The Signature of a Rough Path: Uniqueness

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Abstract

The signature of a path is the collection of all iterated integrals of the path. It plays a fundamental role in rough path theory. B. Hambly and T. Lyons proved that the signature of a bounded variation path is trivial if and only if the path is the image of some closed path in a real tree. Extending their result to general rough paths has been a long standing open problem in rough path theory. We propose a proof for this conjecture in the case of weakly geometric rough paths in finite dimensions.

1 Introduction

The Stieltjes differential equation

$$dy_s = V\left(y_s\right) dx_s,\ y_0 = Y \tag{1}$$

where x is a path and V is a vector field, frequently appears in the mathematical modeling of, for example, electric circuits and stock prices.

We are interested in the information about the driving signal x required to predict y_T, for some time T. For preciseness, assume $x : [0,T] \to \mathbb{R}^d$ has bounded total variation and $V : \mathbb{R}^n \to L\left(\mathbb{R}^d, \mathbb{R}^{d'}\right)$. Under some regulairty conditions on V, y_t depends on x only through the iterated integrals of x up to time t [9].

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For algebraic reasons, it is useful to collect the sequence of iterated integrals into a single tensor element.

Definition 1. Suppose $x : [0,T] \to \mathbb{R}^d$ has bounded variation. Then the tensor element

$$S(x) = \left(1, \int_0^T dx_{s_1}, \ldots, \int_{0<s_1<\ldots<s_n<T} dx_{s_1} \otimes \ldots \otimes dx_{s_n}, \ldots\right) \in T\left(\left(\mathbb{R}^d\right)\right)$$

is called the *signature* of the path x.

Let $\Delta = \{(s,t) : 0 \leqslant s \leqslant t \leqslant T\}$. For $(s,t) \in \Delta$, let $S(x)_{s,t} = S\left(x|_{[s,t]}\right)$. The signature satisfies the Chen's identity

$$S(x)_{s,t} = S(x)_{s,u} \otimes S(x)_{u,t} \;\; \forall s \leqslant u \leqslant t, \tag{2}$$

which is a non-commutative version of additivity of integral over disjoint intervals.

The signature also preserves the regularity of the original path, in the sense that if x has bounded total variation, then for each n, if $\pi^{(n)}$ is the projection of $T\left(\left(\mathbb{R}^d\right)\right)$ onto $\left(\mathbb{R}^d\right)^{\otimes n}$, then $\pi_n \circ S(x)_{\cdot,\cdot}$ also has bounded total variation.

When x only has bounded p variation, $p \geqslant 2$, the iterated integrals if defined as Riemann-Stieltjes sums will not converge. However, the conditions of additivity and preserving the regularity still makes sense. Indeed, let $x : [0,T] \to G^N\left(\mathbb{R}^d\right)$ be a path in the step N nilpotent Lie group [7], viewed as embedded into its enveloping tensor algebra $T^N\left(\mathbb{R}^d\right)$, with finite p-variation, that is

$$\|x\|_p = \max_{n\leqslant N} \sup_{0<t_1<\ldots<t_n<T} \left(\sum_i \left\|\pi^{(n)}\left(S(x)_{t_i,t_{i+1}}\right)\right\|_{\left(\mathbb{R}^d\right)^{\otimes n}}^{\frac{p}{n}}\right)^{\frac{1}{p}} < \infty$$

, where $p < N$, then there exists a unique function $S(x) : \Delta \to T\left(\left(\mathbb{R}^d\right)\right)$ such that $S(x)$ satisfies (2), $\pi_n \circ S(x)$ has finite p variation for all n and $\pi_N\left(S(x)_{s,t}\right) = x_s^{-1} x_t$ [12], where π_N is the projection $T\left(\left(\mathbb{R}^d\right)\right) \to T^N\left(\mathbb{R}^d\right)$.

Let $S_N(x)$ denote $\pi_N(S(x))$. For $p \geqslant 1$, let $WG\Omega_p\left(\mathbb{R}^d\right)$ denote the set of all paths $x : [0,T] \to G^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)$ with finite p variation. The elements of $WG\Omega_p\left(\mathbb{R}^d\right)$ are called p weakly geometric rough paths. For $x \in WG\Omega_p\left(\mathbb{R}^d\right)$, $S(x)_{0,T}$ will be called the signature of the path x. Note that in the case when paths have bounded total variation, this notion of signature coincides with the one we defined before.

