THE TAIL ASYMPTOTICS FOR THE SIGNATURES OF SOME PURE ROUGH PATHS

HORATIO BOEDIHARDJO, XI GENG, NIKOLAOS SOURIS

Abstract. The solutions to linear differential equations driven by a path can be expressed in terms of iterated integrals of the path. While upper bounds for the iterated integrals are well-known, it is known only relatively recently that some asymptotics of the order-n iterated integrals of path, as n tends to infinity, is related to the length of the path for C^1 paths travelling at unit speed. We provide an upper and lower bound for the corresponding asymptotics for rough paths whose signature (sequence of iterated integrals) is the exponential of a degree four Lie polynomial generated by two letters. The methodology, which we believe is new for this problem, involves using the eigenvalues for certain Lie-algebraic development of the signature as a lower bound for the aforementioned asymptotics.

1. Introduction

Controlled differential equation of the form

$$\mathrm{d}Y_t = \sum_{i=0}^{d} V_i(Y_t)\mathrm{d}X_t^i, \tag{1.1}$$

where $Y : [0,1] \to \mathbb{R}^n$, $V_i : \mathbb{R}^n \to \mathbb{R}^n$ and $X : [0,1] \to \mathbb{R}^d$, frequently appears in stochastic analysis. The example when X is a Brownian motion is perhaps the most well-known. Rough path theory, initiated by Lyons' [10], identified a wide class of "rough paths" X for which the equation (1.1) is well-defined. The theory motivates a study of the properties of (1.1) driven by general rough paths. One particularly tractable class of examples is when the functions $\{V_i\}_{i=1}^d$ are linear. In this case, Y can be represented explicitly as

$$Y_t = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} V_{i_n} \ldots V_{i_1}(Y_0) \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1}^{i_1} \ldots \mathrm{d}X_{t_n}^{i_n}. \tag{1.2}$$

In this particular case, the solution Y depends on driving path X through the collection of iterated integrals

$$\{\int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1}^{i_1} \ldots \mathrm{d}X_{t_n}^{i_n}\}_{i_1,\ldots,i_n=1}^{d}.$$

This article is motivated by the study of the properties of these iterated integrals. For algebraic reasons, it is useful to think of this collection as an element of the tensor algebra, called the signature of X and denoted as $S(X)_{0,1}$. Let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^d$. For a bounded variation path X, the signature of X is defined to be

$$1 + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1}^{i_1} \ldots \mathrm{d}X_{t_n}^{i_n} e_{i_1} \otimes \ldots \otimes e_{i_n},$$

or more succinctly as

$$S(X)_{0,1} = \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n},$$

where we use the convention that the 0-th order iterated integral is equal to zero. An interesting question about the signature is how does

$$\| \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \|$$

with a suitably defined tensor norm $\|\cdot\|$ decay as $n \to \infty$. This affects, in particular, the convergence of (1.2), and is also related to the problem of reconstructing a path X from its signature ([5], [13]), because the algebraic properties of signature guarrantees that all the information about the signature is stored at its tail. Lyons [10] showed that the following uniform upper bound, for rough paths with finite p-variation,

$$\| \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \| \leq c_p \frac{\|X\|_{p-var}^n}{(\frac{n}{p})!}, \tag{1.3}$$

where $(\frac{n}{p})!$ is defined using the gamma function, and c_p depends only on p (but not X). More recently, Hambly and Lyons [6] , and subsequently Lyons and Xu [12], showed that it is possible to obtain the exact asymptotics for $\| \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \|$ as $n \to \infty$. They showed, in particular, that if X is a C^1-path with respect to the unit speed parametrisation, then

$$\limsup_{n\to\infty} \|n! \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \|^{\frac{1}{n}} = \|X\|_{1-var}. \tag{1.4}$$

Recall here that

$$\|X\|_{1-var} = \lim_{\max_i |t_{i+1}-t_i| \to 0} \sum_{i=0}^{n-1} |X_{t_{1+i}} - X_{t_i}|,$$

where the limit $\max_i |t_{i+1} - t_i| \to 0$ is taken as the maximum gap between the partition points $(0 = t_0 < \ldots < t_n = 1)$ tend to zero.

In the more general setting of rough paths, where the iterated integrals may be defined in the sense of Lyons [10], the upper bound (1.3) suggests that the normalisation for p-rough path would be $\left(\frac{n}{p}\right)!$. It is however difficult to see what the analogous limit

$$\limsup_{n\to\infty} \| \left(\frac{n}{p}\right)! \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \|^{\frac{p}{n}}. \tag{1.5}$$

It certainly is not the p-variation norm, since (1.4) implies that the limit (1.5) is zero for any C^1 unit speed path. It is useful therefore to explore the limit (1.5) in some special cases. When X is Brownian motion, and the differential $\mathrm{d}X$ is defined in terms of the Stratonovich integral, the following limiting asymptotics is available [2]

$$\limsup_{n\to\infty} \| \left(\frac{n}{2}\right)! \int_{0<t_1<\ldots<t_n<t} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n} \|^{\frac{2}{n}} = Ct, \tag{1.6}$$

where C is a positive and finite deterministic constant. Since the exact value of C is not known, it is unclear how to use (1.6) to make general guess about the limit (1.5). For instance the constant Ct can be interpreted as a constant multiple of the quadratic variation

$$[X,X]_t = \lim_{\max_i |t_{i+1}-t_i|\to 0} \sum_{i=0}^{n-1} |X_{t_{1+i}} - X_{t_i}|^2,$$

which is exactly equal to t for the sample paths of Brownian motion; the constant Ct could also depend on the Levy area of Brownian motion. To get a clearer idea of what the limit (1.5) would be, we investigate in this article, a different class of rough paths. The signature $S(X)_{0,1}$ of a rough path X can be informally interpreted as the formal series of iterated integrals

$$S(X)_{0,1} = 1 + \int_0^1 \mathrm{d}X_{t_1} + \int_0^1 \int_0^{t_1} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2} + \dots.$$

The signature of a p weakly geometric rough path is equal to either the exponential of a Lie series, or equal to the exponential of a degree p polynomial $P = \sum_{i=1}^{p} P_i$, where P_i is a homogeneous Lie polynomial of degree i and $P_p \neq 0$. We investigate the latter case. In the appendix, we showed that certain variational norm of X is equal to $\|P_p\|$. Our main result is that for some values of p, the asymptotic (1.5) is also equal to $\|P_p\|$.

Theorem 1.1. *Let X be a rough path over $\mathbb{R}^d$ whose signature $S(X)_{0,1}$ can be expressed as*

$$S(X)_{0,1} = \exp(P), \tag{1.7}$$

for some $P = \sum_{i=1}^{p} P_i$, P_i is a homogeneous Lie polynomial of degree i. Then for any admissible norms $\|\cdot\|$ (see Section 2.2 for the definition of admissible norms)

$$\limsup_{n\to\infty} \|\left(\frac{n}{p}\right)! \pi_n(S(X)_{0,1})\|^{\frac{p}{n}} \leq \|P_p\|.$$

Moreover when $d=2$, $p\leq 3$ and $\|\cdot\|_{proj}$ is the projective norm induced by the Euclidean norm on $\mathbb{R}^d$, then

$$\limsup_{n\to\infty} \|\left(\frac{n}{p}\right)! \pi_n(S(X)_{0,1})\|_{proj}^{\frac{p}{n}} = \|P_p\|_{proj},$$

and when $d=2$ and $p=4$,

$$\frac{1}{4}\|P_4\| \leq \limsup_{n\to\infty} \|\left(\frac{n}{4}\right)! \pi_n(S(X)_{0,1})\|_{proj}^{\frac{p}{n}} \leq \|P_4\|_{proj}.$$

In the appendix, we details which paths satisfy the relation (1.7), as well as other properties of such a path.

Our proof of Theorem 1.1 considers a finite dimensional projection of the signature $S(X)_{0,1}$, in the following way: We start with a map $\Phi : \mathbb{R}^d \to M_{n\times n}(\mathbb{R})$; then extend Φ naturally to a map $\tilde{\Phi}$ on the tensor algebra of $\mathbb{R}^d$ and consider $\tilde{\Phi}(S(X)_{0,1})$, which can be thought of as the finite dimensional projection of the signature $S(X)_{0,1}$. A specific choice of Φ was used in the work of Hambly-Lyons [6] to prove the C^1 unit speed case (1.4); and other choices of Φ were used by Chevyrev and Lyons [3] and Lyons and Sidorova [11] to prove other properties of the signature. The main new ingredients in this article is that we make a more quantitative use of the development $\tilde{\Phi}$; namely to prove the limiting asymptotics (1.4) through computing the eigenvalues of $\tilde{\Phi}(S(X)_{0,1})$.

The rest of the article is organised in the following way: In Section 2, we recall some essential preliminaries and notations, and formulate our problem in precise terms. In Section

3, we prove the first part of our main result, namely an upper bound on the tail asymptotics for pure rough paths. In Section 4, we prove some general results relating the tail asymptotics of the signature $S(X)_{0,1}$ to the eigenvalues of its projection $\tilde{\Phi}(S(X)_{0,1})$. In Section 5, we prove the lower bounds for tail asymptotics. Finally, in the appendix, we prove some results about the pure rough paths and their signatures, which provides the motivation for our study.

