THE TAIL ASYMPTOTICS OF THE BROWNIAN SIGNATURE

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Abstract. We prove that the order-k multiple Stratonovitch integrals of Brownian motion, suitably normalised, converge to a deterministic limit as k tends to infinity. The proof relies on the hyperbolic development of Brownian sample paths as well as some new results on multiple integrals of rough paths. We discuss a number of open problems in relation to the limiting behaviour of multiple integrals.

1. Introduction

In this article, we aim to prove the following limit theorem for multiple Stratonovitch integrals:

Theorem 1. *Let $(W^1(\omega),\ldots,W^n(\omega))$ be a standard Brownian motion on $\mathbb{R}^n$ equipped with the Euclidean metric. Let $\|\cdot\|$ be the projective tensor norm over tensor products of $\mathbb{R}^n$. Let $\circ$ denote Stratonovitch integration. Then there exists a deterministic constant C such that*

$$\frac{1}{2}(n-1)\leq C\leq\frac{25}{8}n^2$$

and almost surely

$$\limsup_{k\to\infty}\big((\frac{k}{2})!\|\int_{0<s_1<\ldots<s_k<1}\circ\mathrm{d}W_{s_1}\otimes\ldots\otimes\circ\mathrm{d}W_{s_k}\|\big)^{\frac{2}{k}}=C.$$

Our motivation comes from studying the relationship between a path $X:[0,1]\to\mathbb{R}^n$ and its formal series of iterated integrals

$$S(X)=(1,\int_0^1\mathrm{d}X_{s_1},\int_0^1\int_0^{s_2}\mathrm{d}X_{s_1}\otimes\mathrm{d}X_{s_2},\int_0^1\int_0^{s_3}\int_0^{s_2}\mathrm{d}X_{s_1}\otimes\mathrm{d}X_{s_2}\otimes\mathrm{d}X_{s_3},\ldots).$$

The formal series of iterated integrals was first systematically studied by K. T. Chen [3]. The formal series is interesting partly because it is a homomorphism with respect to concatenation, or more specificly that

$$S(X|_{[s,u]})\otimes S(X|_{[u,t]})=S(X|_{[s,t]}),$$

a identity now sometimes known as *Chen*'s identity. The more recent interests in the formal series comes from its role in rough path theory, where the formal series is commonly known as signature. In the rough path context, Hambly and Lyons [7] took Chen's work much further and fully characterised all the bounded variation paths whose signature is the identity sequence

$$(1,0,0\ldots)$$

and showed that there is a unique shortest path with a given signature. These results have natural generalisation in the context of rough paths [2]. Hambly-Lyons' results were in fact quantitative and in particular gives a natural formula for recovering the length of a C^3 path from its signature. More precise, let

$$\triangle_k(0,t) = \{(s_1, \ldots, s_k) : 0 < s_1 < \ldots < s_k < t\}.$$

Hambly-Lyons showed that if X is a C^3 path with respect to the unit speed parametrisation, then

$$\text{Length}(X) = \limsup_{n\to\infty} \|k! \int_{\triangle_k(0,t)} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k}\|^{\frac{1}{k}}, \tag{1.1}$$

where $\|\cdot\|$ will denote the projective tensor norm (defined below in section 2). The key technique Hambly-Lyons used is to develop the path X onto a hyperbolic space and uses the negative curvature. This approach is natural partly because the set of all signatures has a natural hyperbolic structure [2]. The result was extended by Lyons and Xu in [12] to C^1 paths under the unit speed parametrisation, also using the development of the path to the hyperbolic space. Since both sides of the equation (1.1) makes sense for bounded variation paths, it is natural to to ask if (1.1) holds for bounded variation path. However, this has been a surprisingly challenging problem, partly because once the derivative of a path is discontinuous, the hyperbolic development techniques require major modification. This article studies an analogue of (1.1) for the sample paths of Brownian motion, where the integration is defined in the sense of Stratonovitch (or equivalently as geometric rough paths). More precisely, we would like to study for a multidimensional Brownian motion W, the limit

$$\limsup_{n\to\infty} \|(\frac{k}{2})! \int_{\triangle_k(0,t)} \circ\mathrm{d}W_{s_1} \otimes \ldots \otimes \circ\mathrm{d}W_{s_k}\|^{\frac{2}{k}}. \tag{1.2}$$

The different $(\frac{k}{2})!$ normalisation is necessary to ensure we have a non-trivial limit. Indeed, Lyons [10] showed that for all $p > 2$, we have almost surely

$$\|\int_{\triangle_k(0,t)} \circ\mathrm{d}W_{s_1} \otimes \ldots \otimes \circ\mathrm{d}W_{s_k}\| \leq \frac{C(\omega)^k}{(k/p)!}, \tag{1.3}$$

and this motivates our $(\frac{n}{2})!$normalisation factor.

Given Hambly-Lyons' result (1.1), one might expect that the limit (1.2) to be a random variable. It is apriori unclear whether the limit (1.2) is even finite, as Lyons' estimate (1.2) holds only for $p > 2$ while Brownian sample paths have infinite 2-variation almost surely. However, we are able to show that almost surely the limit exists and is deterministic. Moreover, the deterministic limit has non-trivial upper and lower bound in terms of the quadratic variation of Brownian motion. In one dimension, our main result Theorem 1 is trivially true since in this case

$$\int_{\triangle_k(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_k} = \frac{(W_1 - W_0)^k}{k!}$$

and hence the limit (1.2) will be trivially zero.

We use hyperbolic development, similar to that in [12] and [7]. However, our calculation diverges early on from the previous work, which reflects the use of martingales instead of deterministic classical calculus computations. The upper bound is much easier and is a simple manipulation of an L^2 estimate of multiple

Stratonovitch integrals of Brownian motion due to G. Ben-Arous [1]. We will also use, at key places, some new facts about rough paths that we shall also prove.

Theorem 1 give rise to many interesting questions:

(1) What is the exact value of C? This problem is equivalent to computing exactly the expectation

$$\mathbb{E}\big(\|\int_{\triangle_k(0,t)} \circ \mathrm{d}W_{s_1} \otimes \ldots \otimes \circ \mathrm{d}W_{s_k}\|^{\frac{2}{k}}\big).$$

(2) Does Theorem 1 hold for other processes? It seems that our proof should carry forward to Lévy processes. The more interesting question is whether Theorem 1 holds for Markov processes and Gaussian processes, such as the fractional Brownian motion? This is less clear. This will depend on general type of 0-1 laws available to those processes.