The goal of this paper is to investigate the conditions under which two multidimensional paths share the same signature. Note that by (2) and that if $\overleftarrow{x}$ denote the reversal of the path x, then $S(\overleftarrow{x}) = S(x)^{-1}$, the problem is reduced to identifying paths whose signature is trivial.

In [4], K.T. Chen showed that for piecewise regular, irreducible paths, the signatures determines the paths. In [9], B. Hambly and T. Lyons showed that a bounded variation path has trivial signature if and only if it is tree-like in the following sense.

Definition 2. Let V be a topological space. A continuous path $x:[0,T]\rightarrow V$ is *tree-like* if there exists a real tree τ, a continuous map $\phi:[0,T]\rightarrow\tau$ and a map $\psi:\tau\rightarrow V$ such that $\phi(0)=\phi(T)=r$ and $x=\psi\circ\phi$.

The proof in [9] uses the fact that for bounded variation paths x, the map $\phi\rightarrow\int_0^T\phi(dx_s)$ is a continuous function in the uniform norm. As this does not hold for weakly geometric rough paths and we feel the proof in [9] relies on this in a sufficiently fundamental way, we construct a proof that is completely different and independent from that of [9]. The following is the main result of this paper.

Theorem 3. *Let $x\in WG\Omega_p\left(\mathbb{R}^d\right)$ for some $p\geqslant 1$. $S(x)_{0,T}=1$ if and only if x is tree-like.*

There has also been substantial work in proving that the signatures of sample paths determine the sample paths outside the null set of some probability measure. This has been proved for the Wiener measure [11], hypoelliptic diffusions [8], Gaussian measures [3] and the Chordal SLE_κ measure, $\kappa\leqslant 4$ [2].

One important consequence of our main result is that the relation $WG\Omega_p\left(\mathbb{R}^d\right)$ defined by

$$x\sim y\iff x\star\overleftarrow{y}\text{ is tree-like}$$

,where $\star$ denote the concatenation of paths, is an equivalence relation. The main difficulty in proving this directly is the transitivity property, but this can be proved easily using our main result together with the associativity of the tensor product.

Recently, I. Chevyrev [5] proved that under some conditions, the expected signatures of stochastic processes determine the law of the signatures of the processes. Our main result implies that the law of the signatures determine the law of processes as long as the sample paths of the processes do not have tree-like parts.

Finally, as mentioned at the very beginning, let $x,\tilde{x}\in WG\Omega_p\left(\mathbb{R}^d\right)$ and let $y^V,\tilde{y}^V$ be the solution of (1) driven by x and $\tilde{x}$ respectively, then $y_T^V=\tilde{y}_T^V$ for all vector fields V whose uniform norms of the derivatives grow no faster than geometrically with the order, if and only if $S(x)_{0,T}=S(\tilde{x})_{0,T}$.

2 Tree-like paths have trivial signature

In this section, we shall prove one direction of Theorem 3: the signature of a tree-like weakly geometric rough path is trivial.

First recall the definition of a real tree.

Definition 4. A metric space (τ,d) is a *real tree* if for all $x,y\in\tau$, there exists a unique simple curve α starting at x and ending at y and the image of α is isometric to an interval.

Let τ be a real tree. For $a,b\in\tau$, we shall let $[a,b]$ denote the image of the unique simple path in τ from a to b.

Remark 5. If $\sigma \subset \tau$ satisfies $a, b \in \sigma \implies [a, b]$ in σ then σ is also a real tree.

We first recall two important properties of real trees. The first one is an equivalent characterisation of a compact real tree and the second one is about paths in a real tree.

Lemma 6. *([10], Proposition 2.17 and Lemma 2.19) τ is a compact real tree if and only if there exists a continuous function $h : [0, T] \to [0, \infty)$ such that τ is the quotient of $[0, T]$ under the equivalence $s \sim t$ if and only if $h_s = h_t = \inf_{u \in [s,t]} h_u$.*

Lemma 7. *([6], Lemma 2.1)Let τ be a real tree and let $\alpha : [a, b] \to \tau$ be a continuous function. If $x = \alpha (a)$ and $y = \alpha (b)$,then $[x, y] \subseteq \alpha ([a, b])$.*

A consequence of Lemma 6 and Lemma 7 is the following.

