$$

But during $\left[0, \tau_1^{\varepsilon,\delta}\right]$ the path x never visits the interior of $H_{z_1}^{\varepsilon,\delta}$, so the integral on the R.H.S. is zero and hence the extended signature corresponding to w' is zero. If $z_n' = z_1$,

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_n'}^{\varepsilon,\delta}\right]_{0,1}(x) = \int_{\tau_1^{\varepsilon,\delta}}^1 \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'=z_0}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_1}^{\varepsilon,\delta}(dx_t).$$

For $\tau_1^{\varepsilon,\delta} \leqslant t \leqslant 1$, we have

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z'_{n-1}=z_0}^{\varepsilon,\delta}\right]_{0,t}(x)$$

$$= \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z'_{n-1}=z_0}^{\varepsilon,\delta}\right]_{0,\tau_1^{\varepsilon,\delta}}(x) + \int_{\tau_1^{\varepsilon,\delta}}^{1}\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z'_{n-2}=z_1}^{\varepsilon,\delta}\right]_{0,u}(x)\phi_{z_0}^{\varepsilon,\delta}(dx_u)$$

$$= \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z'_{n-1}=z_0}^{\varepsilon,\delta}\right]_{0,\tau_1^{\varepsilon,\delta}}(x).$$

But during $\left[0, \tau_1^{\varepsilon,\delta}\right]$ the path x does not visit the interior of $H_{z_1}^{\varepsilon,\delta}$ and the last term contains the one form $\phi_{z_1}^{\varepsilon,\delta}$, thus it is zero and $\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z'_n}^{\varepsilon,\delta}\right]_{0,1}(x) = 0$. Therefore (2) again follows.

Now assume that the claim is true for all non negative integer less than m, we are going to show that it is true for m. Let $w = (z_0, \cdots, z_m) \in \mathcal{W}_m$ and $x \in E_w^{\varepsilon,\delta}$.

We first show (1). In fact,

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta}\right]_{0,1}(x) = \int_0^{\tau_m^{\varepsilon,\delta}}\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_m}^{\varepsilon,\delta}(dx_t)$$

$$+ \int_{\tau_m^{\varepsilon,\delta}}^{1}\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_m}^{\varepsilon,\delta}(dx_t)$$

$$= \int_0^{\tau_m^{\varepsilon,\delta}}\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_m}^{\varepsilon,\delta}(dx_t)$$

$$+ \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x)\int_{\tau_m^{\varepsilon,\delta}}^{1}\phi_{z_m}^{\varepsilon,\delta}(dx_t),$$

where the last equality comes from the fact that

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x), \ \forall t \in \left[\tau_m^{\varepsilon,\delta}, 1\right],$$

since $z_{m-1} \neq z_m$ and hence during $\left[\tau_m^{\varepsilon,\delta}, 1\right]$ the path does not visit the interior of $H_{z_{m-1}}^{\varepsilon,\delta}$. Now we want to use the induction hypothesis (1) on the term $\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x)$. To this end, let $\widetilde{x}$ be a path in W such that $\widetilde{x} = x$ on $\left[0, \tau_m^{\varepsilon,\delta}\right]$ and $\widetilde{x}$ stays inside the tunnel on $\left[\tau_m^{\varepsilon,\delta}, 1\right]$. It follows that $\widetilde{x} \in E_{\widetilde{w}}^{\varepsilon,\delta}$ where $\widetilde{w} = (z_0, \cdots, z_{m-1}) \in \mathcal{W}_{m-1}$, and

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}).$$

Therefore, by the induction hypothesis (1) and the definition of $\widetilde{x}$ we have

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}) = \left(\prod_{i=1}^{m-1}\int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}}\phi_{z_{i-1}}^{\varepsilon,\delta}(d\widetilde{x}_t)\right)\left(\int_{\tau_{m-1}^{\varepsilon,\delta}}^{1}\phi_{z_{m-1}}^{\varepsilon,\delta}(d\widetilde{x}_t)\right)$$

$$= \left(\prod_{i=1}^{m-1}\int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}}\phi_{z_{i-1}}^{\varepsilon,\delta}(dx_t)\right)\left(\int_{\tau_{m-1}^{\varepsilon,\delta}}^{\tau_m^{\varepsilon,\delta}}\phi_{z_{m-1}}^{\varepsilon,\delta}(dx_t)\right).$$

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Consequently (1) will follow once we show that

$$\int_0^{\tau_m^{\varepsilon,\delta}} \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_m}^{\varepsilon,\delta}(dx_t) = 0.$$

But this is an easy consequence of the fact that

$$\int_0^{\tau_m^{\varepsilon,\delta}} \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_m}^{\varepsilon,\delta}(dx_t) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x})$$

and the induction hypothesis (2).

Now we show (2). Let $w' = (z_0, z_1', \cdots, z_n') \in \mathcal{W}_n$ with $n > m$. As before, the case when there exists some $0 < k \leqslant n$ such that $z_k' \notin \{z_0, \cdots, z_m\}$ is trivial. Otherwise, write

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_n'}^{\varepsilon,\delta}\right]_{0,1}(x) = \sum_i \int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_n'}^{\varepsilon,\delta}(dx_t), \qquad (5.2)$$

where the sum is over those $i \leqslant m+1$ such that $z_{i-1} = z_n'$. Since $z_{n-1}' \neq z_n'$, for each such i we have

$$\int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,t}(x)\phi_{z_n'}^{\varepsilon,\delta}(dx_t)$$

$$= \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,\tau_{i-1}^{\varepsilon,\delta}}(x) \int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \phi_{z_n'}^{\varepsilon,\delta}(dx_t).$$

Define a new path $\widetilde{x} \in W$ such that $\widetilde{x} = x$ on $\left[0, \tau_{i-1}^{\varepsilon,\delta}\right]$ and $\widetilde{x}$ stays inside the tunnel on $\left[\tau_{i-1}^{\varepsilon,\delta}, 1\right]$. Then $\widetilde{x} \in E_{\widetilde{w}}^{\varepsilon,\delta}$ with $\widetilde{w} = (z_0, \cdots, z_{i-2})$. Since during $\left[\tau_{i-1}^{\varepsilon,\delta}, 1\right]$ the path $\widetilde{x}$ does not visit the interior of $H_{z_n'}^{\varepsilon,\delta}$, we have

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,\tau_{i-1}^{\varepsilon,\delta}}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}).$$

Now observe that $i - 2 < m \leqslant n - 1$, and so by the induction hypothesis (2) we know that

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{n-1}'}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}) = 0.$$

Therefore, each term in the R.H.S. is zero and (2) follows.

Finally we show (3). Let $w' = (z_0, z_1', \cdots, z_m') \in \mathcal{W}_m$ with $w' \neq w$. If $z_m' = z_m$, then

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,1}(x)$$

$$= \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x) + \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}'}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x) \int_{\tau_m^{\varepsilon,\delta}}^1 \phi_{z_m}^{\varepsilon,\delta}(dx_t).$$

Define $\widetilde{x} \in W$ by $\widetilde{x} = x$ on $\left[0, \tau_m^{\varepsilon,\delta}\right]$ and staying inside the tunnel on $\left[\tau_m^{\varepsilon,\delta}, 1\right]$. It follows from the induction hypothesis (2) that

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}) = 0.$$

Moreover, in this case we know that $(z_0, \cdots, z_{m-1}') \neq (z_0, \cdots, z_{m-1})$. Therefore, by induction hypothesis (3) we have

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}'}^{\varepsilon,\delta}\right]_{0,\tau_m^{\varepsilon,\delta}}(x) = \left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{m-1}'}^{\varepsilon,\delta}\right]_{0,1}(\widetilde{x}) = 0.$$

Consequently (3) follows. For the case $z_m' \neq z_m$ and there exists some $i \leqslant m+1$ with $z_{i-1} = z_m'$ (otherwise it is trivial), we know that i must be strictly less than $m-1$. By writing $\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m'}^{\varepsilon,\delta}\right]_{0,1}(x)$ as a sum of the form (5.2), the result (3) will follow easily from the induction hypothesis (2) by a similar argument.