(3) Does Theorem 1 hold deterministically for rough paths? If so what would C be? It is known that for C^1 paths parametrised under unit speed, C is the length of the path. However, even for general bounded variation path, this is not a mathematically known fact.

(4) Is it possible to derive the rate of convergence in the lim sup in Theorem 1? Are there central-limit type theorems or law of iterated logarithms? At the moment it is unclear even what would the statement be like.

We would like to also mention that there are major recent progress on recovering not just the length but the entire path in [13] and [6].

The plan for the rest of the paper is as follows. In section 2, we recalled basic properties of signature that we shall use. In section 3, we prove the lower bound in our main result. In section 4, we prove the upper bound in our main result. In section 5, we prove that the limit is deterministic. In section 6, we proved new basic proerties about the limit of signature towards answer some of the open problems stated above.

2. Basics of signature

Let

$$T^{(k)}(\mathbb{R}^n) = 1 \oplus \mathbb{R}^n \oplus \ldots \oplus (\mathbb{R}^n)^{\otimes k}.$$

Let $T((\mathbb{R}^n))$ be the set of all formal sequences of tensors

$$(a_0, a_1, a_2 \ldots),$$

with $a_i \in (\mathbb{R}^n)^{\otimes i}$. We will equip the space $\mathbb{R}^n$ with the Euclidean 2-norm and the tensor power $(\mathbb{R}^n)^{\otimes k}$ with the projective tensor norm $\|\cdot\|$ defined by

$$\|\mathbf{v}\| = \inf\{\sum_{j=1}^{M} \|v_1^{(j)}\| \ldots \|v_k^{(j)}\| : \mathbf{v} = \sum_{j=1}^{M} v_1^{(j)} \otimes \ldots \otimes v_k^{(j)}\ v_i^{(j)} \in \mathbb{R}^n, \forall i,j\}.$$

Let $\triangle = \{(s,t) : 0 \leq s \leq t \leq 1\}$. Let $p \geq 1$. Let $\lfloor p \rfloor$ denote the biggest integer j such that $j \leq p$. Define the p-variation metric between two functions $X, Y : \triangle \to T^{(\lfloor p \rfloor)}(\mathbb{R}^n)$ so that if

$$\begin{aligned} X_{s,t} &= (1, X^1_{s,t}, \ldots, X^{\lfloor p \rfloor}_{s,t}) \\ Y_{s,t} &= (1, Y^1_{s,t}, \ldots, Y^{\lfloor p \rfloor}_{s,t}), \end{aligned}$$

then

$$d_p(X,Y) = \sup_{\mathcal{P}} \max_{1\le k\le \lfloor p\rfloor} \Big(\sum_{i=0}^{r-1} \|X^k_{t_i,t_{i+1}} - Y^k_{t_i,t_{i+1}}\|^{\frac{p}{k}}\Big)^{\frac{k}{p}},$$

where the supremum is taken over all partitions $(t_0 < t_1 < \ldots < t_r)$. For each smooth path X, define

$$\mathbb{X}^k_{s,t} = \int_{\triangle_k(s,t)} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k}.$$

The closure of the space

$$\{(1, \mathbb{X}^1_{s,t}, \ldots, \mathbb{X}^{\lfloor p\rfloor}_{s,t}) : X \text{ is a smooth path}\}$$

under the p-variation metric d_p is called the space of geometric rough paths. For example (c.f. [14]), if W is a multidimensional Brownian motion and $\circ$ denote the Stratonovitch integral, then

$$(s,t) \to (1, \int_s^t \circ \mathrm{d}W_{s_1}, \int_s^t \circ \mathrm{d}W_{s_1} \otimes \circ \mathrm{d}W_{s_2})$$

can be defined almost surely and is a geometric rough path. Therefore, all the results below on geometric rough paths would apply to Stratonovitch integration. For geometric rough paths, there is a canonical way of defining the iterated integrals of the geometric rough paths of all orders, through the following Lyons' extension Theorem [10]. We state a weaker form of the theorem that we need.

Theorem 2. *Let $(s,t) \to \mathbb{X}_{s,t} \in T^{(\lfloor p\rfloor)}(\mathbb{R}^n)$ be a p-geometric rough path. Then there exists a unique extension of $\mathbb{X}$ to $T((\mathbb{R}^n))$, denoted as $(s,t) \to S(\mathbb{X})_{s,t}$ so that*

1. *For all $s \le u \le t$,*

$$S(\mathbb{X})_{s,u} \otimes S(\mathbb{X})_{u,t} = S(\mathbb{X})_{s,t};$$

2. *If $S(\mathbb{X})_{s,t} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t}, \ldots)$, then for all j,*

$$\sup_{\mathcal{P}} \sum_{i=0}^{r-1} \|\mathbb{X}^j_{t_i,t_{i+1}}\|^{\frac{p}{j}} < \infty$$

where the supremum is taken over all partitions $\mathcal{P} = (t_0 < \ldots < t_r)$ or $[0,1]$.

Remark 3. For a geometric rough path $\mathbb{X}$, we will define the iterated integrals

$$\int_{\triangle_k(0,t)} \mathrm{d}\mathbb{X}_{s_1} \otimes \ldots \otimes \mathrm{d}\mathbb{X}_{s_k}$$

to be the tensor $\mathbb{X}^k_{0,t}$ that appeared in the extension Theorem 2.