2. Preliminaries and Notation

2.1. **Tensor algebra.** Let V be a finite dimensional, real vector space and let $T((V))$ be the *tensor algebra of* V over $\mathbb{R}$, that is

$$T((V)) = \mathbb{R} \times V \oplus V^{\otimes^2} \times \cdots \times V^{\otimes^i} \times \cdots = \Pi_{i=0}^{\infty} V^{\otimes^i},$$

where $V^{\otimes^i} = \underbrace{V \otimes \cdots \otimes V}_{i-\text{times}}$ and $V^{\otimes 0} = \mathbb{R}$. The tensor product between two elements in V can be extended to a product $\otimes$ on $T((V))$, making $(T((V)), \otimes)$ an associative algebra. Subsequently, the bracket operation $[\ ,\ ]$ in $T((V))$, with $[x, y] = x \otimes y - y \otimes x$, for $x, y \in T((V))$, induces a Lie algebra structure in $T((V))$. We denote by $L((V)) \subset T((V))$ the *space of Lie formal series over* V, that is

$$L((V)) = V \times [V, V] \times \cdots \times \underbrace{[\cdots [[[V, V], V], \ldots, V]}_{i \text{ times}} \times \cdots = \Pi_{i=1}^{\infty} \underbrace{[\cdots [[[V, V], V], \ldots, V]}_{i \text{ times}}.$$

An element in $\bigoplus_{i=1}^{n} \underbrace{[\cdots [[[V, V], V], \ldots, V]}_{i \text{ times}}$ is called a *Lie polynomial of degree* n, while an element in $[\cdots [[[V, V], V], \ldots, V]$ is called a *homogeneous Lie polynomial.*

2.2. **Admissible norms on the tensor algebra.** A family of norms $\|\cdot\|_n$ in $V^{\otimes^n}$, $n = 1, 2, \ldots$, is called *admissible* if it satisfies the following properties:

(i) $\| \sum_{i=1}^{k} v_1^i \otimes v_2 \otimes \cdots \otimes v_i^n \|_n = \| \sum_{i=1}^{k} v_{\sigma(1)}^i \otimes v_{\sigma(n)}^i \otimes \cdots \otimes v_{\sigma(n)}^n \|_n$, for any $k \in \mathbb{N}$, $v_{i_1}, \ldots v_{i_n} \in V$, and for any permutation σ of the set $\{1, 2, \ldots, n\}$.

(ii) $\|v \otimes w\|_{n+m} \leq \|v\|_n \|w\|_m$, for any $v \in V^{\otimes^n}$, $w \in V^{\otimes^m}$.

For simplicity, we will omit the subscript n, and thus write $\|\cdot\|$, instead of $\|\cdot\|_n$, whenever the value of n is clear from the context. We will consider the following two families of admissible norms.

Definition 2.1. Fix a norm $|\cdot|$ on V. The *projective norm on* $V^{\otimes^k}$ is defined by

$$\|v\|_{proj} = \inf\{\sum_{i=1}^{n} |v_1^i| \cdots |v_k^i| : v = \sum_{i=1}^{n} v_1^i \otimes \cdots \otimes v_k^i\}. \tag{2.1}$$

To define the next family of admissible norms, we let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of V. Then the set $\{e_{i_1} \otimes \cdots \otimes e_{i_k} : i_1, \ldots, i_k = 1, \ldots, d\}$ is a basis for $V^{\otimes^k}$.

Definition 2.2. The l_2*-norm on* $V^{\otimes^k}$ is defined by

$$\| \sum_{i_1,\dots,i_k=1}^{d} a_{i_1,\dots,i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \|_{l_2} := \big(\sum_{i_1,\dots,i_k=1}^{d} a^2_{i_1,\dots,i_k} \big)^{\frac{1}{2}}. \tag{2.2}$$

By considering orthogonal transformations of V and extending them to $V^{\otimes^k}$ we deduce that the l_2-norm is independent of the orthonormal basis of V.

2.3. **Formulation of the conjecture.** For $n \in \mathbb{N}$, let $\pi_n : T((V)) \to V^{\otimes^n}$ be the linear projection. Denote by $\exp : T((V)) \to T((V))$ the *exponential map* given by

$$\exp(v) = \sum_{i=0}^{\infty} \frac{v^i}{i!}, \quad \text{where} \quad v \in T((V)).$$

Assume that $\| \cdot \|$ is a family of admissible norms on $V^{\otimes^n}$, $n \in \mathbb{N}$. We state the following conjecture.

Conjecture 2.1. *Let $P = P_1 + \cdots + P_p \in L((V))$ be a Lie polynomial of degree p, where P_i, $i = 1, \dots, p$, is the homogeneous component of P with degree i . Then*

$$\widetilde{L}_p := \limsup_{n\to\infty} \big((\frac{n}{p})! \|\pi_n(\exp P)\| \big)^{\frac{p}{n}} = \|P_p\|. \tag{2.3}$$

2.4. **Relation of the conjecture to the length of a rough path.** For $n \in \mathbb{N}$, denote by $T^{(n)}(V)$ the *truncated tensor algebra of* V, that is the space $T((V))/I_n$, where I_n is the ideal

$$I_n = \{(a_0, a_1, \dots, a_n, \dots) \in T((V)) : a_0 = a_1 = \cdots = a_n = 0\}.$$

Then the product in $T((V))$ induces naturally a product in $T^{(n)}(V)$. Recall that a *p-rough path* is a multiplicative functional $X : \Delta_{[0,1]} \to T^{[p]}(V)$ with finite p-variation, where $\Delta_{[0,1]} = \{(s,t) \in [0,1] \times [0,1] : 0 \le s \le t \le 1\}$ is the 2-simplex. The *signature functional of a p-rough path* X is the unique extension $S(X) : \Delta_{[0,1]} \to T((V))$ of X, such that $S(X)$ is multiplicative and of finite p-variation. The *signature of* X is the tensor $S(X)_{0,1}$. The signature is the exponential of a Lie series, i.e. there exists a Lie polynomial $P \in L((V))$ such that

$$S(X)_{0,1} = \exp P = \exp(\sum_{i=1}^{\infty} P_i), \tag{2.4}$$

where each P_i is a homogeneous Lie polynomial of degree i. It can be proved that either $\sum_{i=1}^{\infty} P_i$ is a Lie polynomial of degree $[p]$ or at most finitely many terms of $\sum_{i=1}^{\infty} P_i$ are zero. This motivates the following definition.

Definition 2.3. A *pure p-rough path* is a p-rough path X whose signature has the form $S(X)_{0,1} = \exp(\sum_{i=1}^{[p]} P_i)$.

For example, any linear path is a pure 1-rough path. Let $X : \Delta_{[0,1]} \to T^{[p]}(V)$ be a pure p-rough path. Consider a family of admissible norms $\| \cdot \|$ on $V^{\otimes^n}$, $n \in \mathbb{N}$. We set

$$\widetilde{L}_p := \limsup_{n\to\infty} \big((n/p)!\|\pi_n S(X)_{0,1}\|\big)^{p/n} = \limsup_{n\to\infty} \left((n/p)!\|\pi_n(\exp(\sum_{i=1}^{[p]} P_i)\| \right)^{p/n}. \tag{2.5}$$

Then Conjecture 2.1 can be reformulated as follows in terms of pure rough paths.

Conjecture 2.2. *Let $X : \Delta_{[0,1]} \to T^{[p]}(V)$ be a pure p-rough path and let $S(X)_{0,1} = \exp(\sum_{i=1}^{[p]} P_i)$ be the signature of X. Then $\widetilde{L}_p = \|P_{[p]}\|$.*

3. Upper bound of $\widetilde{L_p}$

In this section we will show that one side of the equality in Conjecture 2.2 is true.

Proposition 3.1. *Let $\widetilde{L}_p$ and $P_{[p]}$ be as defined in Definition 2.3 and 2.5. Then for any admissible norm $\|\cdot\|$,*

$$\widetilde{L}_p \le \|P_{[p]}\|. \tag{3.1}$$

To prove this, we first need a lemma.

Lemma 3.1. *Let $0 < \alpha < \beta \le 1$ and let $a, b > 0$. Then*

$$\limsup_k \big((k\alpha)! \sum_{j=0}^{k} \frac{a^{j\alpha} b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\big)^{\frac{1}{k\alpha}} \le b.$$

Proof. Let $\varepsilon > 0$. It follows from Stirling's approximation that for $0 < \gamma \le 1$,

$$(j\gamma)! \sim (\frac{j\gamma}{e})^{j\gamma}\sqrt{2\pi j\gamma},$$

and that for $\beta > \alpha$, there exists J such that for all $j \ge J$,

$$\frac{(j\alpha)!}{(j\beta)!} < \varepsilon^j.$$

Therefore,

$$\begin{aligned} &\sum_{j=0}^{k} \frac{a^{j\alpha} b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!} \\ \le\ &\sum_{j=0}^{J-1} \frac{a^{j\alpha} b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!} + \sum_{j=J}^{k} \frac{(\varepsilon^{1/\alpha} a)^{j\alpha} b^{(k-j)\alpha}}{(j\alpha)!((k-j)\alpha)!}. \end{aligned}$$

We may apply the neoclassical inequality below (See Theorem 1.2 in [7])