Now the proof is complete. $\qquad\square$

Define a map $M^{\varepsilon,\delta}: W \to \mathbb{Z}_+$ by sending a path $x \in W$ to

$$\sup\left\{m \geqslant 0: \exists w = (z_0, z_1, \cdots, z_m) \in \mathcal{W}_m \text{ s.t. } \left[\phi_{z_0}^{\varepsilon,\delta}, \phi_{z_1}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta}\right]_{0,1}(x) \neq 0\right\}.$$

Note that by Lemma 5.1, $M^{\varepsilon,\delta} \leqslant N^{\varepsilon,\delta}$ $\mathbb{P}$-almost surely. Moreover, we are able to prove the following recovery result.

Proposition 5.1. *For each $x \in W$ outside a $\mathbb{P}$-null set, there exists a unique word $w = (z_0, \cdots, z_{M^{\varepsilon,\delta}(x)}) \in \mathcal{W}_{M^{\varepsilon,\delta}(x)}$ such that*

$$\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_{M^{\varepsilon,\delta}(x)}}^{\varepsilon,\delta}\right]_{0,1}(x) \neq 0.$$

This word is exactly given by $M^{\varepsilon,\delta}(x) = N^{\varepsilon,\delta}(x)$, and

$$z_i = \boldsymbol{m}_i^{\varepsilon,\delta}(x), \ i = 0, \cdots, M^{\varepsilon,\delta}(x).$$

Proof. Let $\mathcal{N}^{\varepsilon,\delta}$ be the set

$$\bigcup_{m=0}^{\infty} \bigcup_{w=(z_0,\cdots,z_m)\in\mathcal{W}_m} \bigcup_{i=0}^{m} \bigcup_{\substack{0\leqslant r_1<r_2\leqslant 1 \\ r_1, r_2 \in \mathbb{Q}}} \left(\left\{x \in W: \int_{r_1}^{r_2} \phi_{z_i}^{\varepsilon,\delta}(dx_u) = 0\right\} \bigcap A_{r_1,r_2}^{z_i,\varepsilon,\delta}\right),$$

where $A_{r_1,r_2}^{z_i,\varepsilon,\delta}$ is the set defined in (3.1) associated with the cube $H_{z_i}^{\varepsilon,\delta}$ and the differential one form $\phi_{z_i}^{\varepsilon,\delta}$. It follows from Assumption (C) that $\mathcal{N}^{\varepsilon,\delta}$ is a $\mathbb{P}$-null set.

For any $x \in (\mathcal{N}^{\varepsilon,\delta})^c$, let $w = (z_0, \cdots, z_m)$ be the word in $\mathcal{W}_m$ with $m = N^{\varepsilon,\delta}$ and $z_i = \boldsymbol{m}_i^{\varepsilon,\delta}$, for $i = 0, \ldots, m$, so $x \in E_w^{\varepsilon,\delta}$.

By (5.1) in Lemma 5.1, if $\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta}\right]_{0,1}(x) = 0$, then there exists some $i = 1, \cdots, m+1$ such that $\int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \phi_{z_{i-1}}^{\varepsilon,\delta}(dx_t) = 0$. By the definition of $\tau_k^{\varepsilon,\delta}$ and

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continuity, we can find some rational numbers $r_1 < \tau_{i-1}^{\varepsilon,\delta}$ and $r_2 < \tau_i^{\varepsilon,\delta}$ (if $m = 0$ take $r_1 = 0$ and $r_2 = 1$; otherwise if $i = 1$, take $r_1 = 0$ and if $i = m + 1$, take $r_2 = 1$) such that there exists some $u \in (r_1, r_2)$ with $x_u \in H_{z_{i-1}}^{\varepsilon,\delta}$ and

$$\int_{r_1}^{r_2} \phi_{z_{i-1}}^{\varepsilon,\delta}(dx_t) = \int_{\tau_{i-1}^{\varepsilon,\delta}}^{\tau_i^{\varepsilon,\delta}} \phi_{z_{i-1}}^{\varepsilon,\delta}(dx_t) = 0.$$

This implies that $x \in \mathcal{N}^{\varepsilon,\delta}$, which is a contradiction. Therefore, we have $\left[\phi_{z_0}^{\varepsilon,\delta}, \cdots, \phi_{z_m}^{\varepsilon,\delta}\right]_{0,1}(x) \neq 0$.

By the second and third part of Lemma 5.1, we know that $M^{\varepsilon,\delta}(x) = m$ and w is the unique word in $\mathcal{W}_m$ such that the corresponding extended signature of x is nonzero. $\qquad\square$

Together with the result in Section 4, proposition 5.1 tells us that outside a $\mathbb{P}$-null set, given the signature of a path x we can recover the sequence of open cubes $H_z^{\varepsilon,\delta}$ which x has visited in order.

5.2 An Approximation Result

Now we are going to construct a polygonal approximation of a path based on the ordered sequence of open cubes visited by the path and the corresponding visit times. With probability one, such polygonal approximation converges to the original path under the uniform topology. This result is crucial for the recovery of a path up to reparametrization from its signature.

Let $x \in W$ and define the word $w = (z_0, \cdots, z_m) \in \mathcal{W}_m$ by $m = N^{\varepsilon,\delta}$ and $z_i = \boldsymbol{m}_i^{\varepsilon,\delta}$ for $i = 0, \cdots, m$. Construct a polygonal path $x^{\varepsilon,\delta}$ as follows. If $m = 0$, let $x_t^{\varepsilon,\delta} = 0$ for $t \in [0, 1]$; otherwise for $1 \leqslant k \leqslant m$, define

$$x_t^{\varepsilon,\delta} = \frac{\tau_k^{\varepsilon,\delta} - t}{\tau_k^{\varepsilon,\delta} - \tau_{k-1}^{\varepsilon,\delta}} \varepsilon z_{k-1} + \frac{t - \tau_{k-1}^{\varepsilon,\delta}}{\tau_k^{\varepsilon,\delta} - \tau_{k-1}^{\varepsilon,\delta}} \varepsilon z_k, \ t \in \left[\tau_{k-1}^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta}\right],$$

and

$$x_t^{\varepsilon,\delta} = \varepsilon z_m, \ t \in \left[\tau_m^{\varepsilon,\delta}, 1\right].$$

The approximation scheme is illustrated by Figure 1.

Now we have the following approximation result.

Proposition 5.2. *For each $n \geqslant 1$ and $\varepsilon_n = 1/n$, there exists $\delta_n > 0$, such that for $\mathbb{P}$-almost surely,*

$$\lim_{n\to\infty} \sup_{0 \leqslant t \leqslant 1} \left| x_t^{\varepsilon_n, \delta_n} - x_t \right| = 0. \qquad (5.3)$$

Proof. For each ε, δ, let

$$T^{\varepsilon,\delta} = \mathbb{R}^d \backslash \bigcup_{z \in \mathbb{Z}^d} H_z^{\varepsilon,\delta}$$

be the set of closed tunnels, and define

$$A^{\varepsilon,\delta} = \left\{ x \in W : \ \exists\, [s, t] \subset x^{-1}\left(T^{\varepsilon,\delta}\right), |x_t - x_s| \geqslant \varepsilon \right\}.$$
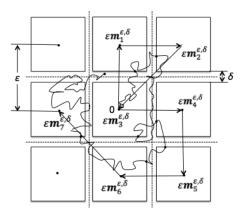
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Figure 1: This figure illustrates the corresponding approximation scheme. The dotted lines represent the degenerate tunnels. According to Assumption (B) on the process, the probability that a path stays in these degenerate tunnels for a positive time period is zero, a crucial fact used in the proof of Proposition 5.2.