3. Lower bound

Let $\circ$ be Stratonovitch and $\bullet$ be Itô integration. Let W_t be a n-dimensional standard Brownian motion and let

$$\begin{aligned} \mathrm{d}X^\lambda_t &= \lambda \sum_{i=1}^n (Y^i_t)^\lambda \circ \mathrm{d}W^i_t,\ X^\lambda_0 = 1 \\ \mathrm{d}(Y^i_t)^\lambda &= \lambda X^\lambda_t \circ \mathrm{d}W^i_t,\ Y^i_0 = 0. \end{aligned} \tag{3.1}$$

For $x, y \in \mathbb{R}^n$ and

$$\begin{aligned} x &= (x^1, \ldots, x^n) \\ y &= (y^1, \ldots, y^n); \end{aligned}$$

we have

$$\langle x, y \rangle = \sum_{i=1}^{n} x^i y^i.$$

Lemma 4. *(Explicit representation) Let X^λ and $((Y^i)^\lambda)_{i=1}^n$ be defined as in (3.1). For all $t \geq 0$,*

$$\begin{aligned} X_t^\lambda &= \sum_{k=0}^{\infty} \lambda^{2k} \int_{\triangle_{2k}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \left\langle \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_{s_{2k}} \right\rangle \\ Y_t^\lambda &= \sum_{k=0}^{\infty} \lambda^{2k+1} \int_{\triangle_{2k+1}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \left\langle \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_{s_{2k}} \right\rangle \circ \mathrm{d}W_{s_{2k+1}}. \end{aligned}$$

Proof. We first prove that

$$\begin{aligned} \alpha_t &= \sum_{k=0}^{\infty} \lambda^{2k} \int_{\triangle_{2k}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \left\langle \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_{s_{2k}} \right\rangle \\ \beta_t &= \sum_{k=0}^{\infty} \lambda^{2k+1} \int_{\triangle_{2k+1}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \left\langle \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_{s_{2k}} \right\rangle \circ \mathrm{d}W_{s_{2k+1}} \end{aligned}$$

is a solution to the system of differential equations (3.1). As the system (3.1) is linear, any solution must be unique (see [10]).

Note that

$$\begin{aligned} \mathrm{d}\alpha_t &= \sum_{k=1}^{\infty} \lambda^{2k} \left\langle \int_{\triangle_{2k-1}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_t \right\rangle \\ &= \sum_{k=0}^{\infty} \lambda^{2k+2} \left\langle \int_{\triangle_{2k+1}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \circ \mathrm{d}W_{s_{2k+1}}, \circ \mathrm{d}W_t \right\rangle \\ &= \lambda \langle \beta_t, \circ \mathrm{d}W_t \rangle. \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{d}\beta_t &= \sum_{k=0}^{\infty} \lambda^{2k+1} \int_{\triangle_{2k}(0,t)} \langle \circ \mathrm{d}W_{s_1}, \circ \mathrm{d}W_{s_s} \rangle \ldots \left\langle \circ \mathrm{d}W_{s_{2k-1}}, \circ \mathrm{d}W_{s_{2k}} \right\rangle \circ \mathrm{d}W_t \\ &= \lambda \alpha_t \circ \mathrm{d}W_t. \end{aligned}$$

Therefore, $\alpha_t = X_t^\lambda$ and $\beta_t = Y_t^\lambda$. □

Lemma 5. *(Hyperbolic length and signature) Let X^λ and Y^λ be defined as in (3.1). Then let $\|\cdot\|$ denote the projective tensor norm. Let*

$$\tilde{L}_t = \limsup_j \|(\frac{j}{2})! \int_{\triangle_j(0,t)} \circ \mathrm{d}W_{s_1} \otimes \ldots \otimes \circ \mathrm{d}W_{s_j} \|^{\frac{2}{j}}.$$

Then for any sequence $\lambda_m \to \infty$,

$$\limsup_{m \to \infty} \frac{1}{\lambda_m^2} \log X_t^{\lambda_m} \leq \tilde{L}_t.$$

Proof. We will use the following property of the projective norm $\|\cdot\|$ (see (2.3) in [9]): for any $2l$-linear functional B such that for all $v_1 \ldots, v_{2l} \in \mathbb{R}^n$,

$$\big|B(v_1,\ldots,v_{2l})\big| \leq \|v_1\|\ldots\|v_{2l}\| \tag{3.2}$$

, we have that for all $\mathbf{v} \in (\mathbb{R}^n)^{\otimes 2l}$

$$|B(\mathbf{v})| \leq \|\mathbf{v}\|.$$

This follows directly from the definition of projective tensor product described in section 2. In particular, as the functional

$$B\big(v_1 \otimes \ldots \otimes v_{2l}\big) = \langle v_1, v_2\rangle \ldots \langle v_{2l-1}, v_{2l}\rangle$$

does satisfy (3.2), and hence we have that

$$B\Big(\int_{\triangle_{2l}(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_{2l}}\Big) \leq \|\int_{\triangle_{2l}(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_{2l}}\|.$$

Let

$$\tilde{L}(j)_t = \sup_{l\geq 2j}((\frac{l}{2})!\|\int_{\triangle_l(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_l}\|)^{\frac{2}{l}}.$$

Using the explicit representation for $X_t^{\lambda_m}$, we have

$$\begin{aligned}
X_t^{\lambda_m} &\leq \sum_{k=0}^{\infty} \lambda_m^{2k} \Big|\int_{\triangle_{2k}(0,t)} \langle \circ\mathrm{d}W_{s_1}, \circ\mathrm{d}W_{s_s}\rangle \ldots \big\langle \circ\mathrm{d}W_{s_{2k-1}}, \circ\mathrm{d}W_{s_{2k}}\big\rangle \Big| \\
&\leq \sum_{k=0}^{\infty} \lambda_m^{2k} \|\int_{\triangle_{2k}(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_{2k}}\| \\
&\leq \sum_{k=0}^{j-1} \lambda_m^{2k} \|\int_{\triangle_{2k}(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_{2k}}\| + \sum_{k=j}^{\infty} \frac{\lambda_m^{2k}\big[\tilde{L}(j)_t\big]^k}{k!} \qquad (3.3) \\
&= \exp(\lambda^2 \tilde{L}(j)_t) + \sum_{k=0}^{j-1} \lambda_m^{2k} a_k, \qquad (3.4)
\end{aligned}$$

with

$$a_k = \|\int_{\triangle_{2k}(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_{2k}}\| - \frac{\tilde{L}(j)_t^k}{k!}$$

Note that for any sequence λ_m such that $\lambda_m \to \infty$, since $\tilde{L}(j)_t \geq 0$, we have

$$\lim_{m\to\infty} \frac{1}{\lambda_m^2} \log\big(\sum_{k=0}^{j-1} \lambda_m^{2k} a_k + \exp(\lambda_m^2 \tilde{L}(j)_t)\big) = \tilde{L}(j)_t.$$

Therefore, from (3.3) we have

$$\limsup_m \frac{1}{\lambda_m^2} \log(X_t^{\lambda_m}) \leq \tilde{L}(j)_t.$$

As this holds for all j, we may take $j \to \infty$ to obtain

$$\begin{aligned}
\limsup_m \frac{1}{\lambda_m^2} \log(X_t^{\lambda_m}) &\leq \lim_{j\to\infty} \tilde{L}(j)_t \\
&= \tilde{L}_t.
\end{aligned}$$