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} \le (x+y)^{\alpha n} \tag{3.2}$$

to conclude that

$$\begin{aligned} &\sum_{j=0}^{k} \frac{a^{j\alpha} b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!} \\ \le\ &\sum_{j=0}^{J-1} \frac{a^{j\alpha} b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!} + \frac{(\varepsilon^{1/\alpha} a + b)^{k\alpha}}{\alpha(k\alpha)!}. \end{aligned} \tag{3.3}$$

We observe that

$$(k\alpha)!\sum_{j=0}^{J-1}\frac{a^{j\alpha}b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\leq\begin{cases}\frac{(k\alpha)!J(a\vee 1)^{J\alpha}b^{(k-J)\alpha}}{((k-j)\alpha)!}, & \text{if } b\leq 1;\\ \frac{(k\alpha)!J(a\vee 1)^{J\alpha}b^{k\alpha}}{((k-j)\alpha)!}, & \text{if } b>1.\end{cases}$$

Noting that once again by Stirling's approximation,

$$\frac{(k\alpha)!}{((k-j)\alpha)!}\leq Ck^{J\alpha}$$

where C is a constant depending only on j and α. Then

$$(k\alpha)!\sum_{j=0}^{J-1}\frac{a^{j\alpha}b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\leq\tilde{C}k^{J\alpha}b^{k\alpha}$$

where $\tilde{C}$ depends on J, a, b and α. Therefore,

$$(k\alpha)!\sum_{j=0}^{k}\frac{a^{j\alpha}b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\leq\tilde{C}k^{J\alpha}b^{k\alpha}+\frac{(\varepsilon^{1/\alpha}a+b)^{k\alpha}}{\alpha}.$$

We then take limsup in k to obtain,

$$\begin{aligned}&\limsup_{k}\big[(k\alpha)!\sum_{j=0}^{k}\frac{a^{j\alpha}b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\big]^{\frac{1}{k\alpha}}\\ \leq\ &\limsup_{k}\big[\tilde{C}k^{J\alpha}b^{k\alpha}+\frac{(\varepsilon^{1/\alpha}a+b)^{k\alpha}}{\alpha}\big]^{\frac{1}{k\alpha}}\\ =\ &\varepsilon^{1/\alpha}a+b.\end{aligned}$$

We then let $\varepsilon\to 0$ to obtain

$$\limsup_{k}\big[(k\alpha)!\sum_{j=0}^{k}\frac{a^{j\alpha}b^{(k-j)\alpha}}{(j\beta)!((k-j)\alpha)!}\big]^{\frac{1}{k\alpha}}\leq b.$$

□

We now prove our main proposition, Proposition 3.1.

Proof. Note that

$$\begin{aligned}&\|\pi_k(\exp(\sum_{i=1}^{p}P_i))\|\\ \leq\ &\sum_{N=0}^{\infty}\|\pi_k\big(\frac{(\sum_{i=1}^{p}P_i)^{\otimes N}}{N!}\big)\|\\ \leq\ &\sum_{N=0}^{\infty}\sum_{i_1+\ldots+i_N=k}\frac{\|P_{i_1}\|\ldots\|P_{i_N}\|}{N!}\\ =\ &\sum_{\sum in_i=k}\frac{\|P_1\|^{n_1}\ldots\|P_p\|^{n_p}}{(n_1)!\ldots(n_p)!}.\end{aligned}$$

If we substitute $\tilde{n}_i = in_i$, we have

$$\begin{aligned} &\sum_{\sum in_i = k} \frac{\|P_1\|^{n_1}\dots\|P_p\|^{n_p}}{(n_1)!\dots(n_p)!} \\ \leq &\sum_{\sum \tilde{n}_i = k} \frac{\|P_1\|^{\tilde{n}_1}\dots\|P_p\|^{\tilde{n}_p/p}}{(\tilde{n}_1)!\dots(\tilde{n}_p/p)!}. \end{aligned} \tag{3.4}$$

Note first that we may use neoclassical inequality (the multi-factor version, see Lemma 1 in [4]) to bound

$$\begin{aligned} &\sum_{\sum_{i=1}^{p-1}\tilde{n}_i = k-\tilde{n}_p} \frac{\|P_1\|^{\tilde{n}_1}\dots\|P_{p-1}\|^{\tilde{n}_{p-1}/(p-1)}}{(\tilde{n}_1)!\dots(\tilde{n}_{p-1}/(p-1))!} \\ \leq &\sum_{\sum_{i=1}^{p-1}\tilde{n}_i = k-\tilde{n}_p} \frac{\|P_1\|^{\tilde{n}_1}\dots\|P_{p-1}\|^{\tilde{n}_{p-1}/(p-1)}}{(\tilde{n}_1/(p-1))!\dots(\tilde{n}_{p-1}/(p-1))!} \\ \leq &\frac{(p-1)^{p-1}(\sum_{i=0}^{p-1}\|P_i\|^{(p-1)/i})^{(k-\tilde{n}_p)/(p-1)}}{((k-\tilde{n}_p)/(p-1))!}. \end{aligned}$$

We now let

$$a = (\sum_{i=0}^{p-1}\|P_i\|^{(p-1)/i})^{p/(p-1)}.$$

Then by continuing the calculation from (3.4),

$$\|\pi_k(\exp(\sum_{i=1}^{p} P_i))\| \leq (p-1)^{p-1}\sum_{\tilde{n}_p=0}^{k}\frac{a^{(k-\tilde{n}_p)/p}\|P_p\|^{\tilde{n}_p/p}}{(\frac{k-\tilde{n}_p}{p-1})!(\frac{\tilde{n}_p}{p})!}.$$

We now apply Lemma 3.1 to see that

$$\limsup_k \|(\frac{k}{p})!\pi_k(\exp(\sum_{i=1}^{p} P_i))\|^{\frac{p}{k}} \leq \|P_p\|.$$

□

Therefore, proving Conjecture 2.2 boils down to establishing the matching lower bound.

4. Estimates for the lower bound of $\widetilde{L}_p$

The main result of this section is an estimate for a lower bound of $\widetilde{L}_p$, in terms of eigenvalues of special matrix representations of the Lie polynomial P.

Let $(\mathcal{A}, +, \cdot)$ be an associative algebra over $\mathbb{R}$. Then $\mathcal{A}$, endowed with the Lie bracket $[X,Y] = XY - YX$, $X, Y \in \mathcal{A}$, becomes a Lie algebra. We consider the space $L(V, \mathcal{A})$ of linear maps from V to $\mathcal{A}$. Let $\Phi \in L(V, \mathcal{A})$. From the universal property of $T((V))$ we obtain a unique algebra homomorphism $\widetilde{\Phi} : T((V)) \to \mathcal{A}$, extending Φ.

Our main example for $\mathcal{A}$ will be the algebra $M_{n\times n}\mathbb{C}$, of $n \times n$ matrices with complex valued entries.

We proceed to define a norm for the space $L(V, M_{n\times n}\mathbb{C})$. We fix norms $|\cdot|$ on V and $\|\cdot\|$ on $\mathbb{C}^n$, and we consider the operator norm $\|\cdot\|_{op}$ on $M_{n\times n}\mathbb{C}$, with respect to $\|\cdot\|$. We define a norm $\|\cdot\|_{L_n}$ on $L(V, M_{n\times n}\mathbb{C})$ by

$$\|\Phi\|_{L_n} := \sup_{|v|=1} \||\Phi(v)\|_{op}.$$

We will show that the limit $\widetilde{L}_p$ can be bounded below, independently of the lower order terms of the Lie polynomial $P = \sum_{i=1}^{p} P_i$. The Euclidean norm $\|\cdot\|$ on $\mathbb{C}^n$ is defined by $\|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. We state the main result of the section.

Proposition 4.1. *Choose a norm $|\cdot|$ on V and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{C}^n$. Let $P = \sum_{i=1}^{p} P_i \in L(V)$ be a Lie polynomial, let $\Phi \in L(V, M_{n\times n}\mathbb{C})$ and denote by $\widetilde{\Phi} : T((V)) \to M_{n\times n}\mathbb{C}$ the unique extension of Φ. Then for any eigenvalue v of $\widetilde{\Phi}(P_p)$ it is*

$$\widetilde{L}_p := \limsup_{n\to\infty} \left((n/p)!\|\pi_n \exp(P)\|_{proj}\right)^{p/n} \geq \frac{\mathrm{Re}(v)}{\|\Phi\|_{L_n}^{p}},$$

where $\|\cdot\|_{proj}$ is the projective norm on $V^{\otimes^n}$ induced from $|\cdot|$, and $\|\Phi\|_{L_n}$ is defined by Equation (4).

In order to prove Proposition 4.1, for $\lambda \in \mathbb{R}$ and $P = \sum_{i=1}^{p} P_i \in L(V)$, we consider the dilated polynomial $\delta_\lambda P = \sum_{i=1}^{p} \lambda^i P_i$. We firstly prove the following.

Lemma 4.1. *Let $P = \sum_{i=1}^{p} P_i \in L(V)$ be a Lie polynomial and let $\Phi \in L(V, M_{n\times n}\mathbb{C})$. Then*

$$\widetilde{L}_p \geq \limsup_{\lambda\to\infty} \frac{\log \|\exp(\widetilde{\Phi}(\delta_\lambda(P))\|_{op}}{\lambda^p \|\Phi\|_{L_n}^{p}}, \tag{4.1}$$

where the exponential on the right-hand side is the matrix exponential in $M_{n\times n}\mathbb{C}$.

Before proving Lemma 4, we need a few intermediate lemmas.