Corollary 8. *Let V be a topological space. A continuous path $x : [0, T] \to V$ is tree-like if and only if there exists a function $h : [0, T] \to \mathbb{R}$ such that $h_0 = h_T = 0$, $h \geqslant 0$ and whenever s, t is such that $h_t = h_s = \inf_{u \in [s,t]} h_u$, we have $x_s = x_t$.*

Proof. The "if" part follows directly from Lemma 6. Let x be tree-like and $x = \phi \circ \psi$ be the decomposition in Definition 2. Then by Lemma 7 and Remark 5, the image $\phi [0, T]$ is a compact real tree. The corollary then follows from Lemma 6. □

An important concept in the study of real tree is partial order.

Lemma 9. *Let τ be a real tree and $r \in \tau$. Define the relation $\precsim$ on τ by $a \precsim b$ if and only if $[r, a] \subseteq [r, b]$. Then*
(1) $\precsim$ is a partial order on τ.
(2) For all $b \in \tau$,$\{a : a \precsim b\}$ is a totally ordered set.

For any $a, b \in \tau$, we define $a \wedge b$ by the unique element of τ such that

$$[r, a \wedge b] \quad = \quad [r, a] \cap [r, b]$$

If $a \preceq b$, then $a \vee b = b$.

Let $x : [0, T] \to G^{\lfloor p \rfloor} (\mathbb{R}^d)$ be a path. For any partition $\mathcal{P}$ of $[0, T]$, $x^{\mathcal{P}}$ denotes a piecewise geodesic interpolation of x with respect to $\mathcal{P}$ in $G^{\lfloor p \rfloor} (\mathbb{R}^d)$.

Lemma 10. *Let τ be a real tree. Let $x = \psi \circ \phi$ be a tree-like path in $G^{\lfloor p \rfloor} (\mathbb{R}^d)$. Let $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition of $[0, T]$. Then there exists a partition $\mathcal{P}'$ such that $\mathcal{P} \subset \mathcal{P}'$ such that $x^{\mathcal{P}'}$ is tree-like and the height function of $x^{\mathcal{P}'}$ is monotone between adjacent partition points in $\mathcal{P}'$.*

Proof. Define a partial order $\precsim$ as in Definition 9 with $r = \phi (0)$.

Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$. Let

$$B = \{\phi (t_{i_1}) \wedge \ldots \wedge \phi (t_{i_k}) \ i_1, \ldots, i_k = 1, \ldots, n, \ k \leqslant n\} .$$

Note that for all $a, b \in B \cup \phi(\mathcal{P})$, $a \wedge b \in B \cup \phi(\mathcal{P})$.

Let $\mathcal{P}' = \phi^{-1}(B \cup \phi(\mathcal{P}))$. Note that the open set $[0,T] \setminus \phi^{-1}(B \cup \phi(\mathcal{P}))$ may be expressed as a union of disjoint open intervals. By the continuity of ϕ and the finiteness of $B \cup \phi(\mathcal{P})$, there can only be finitely many of these intervals where the value of ϕ at the two endpoints are different.

We shall first prove that if (s, s') is a pair of adjacent partition points in $\mathcal{P}'$, then either $\phi(s) \preceq \phi(s')$ or $\phi(s') \preceq \phi(s)$.

Suppose not. Then

$$\phi(s) \wedge \phi(s') \notin \{\phi(s), \phi(s')\}.$$

As $[\phi(s), \phi(s')]$ contains $\phi(s) \wedge \phi(s')$, we have by Lemma 7 that $\phi([s, s'])$ contains $\phi(s) \wedge \phi(s')$. Therefore, there exists $s'' \in [s, s']$ such that $\phi(s'') = \phi(s) \wedge \phi(s')$. As

$$\phi(s), \phi(s') \in B \cup \phi(\mathcal{P})$$

, we have $\phi(s) \wedge \phi(s') \in B \cup \phi(\mathcal{P})$. By our construction, s'' has to lie in $\mathcal{P}'$. This contradicts the adjacency of s and s'.