□

Lemma 6. *Let X^λ and Y^λ be defined as in (3.1). Then for all t,*

$$(X_t^\lambda)^2 - \langle Y_t^\lambda, Y_t^\lambda \rangle = 1.$$

Proof. Note that

$$X_t^\lambda \circ \mathrm{d}X_t^\lambda = \lambda \sum_{i=1}^n X_t^\lambda (Y_t^i)^\lambda \circ \mathrm{d}W_t^i$$

and

$$(Y_t^i)^\lambda \circ \mathrm{d}(Y_t^i)^\lambda = \lambda X_t (Y_t^i)^\lambda \circ \mathrm{d}W_t^i.$$

Therefore,

$$X_t^\lambda \circ \mathrm{d}X_t^\lambda - \sum_{i=1}^n (Y_t^i)^\lambda \circ \mathrm{d}(Y_t^i)^\lambda = 0.$$

Integrating gives

$$\frac{1}{2}(X_t^\lambda)^2 - \frac{1}{2}\langle Y_t^\lambda, Y_t^\lambda \rangle = C$$

for some constant C independent of t. Putting $t = 0$ and substituting in the initial conditions for X_t^λ and Y_t^λ, we have

$$(X_t^\lambda)^2 - \langle Y_t^\lambda, Y_t^\lambda \rangle = 1.$$

□

Corollary 7. *For all λ, $X_t^\lambda \geq 1$.*

Proof. Lemma 6 gives in particular that

$$(X_t^\lambda)^2 \geq \sum_{i=1}^n (Y_t^i)^2 + 1.$$

This implies that $|X_t^\lambda| \geq 1$ and that $X_t^\lambda \neq 0$ for all t. Since $X_0^\lambda = 1 > 0$ and X_t^λ is continuous, $X_t^\lambda \neq 0$ implies that $X_t^\lambda \geq 0$ for all t. Therefore, $X_t^\lambda = |X_t^\lambda| \geq 1$. □

Lemma 8. *Let X^λ and Y^λ be defined as in (3.1), then for all $\mu > 0$,*

$$\mathbb{E}\big((X_t^\lambda)^{-\mu}\big) \leq \exp\big(-\frac{\lambda^2 \mu t}{2}(n-1-\mu)\big).$$

Proof. By Itô-Stratonovitch conversion,

$$\begin{aligned} \mathrm{d}X_t^\lambda &= \lambda \sum_{i=1}^n (Y_t^i)^\lambda \bullet \mathrm{d}W_t^i + \lambda \frac{1}{2} \sum_{i=1}^n \mathrm{d}\left[(Y_t^i)^\lambda, W_t^i\right] \\ &= \lambda \sum_{i=1}^n (Y_t^i)^\lambda \bullet \mathrm{d}W_t^i + \frac{\lambda^2}{2} \sum_{i=1}^n X_t^\lambda \mathrm{d}t \\ &= \lambda \sum_{i=1}^n (Y_t^i)^\lambda \bullet \mathrm{d}W_t^i + \frac{\lambda^2}{2} n X_t^\lambda \mathrm{d}t. \end{aligned}$$

As $X_t^\lambda \geq 1$ for all t, $(X_t^\lambda)^{-\mu}$ is well defined. Moreover, by Itô's formula,

$$\begin{aligned} \mathrm{d}(X_t^\lambda)^{-\mu} &= -\mu\lambda\big(X_t^\lambda\big)^{-\mu-1}\big[\sum_{i=1}^n (Y_t^i)^\lambda \bullet \mathrm{d}W_t^i + \frac{\lambda}{2} n X_t^\lambda \mathrm{d}t\big] \\ &\quad + \frac{\mu(\mu+1)}{2}\lambda^2 (X_t^\lambda)^{-\mu-2} \langle Y_t, Y_t \rangle \mathrm{d}t. \end{aligned}$$

By Lemma 6,

$$\langle Y_t, Y_t\rangle = X_t^2 - 1$$

and taking expectation,

$$\begin{aligned}\mathbb{E}(X_t^\lambda)^{-\mu} &= 1-\frac{\lambda^2\mu(n-1-\mu)}{2}\int_0^t \mathbb{E}(X_u^\lambda)^{-\mu}\mathrm{d}u \\ &\quad -\frac{\mu(\mu+1)}{2}\lambda^2\int_0^t \mathbb{E}\big[(X_u^\lambda)^{-\mu-2}\big]\mathrm{d}u.\end{aligned}$$

The interchange of expectation and integral is permitted as $(X_t^\lambda)^{-\mu}\le 1$. Therefore, $\mathbb{E}(X_t^\lambda)^{-\mu}$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(X_t^\lambda)^{-\mu}\le -\frac{\lambda^2\mu(n-1-\mu)}{2}\mathbb{E}(X_t^\lambda)^{-\mu}.$$

Gronwall's Lemma gives that

$$\mathbb{E}(X_t^\lambda)^{-\mu}\le \exp\big[-\frac{\lambda^2\mu t}{2}(n-1-\mu)\big].$$

□

Lemma 9. *Let $n\ge 2$ and $s>t$. For all $\mu>0$, for almost all ω, there exists a $M(\omega,\mu)>0$ such that for all $m\ge M(\omega,\mu)$,*

$$X_t^m\ge \exp(\frac{m^2 s}{2}(n-1-\mu)).$$

Proof. For any $K>0$, by Chebyshev's inequality

$$\begin{aligned}\mathbb{P}(X_t^\lambda\le K) &= \mathbb{P}((X_t^\lambda)^{-\mu}\ge K^{-\mu}) \\ &\le K^\mu\exp(-\frac{\lambda^2\mu t}{2}(n-1-\mu)).\end{aligned}$$

Let $s<t$. Taking $K=\exp(\frac{\lambda^2 s}{2}(n-1-\mu))$. Then we see that

$$\mathbb{P}(X_t^\lambda\le \exp(\frac{\lambda^2 s}{2}(n-1-\mu)))\le \exp(-\frac{\lambda^2(t-s)}{2}(n-1-\mu)).$$

By Borel-Cantelli Lemma

$$\mathbb{P}(X_t^m\le \exp(\frac{m^2 s}{2}(n-1-\mu))\ \text{ for infinitely many } m\in\mathbb{N})=0$$