The estimate below is well-known:

Lemma 4.2. *Let $v \in V^{\otimes p}$. Then*

$$\|\tilde{\Phi}(v)\|_{op} \leq \|\Phi\|_{L_n}^{p} \|v\|_{proj}.$$

Proof. Note that if $v = \sum_{i=1}^{m} v_1^i \otimes \ldots \otimes v_n^i$, then

$$\begin{aligned}
\|\tilde{\Phi}(v)\|_{op} &= \|\sum_{i=1}^{m} \Phi(v_1^i) \ldots \Phi(v_n^i)\|_{op} \\
&\leq \sum_{i=1}^{m} \|\Phi(v_1^i)\|_{op} \ldots \|\Phi(v_n^i)\|_{op} \\
&= \|\Phi\|_{L_n}^{n} \sum_{i=1}^{m} \|v_1^i\| \ldots \|v_n^i\|.
\end{aligned}$$

Taking infinum over all possible representations $\sum_{i=1}^{m} v_1^i \otimes \ldots \otimes v_n^i$ of v, we obtain

$$\|\tilde{\Phi}(v)\|_{op} \leq \|\Phi\|_{L_n}^{n} \|v\|_{proj}.$$

$\square$

Lemma 4.3. *Let p be a natural number. Define $f_p : [0,\infty) \to \mathbb{R}$ by*

$$f_p(x) = \sum_{n=0}^{\infty} \frac{x^{\frac{n}{p}}}{(\frac{n}{p})!}.$$

Then for all $x \in [0,\infty)$,

$$e^x \leq f_p(x) \leq \sum_{r=0}^{p-1} x^{\frac{r}{p}} e^x.$$

Proof. Note that for each $n \in \mathbb{N}$, we may find $q \in \mathbb{N} \cup \{0\}$ and $r < p$ such that

$$n = qp + r.$$

Then for $x \geq 0$,

$$\begin{aligned} f_p(x) &= \sum_{r=0}^{p-1} x^{\frac{r}{p}} \sum_{q=0}^{\infty} \frac{x^q}{(q+\frac{r}{p})!} \\ &\leq \sum_{r=0}^{p-1} x^{\frac{r}{p}} \sum_{q=0}^{\infty} \frac{x^q}{q!} \\ &\leq \sum_{r=0}^{p-1} x^{\frac{r}{p}} e^x. \end{aligned}$$

At the same time, we have by considering only terms in $\sum_{n=0}^{\infty} \frac{x^{\frac{n}{p}}}{(\frac{n}{p})!}$ which are divisible by p,

$$f_p(x) \geq e^x.$$

$\square$

Proof of Lemma 4. Note that

$$\begin{aligned} &\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op} \\ =&\|\tilde{\Phi}\left[\exp(\delta_\lambda(P))\right]\|_{op} \\ =&\|\sum_{n=0}^{\infty} \lambda^n \tilde{\Phi}\left[\pi_n\left(\exp(P)\right)\right]\|_{op} \\ \leq& \sum_{n=0}^{\infty} \lambda^n \|\tilde{\Phi}\left[\pi_n\left(\exp(P)\right)\right]\|_{op}. \end{aligned}$$

Using Lemma 4.2,

$$\begin{aligned} &\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op} \\ \leq& \sum_{n=0}^{\infty} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj}. \end{aligned}$$

Let $m \in \mathbb{N}$. Then

$$\begin{aligned}
&\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op} \\
\leq &\sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj} \\
&+ \sum_{n=m+1}^{\infty} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj} \\
= &\sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj} \\
&+ \sum_{n=m+1}^{\infty} \lambda^n \|\Phi\|_{L_n}^n \frac{\left[\|(\frac{n}{p})!\pi_n\left(\exp(P)\right)\|_{proj}^{\frac{p}{n}}\right]^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.
\end{aligned}$$

Define

$$\hat{L}_m = \sup_{n \geq m} \|(\frac{n}{p})!\pi_n\left(\exp(P)\right)\|_{proj}^{\frac{p}{n}}.$$

Then

$$\begin{aligned}
&\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op} \\
\leq &\sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj} \\
&+ \sum_{n=m+1}^{\infty} \lambda^n \|\Phi\|_{L_n}^n \frac{\hat{L}_m^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op} \\
\leq &\sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n \|\pi_n\left(\exp(P)\right)\|_{proj} \\
&- \sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n \frac{\hat{L}_m^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!} \\
&+ \sum_{n=0}^{\infty} (\lambda^p)^{\frac{n}{p}} (\|\Phi\|_{L_n}^p)^{\frac{n}{p}} \frac{\hat{L}_m^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.
\end{aligned}$$

We define the degree polynomial in λ

$$\mathcal{P}_m(\lambda) = \sum_{n=0}^{m} \lambda^n \|\Phi\|_{L_n}^n (\|\pi_n\left(\exp(P)\right)\|_{proj} - \frac{\hat{L}_m^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!})$$

and recall the definition of f_p from Lemma 4.3. Then

$$\begin{aligned}&\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op}\\ \le&\mathcal{P}_m(\lambda)+f_p(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m).\end{aligned}$$

Taking limit as $\lambda\to\infty$,

$$\begin{aligned}&\limsup_{\lambda\to\infty}\frac{\log\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op}}{\lambda^p\|\Phi\|_{L_n}^p}\\ \le&\limsup_{\lambda\to\infty}\frac{\log\left[\mathcal{P}_m(\lambda)+f_p(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m)\right]}{\lambda^p\|\Phi\|_{L_n}^p}.\end{aligned}\tag{4.2}$$

Note that by Lemma 4.3,

$$\begin{aligned}&\log\left[\mathcal{P}_m(\lambda)+f_p(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m)\right]\\ \le&\log\left[\mathcal{P}_m(\lambda)+\left[\sum_{r=0}^{p-1}\left(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m\right)^{\frac{r}{p}}\right]e^{\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m}\right]\\ \le&\log\left[\mathcal{P}_m(\lambda)e^{-\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m}+\left[\sum_{r=0}^{p-1}\left(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m\right)^{\frac{r}{p}}\right]\right]\\ &+\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m.\end{aligned}$$

Notice that

$$\mathcal{P}_m(\lambda)e^{-\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m}+\left[\sum_{r=0}^{p-1}\left(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m\right)^{\frac{r}{p}}\right]$$

grows at most polynomially fast in λ as $\lambda\to\infty$. Therefore, using the property that for any polynomial Q in λ,

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log|Q(\lambda)|=0,$$

we have

$$\begin{aligned}&\limsup_{\lambda\to\infty}\frac{\log\left[\mathcal{P}_m(\lambda)+f_p(\lambda^p\|\Phi\|_{L_n}^p\hat{L}_m)\right]}{\lambda^p\|\Phi\|_{L_n}^p}\\ \le&\hat{L}_m.\end{aligned}$$

Therefore, returning to (4.2), we have

$$\begin{aligned}&\limsup_{\lambda\to\infty}\frac{\log\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op}}{\lambda^p\|\Phi\|_{L_n}^p}\\ \le&\hat{L}_m.\end{aligned}$$

This holds for all m, and therefore taking $m\to\infty$ yields

$$\begin{aligned}&\limsup_{\lambda\to\infty}\frac{\log\|\exp(\tilde{\Phi}(\delta_\lambda(P)))\|_{op}}{\lambda^p\|\Phi\|_{L_n}^p}\\ \le&\tilde{L}_p.\end{aligned}$$

□

Proof of Proposition 4.1. For a matrix $M\imath n M_{n\times n}\mathbb{C}$, we denote by $\mathrm{Spec}(M)$ its spectrum. By using Lemma 4 we have

$$\begin{aligned}
\|\Phi\|_{L_n}^p \widetilde{L}_p &\geq \limsup_{\lambda\to\infty} \frac{\log \|\exp(\widetilde{\Phi}(\delta_\lambda(P)))\|}{\lambda^p} \geq \limsup_{\lambda\to\infty} \frac{\log(\max\{\|v\| : v \in \mathrm{Spec}(\exp(\widetilde{\Phi}(\delta_\lambda P)))\}}{\lambda^p} \\
&= \limsup_{\lambda\to\infty} \frac{\log(\max\{|e^v| : v \in \mathrm{Spec}(\widetilde{\Phi}(\delta_\lambda P))\}}{\lambda^p} \\
&= \limsup_{\lambda\to\infty} \frac{\log(\max\{e^{\mathrm{Re}(v)} : v \in \mathrm{Spec}(\widetilde{\Phi}(\delta_\lambda P))\}}{\lambda^p} \\
&= \limsup_{\lambda\to\infty} \frac{\max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(\delta_\lambda P))\}}{\lambda^p} \\
&= \limsup_{\lambda\to\infty} \max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(\frac{\delta_\lambda P}{\lambda^p}))\} \\
&= \limsup_{\lambda\to\infty} \max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(\frac{\sum_{i=1}^{p-1} \lambda^i P_i}{\lambda^p} + P_p))\} \\
&= \limsup_{\lambda\to\infty} \max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(\frac{o(\lambda^p)}{\lambda^p}) + \widetilde{\Phi}(P_p))\}. \qquad (4.3)
\end{aligned}$$