Let $\phi' : [0,T] \to \tau$ be defined by $\phi'(t_i) = \phi(t_i)$ for all $t_i \in \mathcal{P}'$ and if $t_i \leqslant s \leqslant t_{i+1}$, then

$$\phi'(s) = \rho_i \left(\frac{s - t_i}{t_{i+1} - t_i} d(\phi(t_i), \phi(t_{i+1})) \right),$$

where ρ_i is the isometric map from $[0, d(\phi(t_i), \phi(t_{i+1}))]$ to $[\phi(t_i), \phi(t_{i+1})]$. By Lemma 7 and Remark 5, we have that $\phi'([0,T])$ is a real tree.

We now show that if $\mu \in \phi'([0,T])$, then there exists a pair of adjacent partition points s, s' in $\mathcal{P}'$ such that $\phi(s) \preceq \mu \preceq \phi(s')$. Let $u \in \phi'^{-1}(\mu)$. There exists a pair of adjacent points s and s' in $\mathcal{P}'$ such that $s \leqslant u \leqslant s'$. Assume without loss of generality that $\phi(s) \preceq \phi(s')$. As $\mu \in [\phi(s), \phi(s')]$, we must have

$$\phi(s) \preceq \mu \preceq \phi(s').$$

We now show that if (s, s') and (v, v') are two pairs of adjacent points in $\mathcal{P}'$ such that $\phi(s') \preceq \mu \preceq \phi(s)$ and $\phi(v') \preceq \mu \preceq \phi(v)$, then $\phi(v) = \phi(s)$ and $\phi(v') = \phi(s')$.

$$\phi(s') \vee \phi(v') \preceq \mu \preceq \phi(v) \wedge \phi(s).$$

If we do not have

$$\phi(s') = \phi(v') = \phi(s') \vee \phi(v') \text{ and } \phi(v) \wedge \phi(s) = \phi(v) = \phi(s),$$

we would again contradict the adjacency of the (s, s')or (v, v').

Define $\psi' : \phi'([0,T]) \to V$ by $\psi'(t) = \psi(t)$ if $t \in \phi(\mathcal{P}')$ and if $\phi(s_i) \preceq t \preceq \phi(s_{i+1})$, then $\psi(t)$ is the unique point on a geodesic from x_{s_i} to $x_{s_{i+1}}$ such that

$$d_V(\psi(t), \psi(s_i)) = \frac{d_V(\psi(s_{i+1}), \psi(s_i))}{d(\phi(s_i), \phi(s_{i+1}))} d(t, \phi(s_i))$$

where d_V is the metric in V. When connecting $x_{s_{i+1}}$ and x_{s_i}, we shall use the reversal of the geodesic from x_{s_i} to $x_{s_{i+1}}$.

Note that $x^{\mathcal{P}'} = \psi'\phi'$ and that since there are only finitely many pairs of adjacent points in $\mathcal{P}'$ where ϕ' takes different values, we may take $\mathcal{P}'$ to be finite without changing $x^{\mathcal{P}'}$. Moreover, $s \to d\left(\phi'\left(s\right), \phi\left(0\right)\right)$ is a height function for the tree-like path $x^{\mathcal{P}'}$ and is by construction monotone between adjacent partition points. □

Lemma 11. *If x is a tree-like piecewise linear path such that the height function is monotone between all adjacent partition points, then x has trivial signature.*

Proof. Proceed by induction on the number n of partition points in the piecewise linear path.

The $n = 2$ case is trivial.

Let h be the height function and let u be such that $h_u = \sup h$. Then either there exists a closed interval $[a, b]$, $a < b$, containing u such that h is constant on $[a, b]$ or for all $\varepsilon > 0$, there exists $s_\varepsilon, t_\varepsilon$,

$$u - \varepsilon < s_\varepsilon < u < t_\varepsilon < u + \varepsilon$$

such that $h_{s_\varepsilon} = h_{t_\varepsilon} = \inf h_u$.