We first show that for any fixed $\varepsilon > 0$,

$$\bigcap_{\delta>0} A^{\varepsilon,\delta} \subset \left\{ x \in W : \ \exists 1 \leqslant i \leqslant d, k \in \mathbb{Z}, q \in \mathbb{Q} \bigcap (0,1) \text{ s.t. } x_q^i = \frac{2k-1}{2}\varepsilon \right\}.$$
(5.4)

Let $x \in \bigcap_{\delta>0} A^{\varepsilon,\delta}$, and δ_n be a sequence such that $\delta_n \downarrow 0$. Then for each $n \geqslant 1$, there exists $0 \leqslant s_n < t_n \leqslant 1$ such that $[s_n, t_n] \subset x^{-1}\left(T^{\varepsilon,\delta_n}\right)$ and $|x_{t_n} - x_{s_n}| \geqslant \varepsilon$. By compactness we can find a subsequence (s_{n_l}, t_{n_l}) of (s_n, t_n) such that (s_{n_l}, t_{n_l}) converges to some (s, t). The condition $\left|x_{t_{n_l}} - x_{s_{n_l}}\right| \geqslant \varepsilon$ then implies that $s < t$. Therefore, for fixed u, v with $s < u < v < t$, there exists some $N \in \mathbb{N}$ such that $[u, v] \subset \bigcap_{l \geqslant N}[s_{n_l}, t_{n_l}]$, and hence

$$[u,v] \quad \subset \quad \bigcap_{l \geqslant N} x^{-1}\left(T^{\varepsilon,\delta_{n_l}}\right)$$

$$= \quad x^{-1}\left(\bigcup_{k \in \mathbb{Z}} \bigcup_{1 \leqslant i \leqslant d} \mathbb{R}^{i-1} \times \left\{ \frac{2k-1}{2}\varepsilon \right\} \times \mathbb{R}^{d-i} \right).$$

In particular, this implies (5.4) and by Assumption (B) we have $\mathbb{P}\left(\bigcap_{\delta>0} A^{\varepsilon,\delta}\right) = 0$.

Now we are going to show that for each ε, δ,

$$\left\{ x \in W : \ \sup_{0 \leqslant u \leqslant 1} \left|x_u^{\varepsilon,\delta} - x_u\right| \geqslant 11\sqrt{d}\varepsilon \right\} \subset A^{\varepsilon,\delta}.$$
(5.5)

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To see this, first notice that if x belongs to the left hand side of (5.5), then either

(1) there exists some $u \in [\tau_{k-1}^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta}]$ for some $1 \leqslant k \leqslant N^{\varepsilon,\delta}$, such that $\left| x_u^{\varepsilon,\delta} - x_u \right| \geqslant 11\sqrt{d}\varepsilon$; or

(2) there exists some $u \in [\tau_{N^{\varepsilon,\delta}}^{\varepsilon,\delta}, 1]$, such that $\left| x_u - \varepsilon \boldsymbol{m}_{N^{\varepsilon,\delta}}^{\varepsilon,\delta} \right| \geqslant 11\sqrt{d}\varepsilon$.

In the first case, we know that x does not visit any cube other than $H_{\boldsymbol{m}_{k-1}^{\varepsilon,\delta}}^{\varepsilon,\delta}$ during $(\tau_{k-1}^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta})$. If the distance between the cubes $H_{\boldsymbol{m}_k^{\varepsilon,\delta}}^{\varepsilon,\delta}$ and $H_{\boldsymbol{m}_{k-1}^{\varepsilon,\delta}}^{\varepsilon,\delta}$ is at least $3\sqrt{d}\varepsilon$, by continuity there exist $\tau_{k-1}^{\varepsilon,\delta} < s < t < \tau_k^{\varepsilon,\delta}$, such that

$$\left| x_s - x_{\tau_{k-1}^{\varepsilon,\delta}} \right| = \sqrt{d}\varepsilon, \; \left| x_t - x_{\tau_{k-1}^{\varepsilon,\delta}} \right| = 2\sqrt{d}\varepsilon,$$

and $[s,t] \subset x^{-1}\left(T^{\varepsilon,\delta}\right)$. Moreover, by the triangle inequality we have $|x_t - x_s| \geqslant \varepsilon$. Therefore, $x \in A^{\varepsilon,\delta}$. If the distance between $H_{\boldsymbol{m}_k^{\varepsilon,\delta}}^{\varepsilon,\delta}$ and $H_{\boldsymbol{m}_{k-1}^{\varepsilon,\delta}}^{\varepsilon,\delta}$ is strictly less than $3\sqrt{d}\varepsilon$, we know that $\left| x_u^{\varepsilon,\delta} - \varepsilon \boldsymbol{m}_{k-1}^{\varepsilon,\delta} \right| \leqslant 4\sqrt{d}\varepsilon$ for all $u \in \left(\tau_{k-1}^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta} \right)$. Since $\sup_{0 \leqslant u \leqslant 1} \left| x_u^{\varepsilon,\delta} - x_u \right| \geqslant 11\sqrt{d}\varepsilon$, there exists $u \in \left(\tau_{k-1}^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta} \right)$ such that

$$\left| x_u - \varepsilon \boldsymbol{m}_{k-1}^{\varepsilon,\delta} \right|, \; \left| x_u - \varepsilon \boldsymbol{m}_k^{\varepsilon,\delta} \right| \geqslant 7\sqrt{d}\varepsilon.$$

It follows again from continuity that there exist $\tau_{k-1}^{\varepsilon,\delta} < s < t < \tau_k^{\varepsilon,\delta}$ such that

$$\left| x_s - \varepsilon \boldsymbol{m}_{k-1}^{\varepsilon,\delta} \right| = 5\sqrt{d}\varepsilon, \; \left| x_t - \varepsilon \boldsymbol{m}_{k-1}^{\varepsilon,\delta} \right| = 6\sqrt{d}\varepsilon,$$

and $[s,t] \subset x^{-1}\left(T^{\varepsilon,\delta}\right)$. Therefore, $|x_s - x_t| \geqslant \varepsilon$ and we have $x \in A^{\varepsilon,\delta}$.

In the second case, there exist $\tau_{N^{\varepsilon,\delta}}^{\varepsilon,\delta} < s < t \leqslant 1$ such that

$$\left| x_s - \varepsilon \boldsymbol{m}_{N^{\varepsilon,\delta}}^{\varepsilon,\delta} \right| = \sqrt{d}\varepsilon, \; \left| x_t - \varepsilon \boldsymbol{m}_{N^{\varepsilon,\delta}}^{\varepsilon,\delta} \right| = 2\sqrt{d}\varepsilon,$$

and $[s,t] \subset x^{-1}(T^{\varepsilon,\delta})$. Again we have $|x_t - x_s| \geqslant \varepsilon$ and hence $x \in A^{\varepsilon,\delta}$.

Now for $\varepsilon_n = 1/n$, if we choose δ_n small enough such that $\mathbb{P}\left(A^{\varepsilon_n,\delta_n}\right) \leqslant \varepsilon_n^2$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left\{ x \in W : \sup_{0 \leqslant u \leqslant 1} |x_u^{\varepsilon_n,\delta_n} - x_u| \geqslant 11\sqrt{d}\varepsilon_n \right\}\right) \;\; \leqslant \;\; \sum_{n=1}^{\infty} \mathbb{P}\left(A^{\varepsilon_n,\delta(\varepsilon_n)}\right)$$
$$< \;\; \infty,$$

It follows from the Borel-Cantelli lemma that

$$\mathbb{P}\left(\limsup_{n \to \infty}\left\{ x \in W : \sup_{0 \leqslant u \leqslant 1} |x_u^{\varepsilon_n,\delta_n} - x_u| \geqslant 11\sqrt{d}\varepsilon_n \right\}\right) = 0,$$

and hence the uniform convergence (5.3) holds $\mathbb{P}$-almost surely. $\qquad\square$

Remark 5.2. From the previous proof, it is not hard to see that the result of Proposition 5.2 holds for all continuous stochastic processes starting at the origin whose law satisfies Assumption (B).

From now on, we will always assume that $\varepsilon_n = 1/n$, and take δ_n as in the previous proof.

5.3 A Variant of the Fréchet Distance on Path Space

Now we are coming to the last step of the proof of Theorem 3.1.