We consider the one-parameter family of operators $A : \mathbb{R} \to M_{n\times n}\mathbb{C}$ given by $A(\lambda) = \widetilde{\Phi}(\frac{o(\lambda^p)}{\lambda^p}) + \widetilde{\Phi}(P_p))$. Then

$$\lim_{\lambda\to\infty} A(\lambda) = \widetilde{\Phi}(P_p). \qquad (4.4)$$

Moreover, Theorem 5.2. in [9] asserts that there exist n continuous functions $v_i(\lambda)$, $i = 1, \ldots, n$, the values of which constitute the n-tuple $\mathrm{Spec}(A(\lambda))$ for any $\lambda \in \mathbb{R}$. The continuity of v_i as well as Equation (4.4) imply that $\mathrm{Spec}(\widetilde{\Phi}(P_p)) = \{\lim_{\lambda\to\infty} v_1(\lambda), \ldots, \lim_{\lambda\to\infty} v_n(\lambda)\}$, thus the limit $\lim_{\lambda\to\infty} \max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(A(\lambda))\}$ exists and is equal to $\max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(P_p))\}$. Therefore, if v is an eigenvalue of $\widetilde{\Phi}(P_p)$, then inequality (4.3) yields

$$\widetilde{L}_p \geq \frac{1}{\|\Phi\|_{L_n}^p} \max\{\mathrm{Re}(v) : v \in \mathrm{Spec}(\widetilde{\Phi}(P_p))\}.$$

Therefore,

$$\widetilde{L}_p \geq \frac{\mathrm{Re}(v)}{\|\Phi\|_{L_n}^p},$$

for any eigenvalue v of $\widetilde{\Phi}(P_p)$. ◻

By virtue of Proposition (4.1), and upon taking into account the upper bound of $\widetilde{L}_p$, we deduce the following.

Corollary 4.1. *Choose a norm $|\cdot|$ on V and endow $\mathbb{C}$ with the Euclidean norm. Let $P = \sum_{i=1}^p P_i \in L(V)$ and assume that there exists an $n \in \mathbb{N}$ and a map $\Phi \in L(V, M_{n\times n}\mathbb{C})$ with the following properties:*

1) $\|\Phi\|_{L_n} = 1$ *and*
2) $\widetilde{\Phi}(P_p)$ *has an eigenvalue* v *such that* $\mathrm{Re}(v) \geq \|P_p\|_{proj}$.

Then $\widetilde{L}_p \geq \|P_p\|_{proj}$, *that is Conjecture 2.1 is true for the polynomial* P.

5. A diagonal representation for $\widetilde{\Phi}(P_p)$

In this section we prove that for any $p \in \mathbb{N}$, there exists a subspace $\mathfrak{s}$ of $M_{p\times p}\mathbb{C}$ such that for any $\Phi \in L(V, \mathfrak{s})$ the matrix $\widetilde{\Phi}(P_p)$ is diagonal.
Denote by $\mathfrak{sl}_p\mathbb{C}$ the algebra of traceless matrices in $M_{p\times p}\mathbb{C}$. For any $i, j = 1, \ldots, p$, denote by $E_{ij} \in \mathfrak{sl}_n\mathbb{C}$ the matrix whose (i, j) entry is 1, and all the other entries are zero. Denote by H_{ij} the matrix $E_{ii} - E_{jj}$, $1 \leq i < j \leq p$. Then the set $B = \{E_{ij}, H_{kl} : i \neq j = 1, \ldots, p, \quad 1 \leq k < l \leq p\}$ is a basis of $\mathfrak{sl}_p\mathbb{C}$. Moreover, the following relations are valid.

$$\begin{aligned} [E_{ij}, E_{kl}] &= \delta_{jk}E_{il} - \delta_{il}E_{kj}, \\ [H_{ij}, E_{ij}] &= 2E_{ij}. \end{aligned} \tag{5.1}$$

We consider the spaces

$$\begin{aligned} \mathfrak{g}^{ij} :&= \mathbb{C}E_{ij}, \quad 1 \leq i < j \leq p, \\ \mathfrak{h} :&= \mathrm{span}_{\mathbb{C}}\{H_{kl} : 1 \leq k < l \leq p\}. \end{aligned} \tag{5.2}$$

The space $\mathfrak{h}$ coincides with the algebra of diagonal matrices in $\mathfrak{sl}_p\mathbb{C}$. We have the decomposition

$$\mathfrak{sl}_p\mathbb{C} = \mathfrak{h} \oplus \sum_{i\neq j} \mathfrak{g}^{ij}. \tag{5.3}$$

Relation (5.3) coincides with the root decomposition of $\mathfrak{sl}_p\mathbb{C}$ (see [8]). Equation (5.1) implies the following.

$$[\mathfrak{g}^{ij}, \mathfrak{g}^{kl}] \subseteq \begin{cases} \mathfrak{g}^{il}, & \text{if} \quad j = k \quad \text{and} \quad i \neq l, \\ \mathfrak{h}, & \text{if} \quad j = k \quad \text{and} \quad i = l, \\ \mathfrak{g}^{kj}, & \text{if} \quad j \neq k \quad \text{and} \quad i = l, \\ \{0\}, & \text{otherwise.} \end{cases} \tag{5.4}$$

The main result of the section is the following.

Lemma 5.1. *For any integer* $p \geq 2$, *let* $\mathfrak{s}$ *be the subspace of* $\mathfrak{sl}_p\mathbb{C}$ *defined by*

$$\mathfrak{s} := (\sum_{i=1}^{p-1} \mathfrak{g}^{i(i+1)}) + \mathfrak{g}^{p1}, \tag{5.5}$$

Then any matrix in $\underbrace{[\cdots[[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}], \ldots, \mathfrak{s}]}_{p \ \text{times}}$ *is diagonal.*

Remark 5.1. In matrix form, the space $\mathfrak{s}$, defined by Equation (5.5), is given by

$$\mathfrak{s} = \{ \begin{pmatrix} 0 & z_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & z_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & z_{p-1} \\ z_p & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : z_1, \ldots, z_p \in \mathbb{C}\}. \tag{5.6}$$

Proof of Lemma 5.1 It suffices to prove that $\mathfrak{s}$ satisfies the relation

$$\underbrace{[\cdots[[[\mathfrak{s},\mathfrak{s}],\ldots,\mathfrak{s}],\mathfrak{s}]}_{p \text{ times}} \subseteq \mathfrak{h}, \tag{5.7}$$

where $\mathfrak{h}$ is the algebra of diagonal matrices, given by relation (5.2). By taking into account relation (5.4), we obtain

$$\begin{aligned}
[\mathfrak{s},\mathfrak{s}] &\subseteq (\mathfrak{g}^{13} + \mathfrak{g}^{24} + \cdots + \mathfrak{g}^{(p-2)p}) + (\mathfrak{g}^{p2} + \mathfrak{g}^{(p-1)1}), \\
[[\mathfrak{s},\mathfrak{s}],\mathfrak{s}] &\subseteq (\mathfrak{g}^{14} + \mathfrak{g}^{25} + \cdots + \mathfrak{g}^{(p-3)p}) + (\mathfrak{g}^{p3} + \mathfrak{g}^{(p-1)2} + \mathfrak{g}^{(p-2)1}), \\
&\vdots \\
\underbrace{[\cdots[[[\mathfrak{s},\mathfrak{s}],\ldots,\mathfrak{s}],\mathfrak{s}]}_{k \text{ times}} &\subseteq (\mathfrak{g}^{1(k+1)} + \mathfrak{g}^{2(k+2)} + \cdots + \mathfrak{g}^{(p-k)p}) + (\mathfrak{g}^{pk} + \mathfrak{g}^{(p-1)(k-1)} + \cdots + \mathfrak{g}^{(p-k+1)1}), \\
&\vdots \\
\underbrace{[\cdots[[[\mathfrak{s},\mathfrak{s}],\ldots,\mathfrak{s}],\mathfrak{s}]}_{(p-1) \text{ times}} &\subseteq (\mathfrak{g}^{1p}) + (\mathfrak{g}^{p(p-1)} + \mathfrak{g}^{(p-1)(p-2)} + \cdots + \mathfrak{g}^{21}).
\end{aligned} \tag{5.8}$$

Moreover, relation (5.4) implies that

$$[\mathfrak{g}^{ij}, \mathfrak{g}^{ji}] \subseteq \mathfrak{h}. \tag{5.9}$$

Taking into account relation (5.9), relation (5.8) along with the definition (5.5) of $\mathfrak{s}$ yield $\underbrace{[\cdots[[[\mathfrak{s},\mathfrak{s}],\mathfrak{s}],\ldots,\mathfrak{s}]}_{p \text{ times}} \subseteq \mathfrak{h}$. □

Corollary 5.1. *Let $p \in \mathbb{N}$ with $p \geq 2$, and let $P = P_1 + \cdots + P_p \in L((V))$ be a Lie polynomial of degree p. Then there exists a subspace $\mathfrak{s}$ of $M_{p\times p}\mathbb{C}$ such that, for any $\Phi \in L(V,\mathfrak{s})$, the matrix $\widetilde{\Phi}(P_p)$ is diagonal, where $\widetilde{\Phi} : T((V)) \to M_{p\times p}\mathbb{C}$ is the unique homomorphism extending Φ.*

Proof. Let $\mathfrak{s}$ be the space given in Lemma 5.1. Let $V = \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_d\}$, and write

$$P_p = \sum_{i_1,\ldots,i_p=1,\ldots,d} a_{i_1\ldots i_p}[\cdots[[[e_{i_p}, e_{i_{p-1}}], e_{i_{p-2}}], \ldots, e_{i_1}].$$

Then

$$\widetilde{\Phi}(P_p) = \sum_{i_1,\ldots,i_p=1,\ldots,d} a_{i_1\ldots i_p}[\cdots[[[\Phi(e_{i_p}),\Phi(e_{i_{p-1}})],\Phi(e_{i_{p-2}})],\ldots,\Phi(e_{i_1})] \in \underbrace{[\cdots[[[\mathfrak{s},\mathfrak{s}],\mathfrak{s}],\ldots,\mathfrak{s}]}_{p\ \text{times}},$$

which, by virtue of Lemma 5.1 is diagonal. □

6. Conjecture 2.1 for Lie polynomials of degree $p \leq 3$

In this section, we prove Conjecture 2.1 in the case where $V = \mathbb{R}^2$ and $P \in L((V))$ is any Lie polynomial of degree $p \leq 3$. We endow V with the Euclidean norm $\|\cdot\|_{l_2}$. We also consider the projective norm $\|\cdot\|_{proj}$ on $V^{\otimes^n}$, $n = 1, 2, \ldots$, induced by the norm $\|\cdot\|_{l_2}$. We have the following result.