Therefore, for almost all ω, there exists $M(\omega)$ such that for all $m\ge M(\omega)$,

$$X_t^m\ge \exp(\frac{m^2 s}{2}(n-1-\mu)).$$

□

Corollary 10. *For almost all ω,*

$$\tilde{L}_t\ge \frac{t}{2}(n-1).$$

Proof. If $n=1$, then the Corollary is trivial. By Lemma 5, for almost all ω,

$$\tilde{L}_t\ge \limsup_m \frac{1}{m^2}\log X_t^m.$$

Hence by Lemma 9, for almost all ω,

$$\tilde{L}_t\ge \frac{s}{2}(n-1-\mu). \tag{3.5}$$

As this holds for all rational number $s < t$ and $\mu > 0$, we can let s tends to t and μ tends to 0 by enlarging the null set to become another null set so that the inequality (3.5) holds with $s = t$ and $\mu = 0$. □

4. Upper bound

Lemma 11. *Let W be n-dimensional standard Brownian motion and*

$$\mathbb{W}_{0,t}^k = \int_{\triangle_k(0,t)} \mathrm{d}W_{s_1} \otimes \ldots \otimes \mathrm{d}W_{s_k}.$$

Then almot surely,

$$\limsup_k \big((\frac{k}{2})!\|\mathbb{W}_{0,t}^k\|\big)^{\frac{2}{k}} \leq \frac{25}{8} n^2 t.$$

Proof. Let W^{i_j} denote the i_j-th component of W and define

$$\mathbb{W}_{0,t}^{i_1,\ldots,i_k} = \int_{\triangle_k(0,t)} \mathrm{d}W_{s_1}^{i_1} \ldots \mathrm{d}W_{s_k}^{i_k}.$$

If $\{e_1, \ldots e_n\}$ is the standard basis for $\mathbb{R}^n$,

$$\begin{aligned} \|\mathbb{W}_{0,t}^k\| &= \|\sum_{i_1,\ldots,i_k} \mathbb{W}_{0,t}^{i_1,\ldots,i_k} e_{i_1} \otimes \ldots \otimes e_{i_k}\| \\ &\leq \sum_{i_1,\ldots,i_k} \big|\mathbb{W}_{0,t}^{i_1,\ldots,i_k}\big|. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}(\|\mathbb{W}_{0,t}^k\|) &\leq \sum_{i_1,\ldots,i_k} \mathbb{E}\big|\mathbb{W}_{0,t}^{i_1,\ldots,i_k}\big| \\ &\leq \sum_{i_1,\ldots,i_k} \sqrt{\mathbb{E}(\mathbb{W}_{0,t}^{i_1,\ldots,i_k})^2}. \end{aligned}$$

G. Ben Arous ([1], Lemma 3) showed that

$$\mathbb{E}\big((\mathbb{W}_{0,t}^{i_1,\ldots,i_k})^2\big) \leq (\frac{5}{2})^{2k} \frac{1}{k!} t^k.$$

Therefore,

$$\mathbb{E}(\|\mathbb{W}_{0,t}^k\|) \leq n^k (\frac{5}{2})^k \frac{1}{\sqrt{k!}} t^{\frac{k}{2}}.$$

Let $s > t$, then

$$\mathbb{P}(\|\mathbb{W}_{0,t}^k\| > n^k (\frac{5}{2})^k \frac{1}{\sqrt{k!}} s^{\frac{k}{2}}) \leq \left(\frac{t}{s}\right)^{\frac{k}{2}}.$$

Therefore for almost all ω, there exists $M(\omega)$ such that for all $k \geq M(\omega)$,

$$\|\mathbb{W}_{0,t}^k\| \leq n^k (\frac{5}{2})^k \frac{1}{\sqrt{k!}} s^{\frac{k}{2}}.$$

Using the Stirling's approximation, we have

$$\limsup_k (\frac{(k/2)!}{(k!)^{\frac{1}{2}}})^{\frac{2}{k}} \leq \frac{1}{2}.$$

We therefore have

$$\begin{aligned}\limsup_k \left((\frac{k}{2})!\|\mathbb{W}_{0,t}^k\|\right)^{\frac{2}{k}} &\leq \frac{25}{8}n^2 s \times \limsup_k \left((\frac{k}{2})!\frac{1}{\sqrt{k!}}\right)^{\frac{2}{k}} \\ &\leq \frac{25}{8}n^2 s.\end{aligned}$$

As this holds for all ration $s > t$, we may let s tends to t by enlarging the null set. The result now follows. □

5. The limsup is deterministic

We first need a lemma on the behaviour for the factorial.

Lemma 12. *Let α be such that $0 < \alpha < N$. There exists a M and a constant C_4 independent of N such that for all $N \geq M$,*

$$\frac{(N/p)!}{((N-\alpha)/p)!} \leq C_4 N^{\frac{\alpha}{p}}.$$

Proof. By Stirling's formula, we have that there exists $C_1, C_2 > 0$ such that for all sufficiently large N,

$$C_1(\frac{N}{p})^{\frac{N}{p}+\frac{1}{2}}\exp(-\frac{N}{p}) \leq (\frac{N}{p})! \leq C_2(\frac{N}{p})^{\frac{N}{p}+\frac{1}{2}}\exp(-\frac{N}{p}).$$

and

$$C_1(\frac{N-\alpha}{p})^{\frac{N-\alpha}{p}+\frac{1}{2}}\exp(-\frac{N-\alpha}{p}) \leq (\frac{N-\alpha}{p})! \leq C_2(\frac{N-\alpha}{p})^{\frac{N-\alpha}{p}+\frac{1}{2}}\exp(-\frac{N-\alpha}{p})$$

and hence there exists C_3 such that for sufficiently large N,

$$\begin{aligned}\frac{(N/p)!}{((N-\alpha)/p)!} &\leq C_3\exp(\frac{\alpha}{p})(\frac{N}{N-p})^{\frac{N-\alpha}{p}+\frac{1}{2}}N^{\frac{\alpha}{p}} \\ &\leq C_4 N^{\frac{\alpha}{p}}.\end{aligned}$$

□

Lemma 13. *(Addivity lemma) Let X be a geometric rough path and*

$$\mathbb{X}_{s,t}^k = \int_{\triangle_k(s,t)} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k}.$$