Under Assumption (A), (B), (C), what we have obtained so far is that there exists some $\mathbb{P}$-null set $\mathcal{N}$, such that for any path $x \in \mathcal{N}^c$, the signature $S(x)_{0,1}$ is well-defined, and for each $n \geqslant 1$, we can recover the ordered sequence of open cubes $H_z^{\varepsilon_n, \delta_n}$ visited by x from its signature. Moreover, the polygonal approximation $x^{\varepsilon_n, \delta_n}$ constructed before converges to x uniformly.

By possibly enlarging the $\mathbb{P}$-null set $\mathcal{N}$ (still a $\mathbb{P}$-null set), we are going to show that for any two paths $x, x' \in \mathcal{N}^c$, if $S(x)_{0,1} = S(x')_{0,1}$, then x and x' defer by a reparametrization $\sigma \in \mathcal{R}$ in the sense of Definition 3.1.

Now we introduce an equivalence relation "$\sim$" on W by

$$x \sim x' \iff (x_t)_{0 \leqslant t \leqslant 1} = \left(x'_{\sigma(t)} \right)_{0 \leqslant t \leqslant 1}, \quad \text{for some } \sigma \in \mathcal{R}.$$

Let $W/_\sim$ be the quotient space consisting of $\sim$-equivalence classes. For any $[x], [x'] \in W/_\sim$, define

$$d\left([x], [x']\right) = \inf_{\sigma \in \mathcal{R}} \sup_{t \in [0,1]} \left| x_t - x'_{\sigma(t)} \right|. \tag{5.6}$$

If we only assume that σ is non-decreasing, the function $d(\cdot, \cdot)$ is usually known as the Fréchet distance. It was originally introduced by Fréchet to study the shape of geometric spaces. Here we emphasize that σ is strictly increasing.

It is easy to see that $d(\cdot, \cdot)$ does not depend on the choice of representatives in the corresponding equivalence classes, and $d(\cdot, \cdot)$ is nonnegative and symmetric. Moreover, $d(\cdot, \cdot)$ satisfies the triangle inequality. In fact, for any $x, x', x'' \in W$ and $\sigma, \theta \in \mathcal{R}$, we have

$$\sup_{t \in [0,1]} \left| x_t - x''_{\sigma(t)} \right| \leqslant \sup_{t \in [0,1]} \left| x_t - x'_{\theta(t)} \right| + \sup_{t \in [0,1]} \left| x'_{\theta(t)} - x''_{\sigma(t)} \right|.$$

It follows that

$$\begin{aligned}
d\left([x], [x'']\right) &= \inf_{\sigma \in \mathcal{R}} \sup_{t \in [0,1]} \left| x_t - x''_{\sigma(t)} \right| \\
&\leqslant \sup_{t \in [0,1]} \left| x_t - x'_{\theta(t)} \right| + \inf_{\sigma \in \mathcal{R}} \sup_{t \in [0,1]} \left| x'_{\theta(t)} - x''_{\sigma(t)} \right| \\
&= \sup_{t \in [0,1]} \left| x_t - x'_{\theta(t)} \right| + d\left([x'], [x'']\right).
\end{aligned}$$

By taking infimum over $\theta \in \mathcal{R}$, we obtain the triangle inequality.

It should be pointed out that unlike the FrÃ©chet distance, $d(\cdot, \cdot)$ is not a metric on $W/_\sim$. For example, consider the case of $d = 1$. Let $x_t = t$, $t \in [0, 1]$, and

$$x'_t = \begin{cases} 2t, & t \in [0, \frac{1}{2}]; \\ 1, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then it is easy to see that $d([x], [x']) = 0$, but obviously x' is not a reparametrization of x in the sense of Definition 3.1. However, if we exclude paths with certain degeneracy, then on the corresponding quotient space $d(\cdot, \cdot)$ is indeed a metric.

Let D be the set of paths $x \in W$ such that there exist some $0 \leqslant s < t \leqslant 1$ with

$$x_u = x_s, \quad \forall u \in [s, t].$$

We first make an important remark that under Assumption (C), D is a $\mathbb{P}$-null set. To see this, let $\{H_n\}_{n \geqslant 1}$ be a covering of $\mathbb{R}^d$ consisting of open cubes, and for each n let ϕ_n be the differential one form associated with H_n according to Assumption (C). It follows that

$$D \subset \bigcup_{r_1, r_2 \in \mathbb{Q} \cap [0,1]} \bigcup_{n \geqslant 1} \left(\left\{ x \in W : \int_{r_1}^{r_2} \phi_n(dx_u) = 0 \right\} \bigcap A_{r_1, r_2}^{H_n} \right).$$

Therefore, by Assumption (C) we know that $\mathbb{P}(D) = 0$.

Now we have the following result.

Proposition 5.3. *Define the equivalence relation "$\sim$" on $W_0 = D^c \subset W$ as before, and let $W_0/_\sim$ be the corresponding quotient space. Then $d(\cdot, \cdot)$, defined in the same way as in (5.6), is a metric on $W_0/_\sim$.*

Proof. It suffices to show that, for any $x, x' \in W_0$, if

$$\inf_{\sigma \in \mathcal{R}} \sup_{t \in [0,1]} \left| x_t - x'_{\sigma(t)} \right| = 0, \tag{5.7}$$

then

$$x_t = x'_{\sigma(t)}, \quad \forall t \in [0, 1], \tag{5.8}$$

for some $\sigma \in \mathcal{R}$.

In fact, by (5.7), for any $n \geqslant 1$, there exists $\sigma_n \in \mathcal{R}$, such that

$$\left| x_t - x'_{\sigma_n(t)} \right| \leqslant \frac{1}{n}, \quad \forall t \in [0, 1]. \tag{5.9}$$

It follows from compactness, denseness, and a standard diagonal selection argument that we can find a subsequence $\{\sigma_{n_k}\}$ such that for any $r \in \mathbb{Q} \cap [0, 1]$,

$$\lim_{k \to \infty} \sigma_{n_k}(r) =: \widetilde{\sigma}(r)$$

exists.

Now define $\sigma : [0,1] \to [0,1]$ by

$$\sigma(t) = \begin{cases} \inf\{\widetilde{\sigma}(r): \ r > t, r \in \mathbb{Q}\bigcap[0,1]\}, & 0 \leqslant t < 1; \\ 1, & t = 1. \end{cases}$$

We want to show that $\sigma \in \mathcal{R}$ and that it satisfies (5.8).

(1) It is easy to see that σ is increasing. Let $0 \leqslant t < 1$. For any $\varepsilon > 0$, there exists some $r > t, r \in \mathbb{Q}\bigcap[0,1]$, such that

$$\sigma(t) \leqslant \widetilde{\sigma}(r) < \sigma(t) + \varepsilon.$$

Therefore, for any $t' \in (t,r)$, if we take some $r' \in \mathbb{Q}\bigcap[0,1]$ with $t' < r' < r$, then

$$\sigma(t) \leqslant \sigma(t') \leqslant \widetilde{\sigma}(r') \leqslant \widetilde{\sigma}(r) < \sigma(t) + \varepsilon.$$

It follows that σ is right continuous.

(2) σ is also left continuous.