Theorem 6.1. *Let V be a two dimensional vector space and let $P \in L((V))$ be a Lie polynomial of degree $p \leq 3$. Then*

$$\widetilde{L}_p := \limsup_{n\to\infty} \big((\frac{n}{p})!\|\pi_n(\exp P)\|_{proj}\big)^{\frac{p}{n}} = \|P_p\|_{proj},$$

where $\|\cdot\|_{proj}$ is the projective norm induced from the Euclidean norm on V. In other words, Conjecture 2.1 is true for P.

Proof. Since $\widetilde{L}_p \leq \|P_p\|_{proj}$ for any $p \in \mathbb{N}$, it suffices to show that

$$\widetilde{L}_p \geq \|P_p\|_{proj}, \quad p = 1, 2, 3. \tag{6.1}$$

If $p = 1$, then $P = P_1 \in V$ and inequality (6.1) holds trivially. Let $\{e_1, e_2\}$ be an orthonormal basis of V. Assume that $p = 2$, that is $P = P_1 + P_2$ with $P_2 \in [V, V]$. The space $[V, V]$ is one-dimensional, and is generated by the vector $[e_1, e_2]$. Therefore $P_2 = c[e_1, e_2]$. From the definition (2.1) of the projective norm we have

$$\|P_2\|_{proj} \leq |c|\big(\|e_1\|_{l_2}\|e_2\|_{l_2} + \|e_2\|_{l_2}\|e_1\|_{l_2}\big) = 2|c|. \tag{6.2}$$

We will verify Corollary 4.1 for P. We choose $\Phi \in L(V, M_{2\times 2}\mathbb{C})$ by

$$\Phi(e_1) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \quad \text{and} \quad \Phi(e_2) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}. \tag{6.3}$$

We have

$$\|\Phi\|_{L_2} = \sup_{\|v\|_{l_2}=1} \|\Phi(v)\|_{op} = \sup_{\|v\|_{l_2}=1, \|z\|=1} \|\Phi(v)z\|. \tag{6.4}$$

We set $v = ae_1 + be_2$ with $\sqrt{a^2+b^2} = \|v\|_{l_2} \leq 1$ and $z = (z_1, z_2) \in \mathbb{C}^2$ with $\sqrt{\|z_1\|^2 + \|z_2\|^2} = \|z\| \leq 1$. By taking into account equations (6.3) and (6.4), we obtain

$$\begin{aligned}\|\Phi\|_{L_2} &= \sup_{a^2+b^2\le 1, \|z_1\|^2+\|z_2\|^2\le 1} \|\big((a+ib)z_2, (ia+b)z_1\big)\| \\ &= \sup_{a^2+b^2\le 1, \|z_1\|^2+\|z_2\|^2\le 1} \sqrt{\|(a+ib)z_2\|^2 + \|(ia+b)z_1\|^2} \\ &= \sup_{a^2+b^2\le 1, \|z_1\|^2+\|z_2\|^2\le 1} \sqrt{\|(a+ib)\|^2\|z_2\|^2 + \|(ia+b)\|^2\|z_1\|^2} \\ &= \sup_{a^2+b^2\le 1, \|z_1\|^2+\|z_2\|^2\le 1} \sqrt{(a^2+b^2)(\|z_2\|^2+|z_1\|^2)} = 1. \end{aligned} \tag{6.5}$$

Moreover,

$$\widetilde{\Phi}(P_2) = c[\Phi(e_1), \Phi(e_2)] = \begin{pmatrix} 2c & 0 \\ 0 & -2c \end{pmatrix},$$

hence $\widetilde{\Phi}(P_2)$ has the eigenvalue $2|c|$, which, by virtue of relation (6.2) is greater than or equal to $\|P_2\|_{proj}$. Taking into account Equation (6.5), Corollary 4.1 implies that Conjecture 2.1 is true for P.
Finally, assume that $p = 3$, that is $P = P_1 + P_2 + P_3$, with $P_3 \in [V, [V, V]]$. We claim the following.

Claim 6.1. *There exists an orthonormal basis $\{v_1, v_2\}$ of V such that*

$$P_3 = c[v_1, [v_1, v_2]]. \tag{6.6}$$

Proof of Claim 6.1. A basis for the space $[V, [V, V]]$ is $\{[e_1, [e_1, e_2]], [e_2, [e_1, e_2]]\}$. Therefore

$$P_3 = c_1[e_1, [e_1, e_2]] + c_2[e_2, [e_1, e_2]], \quad c_1, c_2 \in \mathbb{R}. \tag{6.7}$$

If $c_2 = 0$ the proof is concluded. If $c_2 \neq 0$, we consider the clockwise rotation A_θ by

$$\theta := \arctan \frac{c_1}{c_2}, \tag{6.8}$$

and we set $v_1 := A_\theta e_2$, $v_2 := A_\theta e_1$. Then $\{v_1, v_2\}$ is an orthonormal basis of V. Moreover, $e_1 = (\cos\theta)v_2 + (\sin\theta)v_1$ and $e_2 = -(\sin\theta)v_2 + (\cos\theta)v_1$. By substituting the last equations into Equation (6.7) and by taking into account Equation (6.8), we obtain

$$P_3 = (c_1 \sin\theta + c_2 \cos\theta)[v_1, [v_1, v_2]].$$

□

By taking into account Equation (6.6) as well as the definition (2.1) of the projective norm we obtain

$$\|P_3\|_{proj} \le |c|\big(\|v_1\|_{l_2}\|v_1\|_{l_2}\|v_2\|_{l_2} + 2\|v_1\|_{l_2}\|v_2\|_{l_2}\|v_1\|_{l_2} + \|v_2\|_{l_2}\|v_2\|_{l_2}\|v_1\|_{l_2}\big) \le 4|c|. \tag{6.9}$$

We define $\Phi \in L(V, M_{3\times 3}\mathbb{C}$ by

$$\Phi(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(e_2) = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & -1 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{if} \quad c > 0,$$

$$\Phi(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(e_2) = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}, \quad \text{if} \quad c < 0.$$

With similar calculations as in the case $p = 2$ and by taking into account relation (6.9), we can verify that 1) $\|\Phi\|_{L_3} = 1$ and 2) $\widetilde{\Phi}(P_3)$ has an eigenvalue which is greater that or equal to $\|P_3\|_{proj}$. Therefore, by virtue of Corollary 4.1, Conjecture 2.1 is true for P. □

7. A weaker version of Conjecture 2.1 for Lie polynomials of degree four

Let $V = \mathbb{R}^2$. We prove the following result for Lie polynomials of degree four in $L((V))$.

Proposition 7.1. *Let $\|\cdot\|_{proj} : V^{\otimes^n} \to \mathbb{R}$, $n = 1, 2, \ldots$ denote the family of projective norms induced from the Euclidean norm $\|\cdot\|_{l_2}$ on V. Then for any Lie polynomial $P = P_1 + \cdots + P_4$, of degree four in $L(V)$, it is*

$$\frac{1}{4}\|P_4\|_{proj} \leq \widetilde{L}_p \leq \|P_4\|_{proj}.$$

We will firstly prove the following lemma.