Let $p \geq 1$. If

$$\tilde{L}_{s,t} = \lim\sup_{N\to\infty} \|(\frac{N}{p})!\mathbb{X}_{s,t}^N\|^{\frac{p}{N}},$$

then for all $s \leq u \leq t$,

$$\tilde{L}_{s,t} \leq \tilde{L}_{s,u} + \tilde{L}_{u,t}.$$

Proof. We will assume that there exists $k_1, k_2 > 0$ such that

$$\mathbb{X}_{s,u}^{k_1} \neq 0 \text{ and } \mathbb{X}_{u,t}^{k_2} \neq 0.$$

Otherwise, then we have by Chen's identity that either

$$\mathbb{X}_{s,t}^N = \mathbb{X}_{u,t}^N \text{ or } \mathbb{X}_{s,t}^N = \mathbb{X}_{s,u}^N, \text{ for all } N$$

in which case the present Lemma is trivially true. Let $\alpha \geq 1$.Assume that $N > 2\alpha$. Note that by Chen's identity,

$$\begin{aligned}\|\mathbb{X}^N_{s,t}\| &\leq \sum_{k=0}^{\alpha-1}\|\mathbb{X}^{N-k}_{s,u}\|\|\mathbb{X}^k_{u,t}\| + \sum_{k=N-\alpha+1}^{N}\|\mathbb{X}^{N-k}_{s,u}\|\|\mathbb{X}^k_{u,t}\| \\ &\quad + \sum_{k=\alpha}^{N-\alpha}\|\mathbb{X}^{N-k}_{s,u}\|\|\mathbb{X}^k_{u,t}\|.\end{aligned}$$

Let

$$\tilde{L}^\alpha_{s,t} = \sup_{k\geq\alpha}\|(\frac{k}{p})!\mathbb{X}^k_{s,t}\|^{\frac{p}{k}}.$$

Note that as we assumed that there exists $k_1, k_2 > 0$ such that $\mathbb{X}^{k_1}_{s,u} \neq 0$ and $\mathbb{X}^{k_2}_{u,t} \neq 0$, we may use Corollary 20 in the Appendix to conclude that

$$\tilde{L}^\alpha_{s,u} > 0,\ \tilde{L}^\alpha_{u,t} > 0.$$

Then

$$\|\mathbb{X}^N_{s,t}\| \leq \sum_{k=0}^{\alpha-1}\frac{(\tilde{L}^\alpha_{s,u})^{\frac{N-k}{p}}}{(\frac{N-k}{p})!}\|\mathbb{X}^k_{u,t}\| + \sum_{k=N-\alpha+1}^{N}\|\mathbb{X}^{N-k}_{s,u}\|\frac{(\tilde{L}^\alpha_{u,t})^{\frac{k}{p}}}{(\frac{k}{p})!} + \sum_{k=\alpha}^{N-\alpha}\frac{(\tilde{L}^\alpha_{s,u})^{\frac{N-k}{p}}(\tilde{L}^\alpha_{u,t})^{\frac{k}{p}}}{(\frac{N-k}{p})!(\frac{k}{p})!}.$$

By the neoclassical inequality [8], which states that for all $a, b \geq 0$ and $p \geq 1$,

$$\sum_{i=0}^{N}\frac{a^{i/p}b^{(N-i)/p}}{(i/p)!((N-i)/p)!} \leq p\frac{(a+b)^{\frac{N}{p}}}{(N/p)!}.$$

Therefore,

$$\|\mathbb{X}^N_{s,t}\| \leq \frac{1}{(\frac{N-\alpha}{p})!}\sum_{k=0}^{\alpha-1}(\tilde{L}^\alpha_{s,u})^{\frac{N-k}{p}}\|\mathbb{X}^k_{u,t}\| + \frac{1}{(\frac{N-\alpha+1}{p})!}\sum_{k=N-\alpha+1}^{N}\|\mathbb{X}^{N-k}_{s,u}\|(\tilde{L}^\alpha_{u,t})^{\frac{k}{p}} + p\frac{(\tilde{L}^\alpha_{s,u}+\tilde{L}^\alpha_{u,t})^{\frac{N}{p}}}{(\frac{N}{p})!}.$$

By Lemma 12 on the behaviour of the factorial function,

$$\begin{aligned}(\frac{N}{p})!\|\mathbb{X}^N_{s,t}\| &\leq C_4N^{\frac{\alpha}{p}}\sum_{k=0}^{\alpha-1}(\tilde{L}^\alpha_{s,u})^{\frac{N-k}{p}}\|\mathbb{X}^k_{u,t}\| + C_4N^{\frac{\alpha}{p}}\sum_{k=N-\alpha+1}^{N}\|\mathbb{X}^{N-k}_{s,u}\|(\tilde{L}^\alpha_{u,t})^{\frac{k}{p}} \\ &\quad + p(\tilde{L}^\alpha_{s,u}+\tilde{L}^\alpha_{u,t})^{\frac{N}{p}}.\end{aligned}$$

Note that if $c > a > 0$, $c > b > 0$, and $k > 0$ independent of N, then

$$(\tilde{C}_4N^ka^N + \tilde{C}_4N^kb^N + \tilde{C}_5c^N)^{\frac{1}{N}} \to c$$

as $N \to \infty$. Therefore,

$$\limsup_{N\to\infty}\big((\frac{N}{p})!\|\mathbb{X}^N_{s,t}\|\big)^{\frac{p}{N}} \leq \tilde{L}^\alpha_{s,u} + \tilde{L}^\alpha_{u,t}.$$

We now take limit as $\alpha \to \infty$, we have

$$\tilde{L}_{s,t} \leq \tilde{L}_{s,u} + \tilde{L}_{u,t}.$$

□

Remark 14. A similar additive lower bound would require that the path is reduced (see [2]). For this upper bound, there is no assumption that the path is reduced.

Corollary 15. *Let B be a Brownian motion and*

$$\mathbb{B}^N_{s,t} = \int_{\triangle_N(s,t)} \circ\mathrm{d}B_{s_1} \otimes \ldots \otimes \circ\mathrm{d}B_{s_N},$$

where $\circ$ denote the Stratonovitch integration. Then for each s and t, almost surely

$$\limsup_{N\to\infty} \|(\frac{N}{2})!\mathbb{B}^N_{s,t}\|^{\frac{2}{N}}$$

is deterministic.