In fact, assume on the contrary that for some $0 < t \leqslant 1$, $\sigma(t-) \neq \sigma(t)$. Fix any $\sigma(t-) < s < \sigma(t)$, and define for $k \geqslant 1$, $t_{n_k} = \sigma_{n_k}^{-1}(s)$. It follows that for any $r > t, r \in \mathbb{Q}\bigcap[0,1]$,

$$s < \sigma(t) \leqslant \widetilde{\sigma}(r).$$

Since $\lim_{k\to\infty} \sigma_{n_k}(r) = \widetilde{\sigma}(r)$, we know that when k is large enough, $s < \sigma_{n_k}(r)$, which is equivalent to $t_{n_k} < r$ for k large enough. Therefore, we have $\limsup_{k\to\infty} t_{n_k} \leqslant r$. But this is true for all $r > t, r \in \mathbb{Q}\bigcap[0,1]$, which implies that $\limsup_{k\to\infty} t_{n_k} \leqslant t$. On the other hand, for any $r < t, r \in \mathbb{Q}\bigcap[0,1]$, we have

$$\widetilde{\sigma}(r) \leqslant \sigma(r) \leqslant \sigma(t-) < s,$$

A similar argument yields that $\liminf_{k\to\infty} t_{n_k} \geqslant t$. Therefore, $\lim_{k\to\infty} t_{n_k}$ exists and is equal to t. Now from (5.9) we know that

$$|x_{t_{n_k}} - x'_s| \leqslant \frac{1}{n_k}, \ \forall k \geqslant 1,$$

and hence $x_t = x'_s$. But this is true for all $\sigma(t-) < s < \sigma(t)$, which contradicts the fact that $x' \in W_0$. Therefore, σ is left continuous. A similar argument also shows that $\sigma(0) = 0$.

(3) For any $r \in \mathbb{Q}\bigcap[0,1]$, $\sigma(r) = \widetilde{\sigma}(r)$.

In fact, it is obvious that $\sigma(r) \geqslant \widetilde{\sigma}(r)$. On the other hand, for any $t < r$ we have $\sigma(t) \leqslant \widetilde{\sigma}(r)$, and by the left continuity of σ we have $\sigma(r) \leqslant \widetilde{\sigma}(r)$.

(4) σ is strictly increasing.

In fact, if for some $0 \leqslant s < t \leqslant 1$, $\sigma(s) = \sigma(t)$, then σ remains constant over $[s,t]$. In particular, for any $r \in \mathbb{Q}\bigcap[s,t]$, from (5.9) and the previous step we have

$$x_r = x'_{\widetilde{\sigma}(r)} = x'_{\sigma(r)} = x'_{\sigma(s)},$$

which implies that x is constant over $[s,t]$, contradicting the fact that $x \in W_0$.

Now it is obvious that $\sigma \in \mathcal{R}$, and (5.8) follows. $\qquad\square$

From now on, we shall include D to the $\mathbb{P}$-null set $\mathcal{N}$.

Now we are in a position to complete the proof of Theorem 3.1.

Assume that $x, x' \in \mathcal{N}^c$ and $S(x)_{0,1} = S(x')_{0,1}$. For each $n \geqslant 1$, let $\left(\phi_{z_0}^{\varepsilon_n,\delta_n}, \cdots, \phi_{z_m}^{\varepsilon_n,\delta_n}\right)$ $\left(\left(\phi_{z_0}^{\varepsilon_n,\delta_n}, \cdots, \phi_{z'_{m'}}^{\varepsilon_n,\delta_n}\right)\right.$, respectively) be the unique maximal sequence of differential one forms along which the extended signature of x (x', respectively) is nonzero. It follows from Proposition 4.1 that $m = m'$ and $z_i = z'_i$ for $i = 0, \ldots, m$. Moreover, by Proposition 5.1 we know that

$$N^{\varepsilon_n,\delta_n}(x) = N^{\varepsilon_n,\delta_n}(x') = m,$$

and

$$\boldsymbol{m}_i^{\varepsilon_n,\delta_n}(x) = \boldsymbol{m}_i^{\varepsilon_n,\delta_n}(x') = z_i, \ \forall i = 0, \cdots, m.$$

It follows that in the quotient space $W/_\sim$, $[x^{\varepsilon_n,\delta_n}] = [(x')^{\varepsilon_n,\delta_n}]$, where $x^{\varepsilon_n,\delta_n}$ and $(x')^{\varepsilon_n,\delta_n}$ are the polygonal approximations of x and x' respectively. On the other hand, by Proposition 5.2 we know that

$$x^{\varepsilon_n,\delta_n} \to x, \ (x')^{\varepsilon_n,\delta_n} \to x',$$

under the uniform topology as $n \to \infty$. Therefore, by the triangle inequality of the distance function $d(\cdot, \cdot)$ we have $d([x], [x']) = 0$. Since $D \subset \mathcal{N}$, it follows from Proposition 5.3 that there exists $\sigma \in \mathcal{R}$, such that (5.8) holds.

Now the proof of Theorem 3.1 is complete.

6 A Fundamental Example: Gaussian Processes

As we remarked before, Assumption (A) and (B) are natural for a large class of stochastic processes. However, Assumption (C) is in general difficult to verify. In this section, as a fundamental example of Theorem 3.1, we are going to show that Assumption (A), (B), (C) hold for a class of Gaussian processes including fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge. The main idea of verifying Assumption (C) for Gaussian processes is to apply local regularity results for Gaussian functionals from the Malliavin calculus, based on pathwise integration by parts which is possible due to the regularity of sample paths and Cameron-Martin paths.

The class of Gaussian processes we shall study in this section is specified in the following.

Let $\mathbb{P}$ be the law of a centered, nondegenerate, continuous Gaussian process over $[0,1]$ starting at the origin with independent components. We assume that $\mathbb{P}$ satisfies the following conditions: there exists $H \in \left(\frac{1}{4}, 1\right)$ such that:

(G1) for all $\rho \in \left[\frac{1}{2H} \vee 1, 2\right)$, the ρ-variation of the covariance function (see [12], Definition 5.50) for each component of X is controlled by a 2D Hölder-dominated control (see [12], Definition 5.51);

(G2) there exists δ and $c_\delta > 0$, such that for all $|t - s| \leqslant \delta$ and $1 \leqslant i \leqslant d$, we have

$$\mathbb{E}\left[(X_t^i - X_s^i)^2\right] \geqslant c_\delta(t - s)^{2H};$$

(G3) the Cameron-Martin space $\mathcal{H}$ associated with $\mathbb{P}$ satisfies the property that

$$C_0^{1+H^-}([0,1]; \mathbb{R}^d) \subset \mathcal{H} \subset C_0^{q-var}([0,1]; \mathbb{R}^d), \ \forall q > \left(H + \frac{1}{2}\right)^{-1},$$

where $C_0^{1+H^-}([0,1]; \mathbb{R}^d)$ is the space of differentiable paths in W with Hölder continuous derivatives of any order smaller than H, and $C_0^{q-var}([0,1]; \mathbb{R}^d)$ is the space of paths in W with finite total q-variation.

Now we are going to prove our second main result, namely Theorem 3.2. Note that in this case the verification of Assumption (A) is a standard result for Gaussian rough paths according to (G1) (see [12], Theorem 15.33), and Assumption (B) is trivial. The main difficulty is the verification of Assumption (C).

For any open cube $H_{x_0,\eta}$ with center $x_0 = (x_0^1, \cdots, x_0^d) \in \mathbb{R}^d$ and edges of length 2η, we are going to construct a differential one form ϕ supported on the closure of $H_{x_0,\eta}$, such that for any $0 \leqslant s < t \leqslant 1$,

$$\mathbb{P}\left(\left\{x \in W : \int_s^t \phi(dx_u) = 0\right\} \cap A_{s,t}^{H_{x_0,\eta}}\right) = 0, \tag{6.1}$$

where $A_{s,t}^{H_{x_0,\eta}}$ is the set defined by (3.1). In other words, Assumption (C) holds.

Let $h(t) \in C_c^\infty(\mathbb{R}^1)$ be a function such that

$$\begin{cases} h(t) > 0, & t \in (-1,1); \\ h(t) = 0, & t \notin (-1,1), \end{cases}$$

and $h'(t)$ is everywhere nonzero in $(-1,1)$ except at $t = 0$. For example, the standard mollifier function

$$h(t) = \begin{cases} e^{\frac{-1}{1-|t|^2}}, & t \in (-1,1); \\ 0, & t \notin (-1,1), \end{cases}$$

will satisfy the properties.