Lemma 7.1. *For any homogeneous Lie polynomial P_4 of degree four in $L(V)$ there exists an orthonormal basis $\{v_1, v_2\}$ of V such that*

$$P_4 = d_1[v_1, [v_1, [v_1, v_2]] + d_2[v_2, [v_2, [v_1, v_2]]], \quad d_1, d_2 \in \mathbb{R}. \tag{7.1}$$

Proof. We fix the standard basis $\{e_1, e_2\}$ on V. A basis for the space $[V, [V, [V, V]]$, of homogeneous Lie polynomials of degree four, is the set
$\{[e_1, [e_1, [e_1, e_2]]], [e_1, [e_2, [e_1, e_2]]], [e_2, [e_2, [e_1, e_2]]]\}$. Therefore, P_4 has the form

$$P_4 = c_1[e_1, [e_1, [e_1, e_2]] + c_2[e_1, [e_2, [e_1, e_2]]] + c_3[e_2, [e_2, [e_1, e_2]]], \quad c_1, c_2, c_3 \in \mathbb{R}. \tag{7.2}$$

If $c_2 = 0$ then the proof is concluded. Assume that $c_2 \neq 0$. We set

$$\theta \triangleq -\frac{1}{2} \operatorname{arccot} \frac{c_1 + c_3}{c_2}, \tag{7.3}$$

and we rotate the basis $\{e_1, e_2\}$ by θ, thus obtaining an orthonormal basis $\{v_1, v_2\}$ satisfying the equations

$$e_1 = (\cos\theta)v_1 + (\sin\theta)v_2 \quad \text{and} \quad e_2 = -(\sin\theta)v_1 + (\cos\theta)v_2. \tag{7.4}$$

By substituting Equations (7.4) into Equation (7) and after straightforward calculations, we obtain

$$P_4 = d_1[e_1, [e_1, [e_1, e_2]] + d_2[e_2, [e_2, [e_1, e_2]]] + d_3[e_1, [e_2, [e_1, e_2]]],$$

where

$$\begin{aligned} d_3 &= 2c_1\cos\theta\sin\theta + c_2(\cos^2\theta - \sin^2\theta) + 2c_3\cos\theta\sin\theta \\ &= (c_1+c_3)\sin(2\theta) + c_2\cos(2\theta) = 0, \end{aligned}$$

the last equality following from Equation (7.3). □

By expressing explicitly the Lie brackets in Equation (7.1), by using the fact that $\{v_1, v_2\}$ is an orthonormal basis as well as the definition (2.1) of the projective norm, we obtain

$$\|P_4\|_{proj} \leq 8(|d_1| + |d_2|). \tag{7.5}$$

We proceed to prove Proposition 7.1.

Proof of Proposition 7.1 It suffices to prove that $\widetilde{L}_p \geq \|P_4\|_{proj}$. Let $\{v_1, v_2\}$ be the basis of V constructed in Lemma 7.1. We consider the following two cases for P_4: (a) $d_1, d_2 \geq 0$ and (b) $d_1 \geq 0$, $d_2 < 0$. The other two cases are similar.
For Case (a) we define $\Phi \in L(V, M_{2\times 2}\mathbb{C})$ by

$$\Phi(v_1) = \begin{pmatrix} 0 & 1 \\ e^{\frac{i\pi}{4}} & 0 \end{pmatrix} \quad \text{and} \quad \Phi(v_2) = \begin{pmatrix} 0 & i \\ e^{-\frac{i\pi}{4}} & 0 \end{pmatrix}. \tag{7.6}$$

We will prove that 1) $\|\Phi\|_{L_2} = 1$ and 2) $\widetilde{\Phi}(P_4)$ has an eigenvalue equal to $8(d_1+d_2)$, which, in view of Proposition 4.1 and relation (7.5), will imply that $\widetilde{L_4} \geq \frac{1}{2}\|P_4\|_{proj}$ for Case (a). For the first fact, we endow $\mathbb{C}^2$ with the Euclidean norm $\|\cdot\|$, and we have

$$\|\Phi\|_{L_2} = \sup_{\|v\|_{l_2}=1, \|z\|=1} \|\Phi(v)z\|. \tag{7.7}$$

Setting $v = av_1 + bv_2$ and $z = (z_1, z_2) \in \mathbb{C}^2$, equations (7.6) and (7.7) yield

$$\begin{aligned} \|\Phi\|_{L_2} &= \sup_{a^2+b^2\leq 1, \|z_1\|^2+\|z_2\|^2\leq 1} \left\| \left((a+bi)z_2, \frac{\sqrt{2}}{2}\big((a+b)+(a-b)i\big)z_1 \right) \right\| \\ &= \sup_{a^2+b^2\leq 1, \|z_1\|^2+\|z_2\|^2\leq 1} \sqrt{\|(a+bi)z_2\|^2 + \frac{1}{2}\|\big((a+b)+(a-b)i\big)z_1\|^2} \\ &= \sup_{a^2+b^2\leq 1, \|z_1\|^2+\|z_2\|^2\leq 1} \sqrt{\|a+bi\|^2\|z_2\|^2 + \frac{1}{2}\|(a+b)+(a-b)i\|^2\|z_1\|^2} \\ &= \sup_{a^2+b^2\leq 1, \|z_1\|^2+\|z_2\|^2\leq 1} \sqrt{(a^2+b^2)(\|z_2\|^2+\|z_1\|^2)} = 1. \end{aligned}$$

The second fact is straightforward to verify, therefore the proof of Proposition 7.1 is concluded for Case (a).
For Case (b) we set

$$\Phi(v_1) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \text{and} \quad \Phi(v_2) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{7.8}$$

By using similar calculations as in Case (a), we deduce that $\|\Phi\|_{L_2} = 1$. Moreover, after some straightforward calculations, it can be shown that $\widetilde{\Phi}(P_4)$ has an eigenvalue v such that $\mathrm{Re}(v) = 2(d_1 - d_2)$. By virtue of relation (7.5), we have $\mathrm{Re}(v) \geq \frac{1}{4}\|P_4\|$. Since $\|\Phi\|_{L_2} = 1$, Proposition 4.1 implies that $\widetilde{L}_4 \geq \frac{1}{4}\|P_4\|$, which concludes the proof of Proposition 7.1 for Case (b). □

8. Appendix: Some properties of pure rough paths

This article focuses on studying rough paths X whose signature can be expressed as

$$S(X)_{0,1} = \exp(P), \tag{8.1}$$

where $P = \sum_{i=1}^{p} P_i$, P_i is a Lie polynomial of degree i and $P_p \neq 0$.

In the first subsection, we will characterise all the rough paths for which (8.1) holds. In the second subsection, we will show that certain variational norms of X is exactly equal to $\|P_p\|$. Since the quantity $\|P_p\|$ appears in our main result Theorem 1.1, the result of the second subsection provides a geometric intuition for the limit studied in Theorem 1.1 in terms of the path X.

8.1. Paths whose signatures are the exponential of Lie polynomials.

Given a Lie polynomial P of degree p, we claim that the function $X : \Delta_{[0,1]} \to T^{[p]}(V)$ defined by

$$X : (s,t) \to \exp((t-s)P)$$

is the only rough paths whose signature is $\exp(P)$. This consists of three claims:

(a) The function X defines a rough path, and hence its signature is well-defined;

(b) The signature of X is $\exp(P)$;

(c) The path X is the only reduced rough path with $\exp(P)$ as its signature.

We start with showing (a). A rough path must satisfy two conditions:

(i) The function X must be multiplicative, in the sense that for any $0 \leq s \leq u \leq t$,

$$X_{s,u} \otimes X_{u,t} = X_{s,t}.$$

Note that $(t-u)P$ and $(u-s)P$ is commutative with respect to the tensor product. Therefore,

$$\begin{aligned}
&X_{s,u} \otimes X_{u,t} \\
=&\exp((u-s)P) \otimes \exp((t-u)P) \\
=&\exp((u-s)P + (t-u)P) \\
=&\exp((t-s)P). \\
=&X_{s,t}.
\end{aligned}$$

(ii) The function X must have finite p-variation, which is shown in the lemma below:

Lemma 8.1. *Let $P = \sum_{i=1}^{p} P_i$, P_i is a homogeneous Lie polynomial of degree i. For every n, the function*

$$X : (s,t) \to \pi_n(\exp((t-s)P))$$

is finite p-variation.

Proof. Let $0 = t_0 < t_1 < \ldots < t_N = 1$ be a partition.

$$\begin{aligned}
&\sum_{i=0}^{N-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{n}} \\
&= \sum_{i=0}^{N-1} \|\pi_n(\exp((t_{i+1}-t_i)P))\|^{\frac{p}{n}} \\
&= \sum_{i=0}^{N-1} \| \sum_{k=0}^{\infty} \frac{((t_{i+1}-t_i)P))^{\otimes k}}{k!} \|^{\frac{p}{n}} \\
&= \sum_{i=0}^{N-1} \| \sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!} \|^{\frac{p}{n}}.
\end{aligned}$$

Note that for each term inside the sum $\sum_{i_1+\ldots+i_k=n}$, since $i_j \leq p$ for all j, we have that $k \geq \frac{n}{p}$. This together with $t_{i+1} - t_i \leq 1$ implies that $(t_{i+1} - t_i)^k \leq (t_{i+1} - t_i)^{\frac{n}{p}}$.

$$\begin{aligned}
&\sum_{i=0}^{N-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{n}} \\
&\leq \sum_{i=0}^{N-1} (t_{i+1}-t_i) \| \sum_{i_1+\ldots+i_k=n} \frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!} \|^{\frac{p}{i}} \\
&= \sum_{i_1+\ldots+i_k=n} \frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!} \|^{\frac{p}{i}}.
\end{aligned}$$

Since the quantity $\| \sum_{i_1+\ldots+i_k=n} \frac{1}{k!} P_{i_1} \otimes \ldots \otimes P_{i_k} \|^{\frac{p}{i}}$ is independent of the partition $(0 < t_1 < \ldots < t_N = 1)$,

$$\begin{aligned}
&\sup_{0<t_1<\ldots<t_N=1} \sum_{i=0}^{N-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{n}} \\
&\leq \| \sum_{i_1+\ldots+i_k=n} \frac{1}{k!} P_{i_1}\otimes\ldots\otimes P_{i_k} \|^{\frac{p}{i}}.
\end{aligned}$$

Therefore, X has finite p-variation. Since X is by definition expressed as the exponential of Lie polynomial, X is a p weakly geometric rough path. □

We now show (b).