In the former case, there exists a partition point t_i such that x is constant on $[t_i, t_{i+1}]$. We remove the partition point t_i. The remaining path is tree-like and hence by induction has trivial signature. Hence by (2), the original path x also has trivial signature.

In the latter case, take $\varepsilon < \min\left(t_{j+1} - t_j\right)$ where the minimum is taken over all partition points. Then either we have as before a degenerate partition point, or there exists a partition points $t_i \in [s_\varepsilon, t_\varepsilon]$ such that $x|_{[t_{i-1}, t_i]}$ and $x|_{[t_i, t_{i+1}]}$ is colinear and in opposite direction. In particular, we have by (2) that x has the same signature as the path obtained by removing t_i from the partition. . Remove t_i and by induction hypothesis, the remaining path has trivial signature and hence x has trivial signature. □

Remark 12. The difficulty in proving this for piecewise geodesics in $G^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)$ is that it is still unkown whether the geodesics in $G^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)$ are unique.

Corollary 13. *A tree-like weakly geometric p-rough path x has trivial signature.*

Proof. We first prove the $p = 1$ case. Let $\mathcal{P}_n$ be a sequence of partitions such that $\|\mathcal{P}_n\| \to 0$. Let $\mathcal{P}'_n$ be the corresponding sequence of approximations given by Lemma 10 and $x^{\mathcal{P}'_n}$ be the piecewise linear path in V constructed as in Corollary 10. By Lemma 11, $\gamma^{\mathcal{P}'_n}$ has trivial signature. Since by Theorem [7] $x^{\mathcal{P}'_n}$ converge in p'-variation for all $p' > 1$ to x, so we have by Theorem 3.1.2 in [12] that x has trivial signature.

In particular, piecewise geodesic tree-like paths in $G^{\lfloor p \rfloor}$ have trivial signatures. It now suffices to use this fact and repeats the argument above with p instead of 1 and geodesics in $G^{\lfloor p \rfloor}$ instead of linear paths. □

3 Existence and Uniqueness of Tree-reduced Paths

The rest of this paper is devoted to the proof of another direction of Theorem 3: if a weakly geometric p-rough path has trivial signature, then it is tree-like.

We wish to construct directly the required height function h. The intuition is that h_t should be the p variation of the "tree-reduction" of $x|_{[0,t]}$. In [9], the notion of "tree reduction" is defined using the tree structure *after* proving paths with trivial signature is tree-like. We observe that it is possible to define tree-reduction *before* proving our main result.

Definition 14. 1. We say $x \in WG\Omega_p$ is tree-reduced if the path $t \to S(x)_{0,t}$ has no self-intersection.

2. We say $\tilde{x} \in WG\Omega_p$ is a tree-reduction of $x \in WG\Omega_p$ if $\tilde{x}$ is tree-reduced and $S(\tilde{x})_{0,T} = S(x)_{0,T}$.

It is easy to see our definition is equivalent to that of [9] in the $p = 1$ case, although we shall not need it. In other words, tree-reducing x is the same as erasing loops from the signature path $t \to S(x)_{0,t}$. If we were to define h_t as the p variation of the tree reduction of $x|_{[0,t]}$, we need to prove two lemmas:

1. The tree reduction of $x|_{[0,t]}$ exists and is unique for each t.
2. The height defined as such is indeed a height function.

We first show the existence result for tree-reduced paths.

Proposition 15. *Let X be a Hausdorff topological space, and$x : [0, T] \to X$ be a continuous function. Then there exists disjoint open intervals $\{I_i : 1 \leqslant i \leqslant \infty\}$ such that the continuous path $\tilde{x}$ defined by*

$$\tilde{x}_t = \begin{cases} x_t, & t \in \left(\cup_{i=1}^{\infty} I_i\right)^c, \\ x_{\inf I_i}, & t \in I_i. \end{cases}$$

satisfies the property that if $\tilde{x}_t = \tilde{x}_s$ then there exists i such that $t, s \in \overline{I_i}$.

Proof. Let $\mathcal{P}$ be the set

$$\left\{\cup_i I_i : \ I_i \text{ disjoint open intervals, } x_{\inf I_i} = x_{\sup I_i}\right\}.$$

Define an order on $\mathcal{P}$ to be such that $I \leqslant J$ if and only if $I \subseteq J$.