Define a differential one form $\phi(x) = \sum_{i=1}^d \phi_i(x)dx^i$ on $\mathbb{R}^d$ by

$$\begin{aligned} \phi_1(x) &= h\left(\frac{x^1 - x_0^1}{\eta}\right) \cdots h\left(\frac{x^d - x_0^d}{\eta}\right) \exp\left(h^2\left(\frac{x^2 - x_0^2}{\eta}\right)\right), \ x \in \mathbb{R}^d, \\ \phi_i &= 0, \text{ for all } i = 2, \cdots, d. \end{aligned} \tag{6.2}$$

It is easy to see that the support of ϕ is exactly the closure of the $H_{x_0,\eta}$. Moreover, we have

$$\begin{aligned} \frac{\partial \phi_1}{\partial x^2}(x) &= \frac{1}{\eta}\left(\prod_{i \neq 2} h\left(\frac{x^i - x_0^i}{\eta}\right)\right) h'\left(\frac{x^2 - x_0^2}{\eta}\right) \\ &\quad \cdot \exp\left(h^2\left(\frac{x^2 - x_0^2}{\eta}\right)\right)\left(1 + 2h^2\left(\frac{x^2 - x_0^2}{\eta}\right)\right), \end{aligned}$$

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which is everywhere nonzero in $H_{x_0,\eta}$ except on the slice $\left\{ x \in H_{x_0,\eta} : \ x^2 = x_0^2 \right\}$.

To verify Assumption (C) for such differential one form ϕ, we need the following Lemma.

Lemma 6.1. *Fix $0 \leqslant s < t \leqslant 1$. Let f be a smooth function on $\mathbb{R}^d$ with compact support. Then there exists a $\mathbb{P}$-null set $\mathcal{N}_1$ such that for any $x \in (\mathcal{N}_1)^c$, if $\int_u^v f(x_r)dx_r^1 = 0$ for all u, v with $[u,v] \subset [s,t]$, then $f(x_u) = 0$ for all $u \in [s,t]$.*

Proof. Fix $\frac{1}{2H} < \rho < \frac{1}{H}$. According to (G1) and [12], Theorem 15.33, outside some $\mathbb{P}$-null set $\mathcal{N}_0'$, a sample path x admits a canonical lifting to a geometric 2ρ-rough path $\mathbf{X}$ as well as a $G^{\lfloor 2\rho \rfloor}(\mathbb{R}^d)$-valued $\frac{1}{2\rho}$-HÃ¶lder continuous path ($G^N(\mathbb{R}^d)$ is the free nilpotent group of step N over $\mathbb{R}^d$, see [12], Theorem 7.30). Since the path integral $\int_u^v f(x_r)dx_r^1$ can be regarded as the projection of the solution to the rough differential equation

$$
\begin{cases}
dx_r^1 = dx_r^1, \\
\cdots, \\
dx_r^d = dx_r^d, \\
dx_r^{d+1} = f(x_r^1, \cdots, x_r^d)dx_r^1
\end{cases}
$$

over $[u,v]$ with initial condition $(x_u^1, \cdots, x_u^d, x_u^{d+1}) = (x_u^1, \cdots, x_u^d, 0)$, according to [12], Corollary 10.15, we know that pathwisely

$$
\left| \int_u^v f(x_r)dx_r^1 - f(x_u)\mathbf{X}_{u,v}^{1;1} - \sum_{i=1}^d \frac{\partial f}{\partial x^i}(x_u)\mathbf{X}_{u,v}^{2;i,1} - \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}(x_u)\mathbf{X}_{u,v}^{3;i,j,1} \right|
$$
$$
\leqslant C_1 \|\mathbf{X}\|_{\frac{1}{2\rho}-\text{Höl};[u,v]}^{2\rho\theta} |u-v|^\theta,
$$

where $\theta > 1$ and C_1 is some positive constant depending only on ρ, θ and the uniform bounds on the derivatives of f. If $\int_u^v f(x_r)dx_r^1 = 0$, then we have

$$
\left| f(x_u) \left(x_v^1 - x_u^1 \right) \right|
$$
$$
\leqslant C_1 \|\mathbf{X}\|_{\frac{1}{2\rho}-\text{Höl};[u,v]}^{2\rho\theta} |u-v|^\theta + \|Df\|_\infty |\pi_2(\mathbf{X}_{u,v})| + \|D^2 f\|_\infty |\pi_3(\mathbf{X}_{u,v})| \quad (6.3)
$$

On the other hand, according to (G1) and [12], Proposition 15.19, Corollary 15.21 and Theorem 15.33, we know that

$$
\mathbb{E} \left| \pi_j(\mathbf{X}_{u,v}) \right|^2 \ \leqslant \ C_2 |u-v|^{j/\rho} \tag{6.4}
$$

for each level j, where C_2 is some positive constant depending only on ρ. Now we choose α, γ such that $H < \alpha < \gamma < \frac{1}{\rho}$. According to (G2) and (6.4), it follows from Borel-Cantelli's lemma that

$$
\mathcal{N}(u) \ := \ \left\{ x \in W : \left| x_{u+\frac{1}{2^n}}^1 - x_u^1 \right| \leqslant \frac{1}{2^{\alpha n}}, \text{ for infinitely many } n \right\}
$$
$$
\bigcup \left\{ x \in W : \left| \pi_2 \left(\mathbf{X}_{u,u+\frac{1}{2^n}} \right) \right| \geqslant \frac{1}{2^{\gamma n}}, \text{ for infinitely many } n \right\}
$$
$$
\bigcup \left\{ x \in W : \left| \pi_3 \left(\mathbf{X}_{u,u+\frac{1}{2^n}} \right) \right| \geqslant \frac{1}{2^{\gamma n}}, \text{ for infinitely many } n \right\}
$$

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is a $\mathbb{P}$-null set.

Let $x \in (\mathcal{N}_0' \bigcup \mathcal{N}(u))^c$. Then there exists some $N \geqslant 1$, such that

$$\left| x^1_{u+\frac{1}{2^n}} - x^1_u \right| > \frac{1}{2^{\alpha n}}, \ \left| \pi_2 \left(\mathbf{X}_{u,u+\frac{1}{2^n}} \right) \right| < \frac{1}{2^{\gamma n}}, \ \left| \pi_3 \left(\mathbf{X}_{u,u+\frac{1}{2^{n_k}}} \right) \right| < \frac{1}{2^{\gamma n}},$$

for all $n > N$. Therefore, by (6.3) with $v = u + \frac{1}{2^n}$, for any $n > N$ we have

$$|f(x_u)|$$
$$\leqslant \frac{1}{2^{n(\theta-\alpha)}} C_1 \|\mathbf{X}\|^{2\rho\theta}_{\frac{1}{2\rho}-\text{Höl};[0,1]} + \frac{1}{2^{n(\gamma-\alpha)}} \left(\|Df\|_\infty + \|D^2 f\|_\infty \right).$$

By taking $n \to \infty$, we have $f(x_u) = 0$.

Now the result follows easily if we take

$$\mathcal{N}_1 = \mathcal{N}_0' \bigcup \bigcup_{u \in \mathbb{Q} \bigcap [s,t]} \mathcal{N}(u).$$

$\square$

Remark 6.1. By the denseness argument, it is easy to see that the $\mathbb{P}$-null set $\mathcal{N}_1$ can be taken uniformly in s, t.

Now we are going to complete the proof of Theorem 3.2.

In what follows, for simplicity we will use Einstein's summation convention: repeated indices of superscript and subscript are automatically summed over from 1 to d.

Let $F(x) = \int_s^t \phi(dx_u) = \int_s^t \phi_i(x_u) dx^i_u$. It follows that $F \in \mathbb{D}^{\infty,\infty}$ in the sense of Malliavin. Since F is a random variable on the abstract Wiener space $(W, \mathcal{H}, \mathbb{P})$, it suffices to show that outside a $\mathbb{P}$-null set, for any $x \in A^{H_{x_0,\eta}}_{s,t}$ the Malliavin derivative $DF(x)$ is a nonzero element in the Cameron-Martin space $\mathcal{H}$. It will then follow from standard local regularity results from the Malliavin calculus (see for example [23], Theorem 2.1.1 and the remark on p.93) that the measure

$$\lambda(B) = \mathbb{P}\left(\{F \in B\} \cap A^{H_{x_0,\eta}}_{s,t} \right), \ B \in \mathcal{B}(\mathbb{R}^1),$$

is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^1$. In particular, (6.1) holds.