Proof. By our apriori lower bound, Lemma 10, for each s and t, almost surely

$$\tilde{L}_{s,t} = \limsup_{N\to\infty} \|(\frac{N}{2})!\mathbb{B}^N_{s,t}\|^{\frac{2}{N}} > 0.$$

Therefore, by the additivity Lemma 13, if $t^m_i = s + \frac{i}{2^m}(t-s)$,

$$\begin{aligned} \tilde{L}_{s,t} &\leq \sum_{i=0}^{2^m-1} \tilde{L}_{t^m_i, t^m_{i+1}} \\ &= \frac{1}{2^m}\sum_{i=0}^{2^m-1}\left(2^m \tilde{L}_{t^m_i, t^m_{i+1}}\right). \end{aligned} \tag{5.1}$$

Note first that by Brownian scaling,

$$2^m \tilde{L}_{t^m_i, t^m_{i+1}} =_D \tilde{L}_{s,t}.$$

Therefore, each random variable $2^m \tilde{L}_{t^m_i, t^m_{i+1}}$ is almost surely bounded above by $(\frac{5}{8})n^2(t-s)$. Moreover,

$$(\tilde{L}_{t^m_i, t^m_{i+1}})_{i=0}^{2^m}$$

are independent.

By the weak law of large numbers, we have the convergence

$$\frac{1}{2^m}\sum_{i=0}^{2^m-1}\left(2^m \tilde{L}_{t^m_i, t^m_{i+1}}\right) \to \mathbb{E}(\tilde{L}_{s,t})$$

in probability. This allows us to take a subsequence m_k so that almost surely,

$$\frac{1}{2^{m_k}}\sum_{i=0}^{2^{m_k}-1}\left(2^{m_k} \tilde{L}_{t^{m_k}_i, t^{m_k}_{i+1}}\right) \to \mathbb{E}(\tilde{L}_{s,t})$$

Therefore by (5.1), we have almost surely,

$$\tilde{L}_{s,t} \leq \mathbb{E}(\tilde{L}_{s,t})$$

However, any random variable X that satisfies

$$X \leq \mathbb{E}(X)$$

must satisfy almost surely that

$$X = \mathbb{E}(X).$$

Therefore, $\tilde{L}_{s,t}$ is deterministic. □

Corollary 16. *For each $s \leq t$, almost surely,*

$$\tilde{L}_{s,t} = \mathbb{E}(\tilde{L}_{0,1})(t-s).$$

In particular, for each $s \leq u \leq t$, we have almost surely

$$\tilde{L}_{s,u} + \tilde{L}_{u,t} = \tilde{L}_{s,t}.$$

Proof. Since for each $s \leq t$ almost surely

$$\tilde{L}_{s,t} = \mathbb{E}(\tilde{L}_{s,t}),$$

we have by Brownian scaling that

$$\tilde{L}_{s,t} = (t-s)\mathbb{E}(\tilde{L}_{0,1}).$$

The second statement of the Corollary follows directly from the first statement. □

6. Appendix: some useful properties on Signature

We define the set $Sh(m_1, \ldots, m_k)$ as the set of all permutations σ on

$$\{1, \ldots, m_1 + \ldots + m_k\}$$

such that

$$\begin{aligned} \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m_1); \\ \sigma^{-1}(m_1+1) < \sigma^{-1}(m_1+2) < \ldots < \sigma^{-1}(m_2); \\ \ldots \\ \sigma^{-1}(\sum_{i=1}^{k-1} m_i + 1) < \ldots < \sigma^{-1}(\sum_{i=1}^{k} m_i). \end{aligned}$$

For each permutation σ, we will define a linear map on $(\mathbb{R}^n)^{\otimes k}$ by extending linearly the map

$$\sigma(v_1 \otimes \ldots \otimes v_k) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}.$$

Note that the projective tensor norm has the symmetric property [11] that for any permutation σ and any $\mathbf{v} \in (\mathbb{R}^n)^{\otimes k}$,

$$\|\sigma \mathbf{v}\| = \|\mathbf{v}\|.$$

We will use the following property of iterated integral.

Lemma 17. *Let X be a bounded variation path and let*

$$\mathbb{X}^k_{0,t} = \int_{\triangle_k(0,t)} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_k}.$$

Then for all $m_1, \ldots, m_k$,

$$\mathbb{X}^{m_1}_{0,t} \otimes \ldots \otimes \mathbb{X}^{m_k}_{0,t} = \sum_{\sigma \in Sh(m_1,\ldots,m_k)} \sigma^{-1} \mathbb{X}^{m_1+\ldots+m_k}_{0,t}. \tag{6.1}$$

Proof. Note that

$$\begin{aligned}
&\mathbb{X}_{0,t}^{m_1} \otimes \ldots \otimes \mathbb{X}_{0,t}^{m_k} \\
= &\int_{0<s_1<\ldots<s_{m_1}<t} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_{m_1}} \otimes \ldots \\
&\otimes \int_{0<s_{m_1+\ldots+m_{k-1}+1}<\ldots<s_{m_1+\ldots+m_k}<t} \mathrm{d}X_{s_{m_1+\ldots+m_{k-1}+1}} \otimes \ldots \otimes \mathrm{d}X_{s_{m_1+\ldots+m_k}}. \\
= &\int_A \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_{m_1+\ldots+m_k}},
\end{aligned}$$

where

$$\begin{aligned}
A \quad = \quad &\{(s_1, \ldots, s_{m_1+\ldots+m_k}) : 0 < s_1 < \ldots < s_{m_1} < t \\
&, \ldots 0 < s_{m_1+\cdots+m_{k-1}+1} < \ldots < s_{m_1+\ldots+m_k<t} < t\}.
\end{aligned}$$

We may rewrite A as the disjoint union

$$A = \cup_{\sigma \in Sh(m_1,\ldots,m_k)} \{(s_1, \ldots, s_{m_1+\ldots+m_k}) : 0 < s_{\sigma(1)} < \ldots < s_{\sigma(m_1+\ldots+m_k)} < t\}.$$