Lemma 8.2. *The signature $S(X)_{0,1}$ of the p-rough path X is of the form*

$$S(X)_{0,1} = \exp(P).$$

Proof. The signature functional

$$(s,t) \to S(X)_{s,t}$$

is defined to be the unique function $(s,t) \to T((\mathbb{R}^d))$ such that

$$S(X)_{s,u} \otimes S(X)_{u,t} = S(X)_{s,t}$$

for all $s \le u \le t$; $(s,t) \to \pi_n(S(X)_{s,t})$ has finite p-variation; and

$$\pi_n(S(X)_{s,t}) = \pi_n(X_{s,t})$$

for all $n \le p$ (see Theorem 2.2.1 in [10]). Define

$$\tilde{S}(X)_{s,t} = \exp((t-s)P).$$

Note that $\tilde{S}(X)_{s,t}$ is an extension of X in the sense that

$$\pi_n(\tilde{S}(X)_{s,t}) = \pi_n(X_{s,t})$$

for all $s \le t$.

Since $(t-s)P$ and $(u-s)P$ commutes with respect to the tensor product,

$$\begin{aligned}
&\tilde{S}(X)_{s,u} \otimes \tilde{S}(X)_{u,t} \\
=&\exp((u-s)P) \otimes \exp((t-u)P) \\
=&\exp((t-s)P) \\
=&\tilde{S}(X)_{s,t},
\end{aligned}$$

and therefore $\tilde{S}(X)$ is a multiplicative extension of X. By Lemma 8.1, $(s,t) \to \tilde{S}(X)_{s,t}$ has finite p-variation.

Therefore, by the uniqueness of multiplicative extension for rough paths, Theorem 2.2.1 in [10],

$$S(X)_{s,t} = \exp((t-s)P).$$

□

Finally, to show Claim (c) at the beginning of this subsection, we simply note that X is the unique reduced path such that $S(X)_{0,1} = \exp(P)$. This is a direct consequence of the uniqueness theorem (Lemma 4.6 in [1]): The signature of a reduced weakly geometric rough path uniquely determines the path.

8.2. **Interpretation of $\|P_p\|$ in terms of X.**

Lemma 8.3. *Let $\lim_{\max_i |t_{i+1}-t_i| \to 0}$ denote the limit as the maximum gap, $\max_i |t_{i+1}-t_i| \to 0$, between adjacent partition points of $(t_0 < t_1 < \ldots < t_n)$ tends to zero. Let $\pi_n(X_{t_i,t_{i+1}})$ be the projection onto the degree-n component of X. Then*

$$\lim_{\max_i |t_{i+1}-t_i| \to 0} \sum_{i=0}^{n-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{i}} = \begin{cases} \|P_p\|, & \text{if } n = p; \\ 0, & \text{if} n < p. \end{cases}$$

Proof. Recall from the proof of Lemma 8.1 that

$$\begin{aligned}
&\sum_{i=0}^{N-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{n}} \\
=&\sum_{i=0}^{N-1} \| \sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \|^{\frac{p}{n}}.
\end{aligned}$$

Suppose that $n < p$. Note that as $k \geq 1$ and $t_{i+1} - t_i < 1$,

$$\begin{aligned}&\sum_{i=0}^{N-1} \| \sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \|^{\frac{p}{n}} \\ \leq &\sum_{i=0}^{N-1} (t_{i+1}-t_i)^{\frac{p}{n}} \| \sum_{i_1+\ldots+i_k=n} \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \|^{\frac{p}{n}}.\end{aligned}$$

As $n < p$,

$$\lim_{\max_i |t_{i+1}-t_i| \to 0} \sum_{i=0}^{N-1} (t_{i+1}-t_i)^{\frac{p}{n}} = 0.$$

Therefore if $n < p$, then

$$\lim_{\max_i |t_{i+1}-t_i| \to 0} \sum_{i=0}^{N-1} \|\pi_n(X_{t_i,t_{i+1}})\|^{\frac{p}{n}} = 0.$$

Now suppose that $n = p$. Since the only term in

$$\sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!}$$

such that $k = 1$ is P_p. Therefore, we have

$$\begin{aligned}&\sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \\ =&(t_{i+1}-t_i)P_p + (t_{i+1}-t_i)^2 R,\end{aligned}$$

where

$$R = \sum_{i_1+\ldots+i_k=n, k\geq 2} (t_{i+1}-t_i)^{k-2} \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!}.$$

Note that R can be bounded so that it is independent of the partition points, through

$$\begin{aligned}\|R\| &\leq \| \sum_{i_1+\ldots+i_k=n, k\geq 2} (t_{i+1}-t_i)^{k-2} \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \| \\ &\leq \sum_{i_1+\ldots+i_k=n, k\geq 2} (t_{i+1}-t_i)^{k-2} \frac{\|P_{i_1} \otimes \ldots \otimes P_{i_k}\|}{k!} \\ &\leq \sum_{i_1+\ldots+i_k=n, k\geq 2} \frac{\|P_{i_1} \otimes \ldots \otimes P_{i_k}\|}{k!}, \text{ as } t_i \leq 1.\end{aligned}$$

Since $(t_i)_{i=0}^N$ are partition points of $[0,1]$, $\sum_{i=0}^{N-1}(t_{i+1}-t_i) = 1$. We have

$$\begin{aligned}&\left| \sum_{i=0}^{N-1} \| \sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \| - \|P_p\| \right| \\ =&\left| \sum_{i=0}^{N-1} \| \sum_{i_1+\ldots+i_k=n} (t_{i+1}-t_i)^k \frac{P_{i_1} \otimes \ldots \otimes P_{i_k}}{k!} \| - \sum_{i=0}^{N-1} (t_{i+1}-t_i)\|P_p\| \right|.\end{aligned}$$

Using the reverse triangle inequality,

$$\begin{aligned}
&\left|\sum_{i=0}^{N-1}\left[\|\sum_{i_1+\ldots+i_k=n}(t_{i+1}-t_i)^k\frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!}\|-(t_{i+1}-t_i)\|P_p\|\right]\right|\\
\leq&\sum_{i=0}^{N-1}\left|\|\sum_{i_1+\ldots+i_k=n}(t_{i+1}-t_i)^k\frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!}-(t_{i+1}-t_i)P_p\|\right|\\
=&\sum_{i=0}^{N-1}\|(t_{i+1}-t_i)^2R\|.
\end{aligned}$$

Since R can be bounded independent of the choice of partition $(t_i)_{i=0}^{N-1}$,

$$\lim_{\max_i|t_{i+1}-t_i|\to 0}\sum_{i=0}^{N-1}\|(t_{i+1}-t_i)^2R\|=0.$$

Therefore,

$$\begin{aligned}
&\lim_{\max_i|t_{i+1}-t_i|\to 0}\sum_{i=0}^{N-1}\|\sum_{i_1+\ldots+i_k=n}(t_{i+1}-t_i)^k\frac{P_{i_1}\otimes\ldots\otimes P_{i_k}}{k!}\|\\
=&\|P_p\|.
\end{aligned}$$

□

9. ACKNOWLEDGEMENT

HB and NS are supported by the EPSRC grant EP/R008205/1 and the Department of Mathematics and Statistics, University of Reading, U.K.. XG is supported by the Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213, United States.

References

[1] [BGLY16] H. Boedihardjo, X. Geng, T. Lyons and D. Yang, The signature of a rough path: uniqueness, *Advances in Mathematics*, 293, 720-737, 2016.

[2] [BG18]H. Boedihardjo and X. Geng, The Tail Asymptotics of Brownian Signature, arXiv:1609.08111, 2018.

[3] [CL16]I. Chevyrev and T. Lyons, Characteristic functions of measures on geometric rough paths, *The Annals of Probability*, 44(6),4049-4082, 2016.

[4] [FR11]P. Friz, S. Riedel, Integrability of Linear rough differential equations and moment estimates for iterated integrals of Gaussian processes, https://arxiv.org/pdf/1104.0577v2.pdf.

[5] [Geng17] X. Geng, Reconstruction for the signature of a rough path, *Proceedings of the London Mathematical Society*, 114(3), 495-526, 2017.

[6] [HL10]B. Hambly and T. Lyons, Uniqueness for the signature of a path of bounded variation and the reduced path group, *Ann. of Math.* 171 (1): 109–167, 2010.

[7] [HH10]K. Hara and M. Hino, Fractional order Taylor's series and the neoclassical inequality, *Bulletin of the London Mathematical Society* 42(3), 467-477, 2010.

[8] [Hum72]J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, third printing, 1980.

[9] [Kat82]T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer-Verlag, 1982.

[10] [Lyons98]T. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoamericana* 14 (2), 215–310, 1998.

[11] [LS06]T. Lyons and N. Sidorova, On the radius of convergence of the logarithmic signature, *Illinois Journal of Mathematics*, 50(1-4), 763-790, 2006.

[12] [LX17]T. Lyons and W. Xu, Hyperbolic development and inversion of signature, *Journal of Functional Analysis*, 272(7), 2933-2955, 2017.

[13] [LX18]T. Lyons and W. Xu, Inverting the signature of a path, *Journal of the European Mathematical Society*, 20(7), 1655-1687, 2018.