Let $\mathcal{N}_1$ be the null set in Lemma 6.1. We know that for $\mathbb{P}$-almost surely sample paths can be lifted as geometric p-rough paths for $1 < p < 4$ with $Hp > 1$, and according to (G3) we have $\mathcal{H} \subset C_0^{q-var}([0,1]; \mathbb{R}^d)$ for any $q > \left(H + \frac{1}{2} \right)^{-1}$. Obviously we can choose such p, q so that $\frac{1}{p} + \frac{1}{q} > 1$. Therefore, in the sense of Young's integrals we know that for any $x \in A^{H_{x_0,\eta}}_{s,t} \cap \mathcal{N}_1^c$ and $h \in \mathcal{H}$,

$$\langle DF(x), h \rangle_\mathcal{H} = \frac{d}{d\varepsilon}|_{\varepsilon=0} F(x + \varepsilon h)$$

$$= \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_s^t \phi_i(x_u + \varepsilon h_u) d(x^i_u + \varepsilon h^i_u)$$

$$= \int_s^t \frac{\partial \phi_i}{\partial x^j}(x_u) h^j_u dx^i_u + \int_s^t \phi_i(x_u) dh^i_u,$$

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where the interchange of differentiation and integration can be verified easily by the geometric rough path nature of x and the continuity of the integration map.

Integration by parts shows that

$$\int_s^t \phi_i(x_u)dh_u^i = \phi_i(x_t)h_t^i - \phi_i(x_s)h_s^i - \int_s^t h_u^i \frac{\partial \phi_i}{\partial x^j}(x_u)dx_u^j.$$

Therefore,

$$\langle DF(x), h \rangle_{\mathcal{H}^H} = (\phi_i(x_t)h_t^i - \phi_i(x_s)h_s^i) + \int_s^t \left(\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} \right)(x_u)h_u^j dx_u^i.$$

Let

$$Y_{u,j} = \int_s^u \left(\frac{\partial \phi_i}{\partial x^j} - \frac{\partial \phi_j}{\partial x^i} \right)(x_v)dx_v^i, \ u \in [0,1], \ j = 1, \cdots, d. \qquad (6.5)$$

It follows from integration by parts again that

$$\begin{aligned}
\langle DF(x), h \rangle_{\mathcal{H}^H} &= (\phi_i(x_t)h_t^i - \phi_i(x_s)h_s^i) + \int_s^t h_u^i dY_{u,i} \\
&= (\phi_i(x_t) + Y_{t,i})h_t^i - (\phi_i(x_s) + Y_{s,i})h_s^i - \int_s^t Y_{u,i}dh_u^i.
\end{aligned}$$

Now we define $h = (h^1, \cdots, h^d)$ by

$$h_u^i = \int_s^u (\phi_i(x_t) + Y_{t,i} - Y_{v,i})dv, \ u \in [0,1], \ i = 1, \cdots, d, \qquad (6.6)$$

then $h_s^i = 0$ for $i = 1, \ldots, d$. Technically if $s > 0$ we modify h^i smoothly on $[0, \frac{s}{2})$ so that $h_0^i = 0$ for all i. Note that the modification does not change the value of $\langle DF(x), h \rangle_{\mathcal{H}^H}$ as it depends only on the value of h on $[s, t]$. By the regularity of sample paths, it is easy to see that $h \in C_0^{1+H^-}([0,1]; \mathbb{R}^d)$, which is also in $\mathcal{H}$ according to (G3). Therefore,

$$\langle DF(x), h \rangle_{\mathcal{H}} = \sum_{i=1}^d \int_s^t (\phi_i(x_t) + Y_{t,i} - Y_{u,i})^2 du.$$

If $DF(x) = 0$, then $\langle DF(x), h \rangle_{\mathcal{H}} = 0$, which implies that for all $i = 1, \cdots, d$, and $u \in [s, t]$, $\phi_i(x_t) + Y_{t,i} - Y_{u,i} = 0$. It follows from taking $i = 2$ and our construction of ϕ that

$$\int_u^v \frac{\partial \phi_1}{\partial x^2}(x_r)dx_r^1 = 0, \ \forall [u, v] \subset [s, t].$$

Therefore, by Lemma 6.1 we have for all $u \in [s, t]$, $\frac{\partial \phi_1}{\partial x^2}(x_u) = 0$.

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On the other hand, since $x \in A_{s,t}^{H_{x_0,\eta}}$, there exists some $u \in (s,t)$ such that $x_u \in H_{x_0,\eta}$. From the construction of ϕ we have already seen that $\frac{\partial \phi_1}{\partial x^2}$ is everywhere nonzero in $H_{x_0,\eta}$ except on the "slice"

$$L_{x_0,\eta} = \{x \in H_{x_0,\eta} : \ x^2 = x_0^2\}.$$

Therefore, by continuity there exists some open interval $(u,v) \subset [s,t]$, such that $x_r \in L_{x_0,\eta}$ for all $r \in (u,v)$. But this implies that there exists some $r \in \mathbb{Q} \bigcap (s,t)$ such that $x_r^2 = x_0^2$. Since for any $r \in (0,1)$, the law of x_r is absolutely continuous with respect to the Lebesgue measure, we know that

$$\mathcal{N}_2 := \bigcup_{r \in Q \bigcap (0,1)} \{x_r^2 = x_0^2\}$$

is a $\mathbb{P}$-null set. By further removing $\mathcal{N}_2$, we will arrive at a contradiction. Therefore, for any $x \in A_{s,t}^H \bigcap \mathcal{N}_1^c \bigcap \mathcal{N}_2^c$, $DF(x)$ a nonzero element in $\mathcal{H}$.

Now the proof of Theorem 3.2 is complete.

In the rest of present paper we will consider three specific examples of Gaussian processes which all verify conditions (G1), (G2) and (G3): fractional Brownian motion with Hurst parameter $H > 1/4$, the Ornstein-Uhlenbeck process and the Brownian bridge.

6.1 Fractional Brownian Motion with Hurst Parameter $H > 1/4$

Let X be the d-dimensional fractional Brownian motion with Hurst parameter H for $H > \frac{1}{4}$. In other words, X is a Gaussian process starting at the origin with i.i.d. components, and the covariance function of X^i is given by

$$R^H(s,t) = \frac{1}{2} \left(s^{2H} + t^{2H} - |t-s|^{2H} \right), \ \ s,t \in [0,1].$$

In this case the parameter H in the conditions (G1), (G2) and (G3) is just the Hurst parameter. The verification of Condition (G1) is the content of [12], Proposition 15.5 if $H \in \left(\frac{1}{4}, \frac{1}{2} \right]$ (the case when $H > 1/2$ is trivial in the rough path setting), and (G2) follows from direct calculation. The verification of (G3) is contained in the following two lemmas.

Let $\mathcal{H}^H$ be the Cameron-Martin space associated with X.

Lemma 6.2. $\mathcal{H}^H$ contains $C_0^\alpha([0,1]; \mathbb{R}^d)$ for all $\alpha > H + \frac{1}{2}$.

Proof. We will assume $H \neq \frac{1}{2}$, as the result is well-known for Brownian motion. According to [9], Theorem 2.1, we have $\mathcal{H}^H = \mathcal{I}_{0+}^{H+\frac{1}{2}} \left(L^2 [0,1] \right)$, where

$$\mathcal{I}_{0+}^\alpha (f) (x) = \int_0^x f(t) (x-t)^{\alpha-1} \, dt$$

is the fractional integral operator.