Therefore, by the additivity of integral,

$$\begin{aligned}
&\mathbb{X}_{0,t}^{m_1} \otimes \ldots \otimes \mathbb{X}_{0,t}^{m_k} \\
= &\sum_{\sigma \in Sh(m_1\ldots,,m_k)} \int_{0<s_{\tau(1)}<\ldots<s_{\tau(m_1+\ldots+m_k)}<t} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_{m_1+\ldots+m_k}} \\
= &\sum_{\sigma \in Sh(m_1\ldots,,m_k)} \int_{0<s_1<\ldots<s_{m_1+\ldots+m_k}<t} \mathrm{d}X_{s_{\sigma^{-1}(1)}} \otimes \ldots \otimes \mathrm{d}X_{s_{\sigma^{-1}(m_1+\ldots+m_k)}} \\
= &\sum_{\sigma \in Sh(m_1,\ldots,m_k)} \sigma^{-1} \mathbb{X}_{0,t}^{m_1+\ldots+m_k}.
\end{aligned}$$

□

Corollary 18. *Equation (6.1) holds even when X is a geometric rough path.*

Proof. This follows directly from Lemma 17 (6.1), that geometric rough paths can be approximated by bounded variations paths in p-variation metric and that the iterated integrals are continuous with respect to the p-variation metric [11]. □

Remark 19. In the case of Brownian motion, one can simply prove equation (6.1) by replacing the Lebesgue-Stieltjes integral with Stratonovitch integral line by line in the proof of Lemma 17.

Corollary 20. *Let X be a geometric rough path. Let $\mathbb{X}_{s,t}^k$ be its k-order iterated integral (defined in Theorem 2). If there exists $k > 0$ such that*

$$\mathbb{X}_{s,t}^k \neq 0, \tag{6.2}$$

then (6.2) hold for infinitely many k.

Proof. Assume for contradiction that there exists $K > 0$ such that $\mathbb{X}_{s,t}^K \neq 0$ and

$$\mathbb{X}_{s,t}^k = 0$$

for all $k > K$. Then

$$\mathbb{X}_{s,t}^K \otimes \mathbb{X}_{s,t}^K \neq 0.$$

However, as $\mathbb{X}^k_{s,t} = 0$ for all $k > K$,

$$\sum_{\sigma \in Sh(K,\ldots,K)} \sigma^{-1}\mathbb{X}^{2K}_{0,t} = 0$$

This contradicts Lemma 17. □

An interesting corollary of Lemma 17 is that the terms in signature are increasing when the N-th term is normalised by $N!$.

Lemma 21. *Let X be any geometric rough path and let*

$$\mathbb{X}^m_{0,t} = \int_{\triangle_m(0,t)} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_m}.$$

Then if $\|\cdot\|$ is the projective norm, we have

$$\left(j!\|\mathbb{X}^j_{0,t}\|\right)^{\frac{1}{j}} \leq ((jk)!\|\mathbb{X}^{jk}_{0,t}\|)^{\frac{1}{jk}}$$

for all $j,k \in \mathbb{N}$.

Proof. Note that by Lemma 17,

$$(\mathbb{X}^j_{0,t})^{\otimes k} = \sum_{\sigma \in Sh(j,\ldots,j)} \sigma^{-1}\mathbb{X}^{jk}_{0,t}.$$

Therefore, using the property of projective norm that $\|\sigma^{-1}\mathbf{v}\| = \|\mathbf{v}\|$ (see [11]),

$$\begin{aligned} \|(\mathbb{X}^j_{0,t})^{\otimes k}\| &\leq \sum_{\sigma^{-1} \in Sh(j,\ldots,j)} \|\mathbb{X}^{jk}_{0,t}\| \\ &= \frac{(jk)!}{(j!)^k}\|\mathbb{X}^{jk}_{0,t}\|. \end{aligned}$$

Therefore, by using the multiplicative property of projective norm, that $\|a \otimes b\| = \|a\|\|b\|$,

$$\|\mathbb{X}^j_{0,t}\|^k \leq \frac{(jk)!}{(j!)^k}\|\mathbb{X}^{jk}_{0,t}\|.$$

Rearranging gives,

$$\left(j!\|\mathbb{X}^j_{0,t}\|\right)^{\frac{1}{n}} \leq ((jk)!\|\mathbb{X}^{jk}_{0,t}\|)^{\frac{1}{jk}}.$$

□

Corollary 22. *(limsup is sup) For any bounded variation path,*

$$\limsup_m \left(m!\|\mathbb{X}^m_{0,t}\|\right)^{\frac{1}{m}} = \sup_m \left(m!\|\mathbb{X}^m_{0,t}\|\right)^{\frac{1}{m}}.$$

Proof. One side of the inequality, namely that

$$\limsup_m \left(m!\|\mathbb{X}^m_{0,t}\|\right)^{\frac{1}{m}} \leq \sup_m \left(m!\|\mathbb{X}^m_{0,t}\|\right)^{\frac{1}{m}},$$

follows directly from the definition of limsup and sup. For the other direction, note that for any j,

$$\limsup_m \left(m!\|\mathbb{X}^m_{0,t}\|\right)^{\frac{1}{m}} \geq \limsup_k \left((jk)!\|\mathbb{X}^{jk}_{0,t}\|\right)^{\frac{1}{jk}}$$

and by Corollary 21, for all k,

$$\left((jk)!\|\mathbb{X}^{jk}_{0,t}\|\right)^{\frac{1}{jk}} \geq \left(j!\|\mathbb{X}^j_{0,t}\|\right)^{\frac{1}{j}}.$$

Therefore,

$$\limsup_{k} \left((jk)!\|\mathbb{X}_{0,t}^{jk}\|\right)^{\frac{1}{jk}} \geq \left(j!\|\mathbb{X}_{0,t}^{j}\|\right)^{\frac{1}{j}}.$$

The Corollary now follows from taking supremum over j on both sides. □

Remark 23. Corollary 22 does not hold for Brownian motion. Note that if $\mathbb{W}^m$ now denotes the m-th term in the iterated integrals of Brownian motion, then

$$\sup_m[(\frac{m}{2})!\|\mathbb{W}_{0,1}^m\|^{\frac{2}{m}}] \geq \|(\frac{1}{2})!\mathbb{W}_{0,1}^1\|^2 \geq |\frac{1}{2}!|\|W_1 - W_0\|^2.$$

Since $|W_T - W_0|$ can be arbitrarily large with positive probability, so is the supremum

$$\sup_m[(\frac{m}{2})!\|\mathbb{W}_{0,1}^m\|^{\frac{2}{m}}].$$

On the other hand, by our main result Theorem 1, the lim sup is bounded above.

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