We claim that $\mathcal{P}$ is inductively ordered. Indeed, let $\mathcal{I}$ be a totally ordered subset of $\mathcal{P}$. Then the set $\cup_{j\in\mathcal{I}} j$ can be expressed in terms of union of disjoint open intervals $\cup_i I_i$

Fix an i. We will prove that $x_{\inf I_i} = x_{\sup I_i}$. Note that $I_i \subseteq \cup_{j\in\mathcal{I}} j$. Let $\varepsilon > 0$. For each $y \in I_i$, there exists $j_y \in \mathcal{I}$ such that $y \in j_y$. Now $\cup_{y\in[\inf I_i+\varepsilon, \sup I_i-\varepsilon]} j_y$ is an open cover for $[\inf I_i + \varepsilon, \sup I_i - \varepsilon]$ and therefore has a finite subcover. $\cup_{i=1}^{n} j_{y_i}$. Let $j_Y = \max\{j_{y_1}, \ldots, j_{y_n}\}$. Then

$$[\inf I_i + \varepsilon, \sup I_i - \varepsilon] \subset j_Y.$$

Note that in particular, $[\inf I_i + \varepsilon, \sup I_i - \varepsilon]$ has to lie in a single connected component of j_Y, which we will call I_Y. Note that we must have $I_Y \subseteq I_i$ as $I_Y \in \cup_i I_i$ and I_i is the maximal connected component of $\cup_i I_i$ containing I_i . Therefore,

$$\begin{aligned} \inf I_i &\leqslant \inf I_Y \leqslant \inf I_i + \varepsilon \\ \sup I_i - \varepsilon &\leqslant \sup I_Y \leqslant \sup I_i \end{aligned}$$

Moreover we have $x_{\inf I_Y} = x_{\sup I_Y}$. Taking limit as $\varepsilon \to 0$ we have $x_{\inf I_i} = x_{\sup I_i}$.

By Zorn's Lemma, $\mathcal{P}$ has a maximal element, which we will denote by J. Let $J = \cup_i J_i$, where $J_i$are open intervals. Now we will prove that if $\tilde{x}_t = \tilde{x}_s$ then $t, s \in \overline{J_i}$ for some i. Let s, t be such that $\tilde{x}_t = \tilde{x}_s$. There are four cases:

1. If both s and t lies in $(\cup_{i=1}^{\infty} J_i)^c$ and $(s,t) \neq J_i$ for all i, then

$$\cup_{i=1}^{\infty} J_i \subset \cup_{i=1}^{\infty} J_i \cup (s,t) \in \mathcal{P}$$

which contradicts the maximality of $\cup_{i=1}^{\infty} J_i$ in $\mathcal{P}$.

2. If $t \in \cup_{i=1}^{\infty} J_i$ and $s \in (\cup_{i=1}^{\infty} J_i)^c$, then by the definition of $\tilde{x}$ we have $x_s = x_{\inf J_i}$. Note that $\inf J_i \in (\cup_i J_i)^c$.

2(a) If $s < \inf J_i$ and $(s, \inf J_i) = J_{i'}$ for some i', then $\inf J_i \notin \cup_i J_i$ and $x_{\sup J_{i'}} = x_{\inf J_i}$. Therefore $\cup_{i'=1}^{\infty} J_{i'} \cup \{\inf J_i\}$ strictly contains $\cup_i J_i$, which is a contradiction. Therefore, $(s, \inf J_i) \neq J_{i'}$ for any i'. We have a contradiction for the maximality of $\cup_i J_i$ as

$$\cup_i J_i \subset \cup_{i=1}^{\infty} J_i \cup (s, \inf J_i) \in \mathcal{P}.$$

2(b) If $\inf J_i < s$ and $(\inf J_i, s) = J_{i'}$ for some i', then $i = i'$ and $s, t \in \overline{J_i}$ so there is nothing to prove. Therefore $(\inf J_i, s) \neq J_{i'}$ for all i'. Again we have a contradiction as

$$\cup_{i'=1}^{\infty} J_{i'} \subset \cup_{i'=1}^{\infty} J_{i'} \cup (\inf J_i, s) \in \mathcal{P}.$$