If $0 < H < \frac{1}{2}$, from fractional calculus (see [24], p.233) we know that $\mathcal{I}_{0+}^{H+\frac{1}{2}}\left(L^2[0,1]\right)$ contains all α-Hölder continuous functions whenever $\alpha > H + \frac{1}{2}$. If $H > \frac{1}{2}$, by the fundamental theorem of calculus we know that $h \in \mathcal{I}_{0+}^{H+\frac{1}{2}}\left(L^2[0,1]\right)$ if and only if h is differentiable with derivative in $\mathcal{I}_{0+}^{H-\frac{1}{2}}\left(L^2[0,1]\right)$. Therefore, in both cases we have $\mathcal{H}^H$ containing $C_0^\alpha\left([0,1];\mathbb{R}^d\right)$ for all $\alpha > H + \frac{1}{2}$. $\qquad\square$

Lemma 6.3. *(1) (see [9], Theorem 2.1, Theorem 3.3 and [24], Theorem 3.6) If $H > \frac{1}{2}$, we have*

$$\mathcal{H}^H \subset C_0^H([0,1];\mathbb{R}^d). \tag{6.7}$$

(2) (see [10], Corollary 1) If $0 < H \leqslant \frac{1}{2}$, then for any $q > \left(H + \frac{1}{2}\right)^{-1}$, we have

$$\mathcal{H}^H \subset C_0^{q-var}([0,1];\mathbb{R}^d).$$

Remark 6.2. From the proof of Theorem 3.2 we can see that the embedding $\mathcal{H}^H \subset C_0^{q-var}([0,1];\mathbb{R}^d)$ is only used for making sense of path integrals in the sense of Young. Therefore, when $H > \frac{1}{2}$, (6.7) will obviously be sufficient for us to carry out all the calculations before as we are also in the setting of Young's integrals.

6.2 The Ornstein-Uhlenbeck Process

Let

$$X_t = \int_0^t e^{-(t-s)} dB_s, \ t \in [0,1],$$

be the standard Ornstein-Uhlenbeck process in $\mathbb{R}^d$ starting at the origin, where B is the standard d-dimensional Brownian motion.

We take $H = \frac{1}{2}$. The verification of Condition (G1) is contained in [12], p.405 and (G2) follows direct calculation. (G3) is a consequence of the fact that the Cameron-Martin space $\mathcal{H}^{\text{OU}}$ associated with X is the same as the one of Brownian motion with a different but equivalent inner product (see [25], Theorem 8.5.4).

Remark 6.3. The uniqueness of signature for the Ornstein-Uhlenbeck process is the direct consequence of the general result in [13], as it is the solution of a (hypo)elliptic SDE.

6.3 The Brownian Bridge

Finally we consider the Brownian bridge

$$X_t = B_t - tB_1, \ t \in [0,1].$$

In this case we also take $H = \frac{1}{2}$. Similar to the case of the Ornstein-Uhlenbeck process, (G1) and (G2) follows quite easily by direct calculations. However, (G3) is not satisfied as the Cameron-Martin space $\mathcal{H}^{\text{Bridge}}$ associated

with X is the one for Brownian motion with vanishing terminal condition: $h_1 = 0$ (see [25], p.334–335). Of course the embedding $\mathcal{H}^{\text{Bridge}} \subset C^{q-var}([0,1];\mathbb{R}^d)$ still holds for any $q > 1$.

The main trouble in the verification of Assumption (C) is that in the explicit construction of our Cameron-Martin path, the h given by (6.6) may not satisfy $h_1 = 0$. However, it is just a technical issue to overcome such difficulty.

Recall that we want to show $DF(x) \neq 0$ for $x \in A_{s,t}^{H_{x_0,\eta}}$, where $F = \int_s^t \phi(dx_u)$ and ϕ is the differential one form given by (6.2). From our proof before it is easy to see that everything follows in the same way if $t < 1$, since we can always modify h^i on $\left(\frac{t+1}{2}, 1\right]$ so that $h_1^i = 0$ and the value of $\langle DF(x), h \rangle$ will not change as it depends only on the value of h on $[s,t]$. Therefore, we only need to consider the case when $t = 1$.

On the path space W let $x \in A_{s,t}^{H_z^{\varepsilon,\delta}}$ and take $\varepsilon > 0$ such that $x|_{[1-\varepsilon,1]} \subset H_0^{\varepsilon,\delta}$ (this is possible since $x_1 = 0$). Define ϕ by (6.2) for the open cube $H_z^{\varepsilon,\delta}$, and define $Y_{u,j}$ by (6.5). Now we need to consider two cases.

(1) If $z \neq 0$, then

$$\phi_i(x_1) + Y_{1,i} - Y_{v,i} = 0, \ \forall v \in [1 - \varepsilon, 1],$$

since ϕ is supported on the closure of $H_z^{\varepsilon,\delta}$. Therefore, for any $h \in \mathcal{H}$,

$$\langle DF(x), h \rangle = \int_s^{1-\varepsilon} (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, dh_u^i.$$

To apply our previous argument, we just define h by (6.6) but modified on $\left(1 - \frac{\varepsilon}{2}, 1\right]$ so that $h_1^i = 0$, and the resulting h will be an element in $\mathcal{H}^{\text{Bridge}}$. By making use of Remark 6.1, the proof follows easily in the same way.

(2) If $z = 0$, based on our argument before, for any $\psi^i \in C^1([1 - \varepsilon, 1])$ $(i = 1, \cdots, d)$ with

$$\psi_{1-\varepsilon}^i = C_i := \int_s^{1-\varepsilon} (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, dv$$

and $\psi_1^i = 0$, the function

$$h_u^i = \begin{cases} \int_s^u (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, dv, & u \in [0, 1 - \varepsilon]; \\ \psi_u^i, & u \in [1 - \varepsilon, 1], \end{cases} \tag{6.8}$$

defines an element $h \in \mathcal{H}^{\text{Bridge}}$. It follows that

$$\begin{aligned} \langle DF(x), h \rangle &= \sum_{i=1}^d \int_s^{1-\varepsilon} (\phi_i(x_1) + Y_{1,i} - Y_{v,i})^2 \, dv \\ &\quad + \sum_{i=1}^d \int_{1-\varepsilon}^1 (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, d\psi_v^i. \end{aligned}$$

Now we take ψ^i of the form

$$\psi_u^i = C_i - \int_{1-\varepsilon}^u \xi_v^i dv, \ u \in [1-\varepsilon, 1],$$

where $\xi^i \in C([1-\varepsilon, 1])$ with $\int_{1-\varepsilon}^1 \xi_v^i dv = C_i$. If $\langle DF(x), h \rangle = 0$, then we have

$$\sum_{i=1}^d \int_s^{1-\varepsilon} (\phi_i(x_1) + Y_{1,i} - Y_{v,i})^2 \, dv - \sum_{i=1}^d \int_{1-\varepsilon}^1 (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, \xi_v^i dv = 0.$$

It follows that for any $\zeta^i \in C([1-\varepsilon], 1)$ with $\int_{1-\varepsilon}^1 \zeta_v^i dv = 0$, we have

$$\sum_{i=1}^d \int_{1-\varepsilon}^1 (\phi_i(x_1) + Y_{1,i} - Y_{v,i}) \, \zeta_v^i dv = 0,$$

which by an elementary argument implies that

$$\phi_i(x_1) + Y_{1,i} - Y_{v,i} = \text{const.}, \ \forall v \in [1-\varepsilon, 1] \text{ and } 1 \leqslant i \leqslant d.$$

It follows from taking $i = 2$ that

$$\int_u^v \frac{\partial \phi_1}{\partial x^2}(x_r) dx_r^1 = 0, \ \forall [u, v] \subset [1-\varepsilon, 1].$$

Now the proof follows again by making use of Remark 6.1 and the fact that $x|_{[1-\varepsilon, 1]} \subset H_0^{\varepsilon, \delta}$.

Remark 6.4. By the same argument with a technical modification of ψ so that the h defined by (6.8) is regular enough to lie in the Cameron-Martin space, the result holds for general Gaussian bridge processes

$$X_t = G_t - tG_1, \ t \in [0, 1],$$

as long as the underlying Gaussian process G itself satisfies conditions (G1), (G2) and (G3).

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