4. Finally, if $s \in J_i$, $t \in J_j$, then

$$\tilde{x}_s = \tilde{x}_{\inf J_i} = \tilde{x}_{\inf J_j} = \tilde{x}_t.$$

If $(\inf J_i, \inf J_j) = J_{i'}$ for some i', then we must have $i' = i$. Therefore,

$$\cup_{i'=1}^{\infty} J_{i'} \cup \{\inf J_j\} \subset \cup_{i'=1}^{\infty} J_{i'}$$

which is a contradiction. Since $(\inf J_i, \inf J_j) \neq J_{i'}$ for all i', so

$$\cup_{i'=1}^{\infty} J_{i'} \in \cup_{i'=1}^{\infty} J_{i'} \cup (\inf J_i, \inf J_j) \in \mathcal{P}$$

contradicting maximality. □

Remark 16. $\tilde{x}$ is in general not unique.

Let X be again a Hausdorff space. Given a continuous path $x: [0,T] \to X$, let $\widetilde{x}$ be the path constructed in Proposition 15. Let $\widehat{x}$ be a continuous simple curve over $[0,T]$ joining $\widetilde{x}_0$ and $\widetilde{x}_1$, with image lying in $\widetilde{x}([0,T])$ (the existence of $\widehat{x}$ follows from a general fact in topology that a compact path connected Hausdorff space is arcwise-connected, see [13]).

Lemma 17. *$\widehat{x}[0,T] = \widetilde{x}[0,T]$, and $\varphi = \widehat{x}^{-1} \circ \widetilde{x} : [0,T] \to [0,T]$ is non-decreasing.*

Proof. Let's first show that if $\widehat{x}([0,T]) = \widetilde{x}([0,T])$, then φ is non-decreasing. Since $\widehat{x}: [0,T] \to \widehat{x}([0,T])$ is a homeomorphism, so $\varphi(s) = \varphi(t)$ if and only if $\widetilde{x}_s = \widetilde{x}_t$, and by the construction of $\widetilde{x}$ this is equivalent to $s,t \in \overline{I}_i$ for some i.

If φ is not non-decreasing, since $\varphi(0) = 0$ and $\varphi(T) = T$, then by continuity there exists some $s < u < t$ such that

$$\varphi(u) < \varphi(s) = \varphi(t). \tag{3}$$

Therefore $s,t \in \overline{I}_i$ for some i and by the construction of $\widetilde{x}$ we know that $\widetilde{x}_u = \widetilde{x}_s = \widetilde{x}_t$, contradicting 3.

Now we show that $\widehat{x}([0,T]) = \widetilde{x}([0,T])$.

We first prove that for any $0 < t < T$, $\widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$ is disconnected.

If $t \notin \cup_i \overline{I}_i$, then $\widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$ can be written as the disjoint union $\widetilde{x}([0,t)) \cup \widetilde{x}((t,T])$ of non-empty sets. By continuity and Hausdorff property, $\widetilde{x}([t,T])^{c}$ is open in X. Since

$$\widetilde{x}([0,t)) = \widetilde{x}([t,T])^{c} \bigcap \widetilde{x}([0,T]),$$

so $\widetilde{x}([0,t))$ is open in $\widetilde{x}([0,T])$. Similarly, $\widetilde{x}((t,T])$ is open in $\widetilde{x}([0,T])$.

If $t \in \overline{I}_i$ for some i, then $\widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$ can be written as the disjoint union $\widetilde{x}([0,\inf I_i)) \cup \widetilde{x}((\sup I_i, T])$ of non-empty sets. Since

$$\widetilde{x}([0,\inf I_i)) = \widetilde{x}([\inf I_i, T])^{c} \bigcap \widetilde{x}([0,T]),$$

so $\widetilde{x}([0,\inf I_i))$ is open in $\widetilde{x}([0,T])$, and similarly for $\widetilde{x}((\sup I_i, T])$.

Therefore, $\widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$ is disconnected.

Now assume there exists some $0 < t < T$ such that $\widetilde{x}_t \notin \widehat{x}([0,T])$, then $\widehat{x}([0,T]) \subset \widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$, but this is a contradiction to connectedness since $\widehat{x}(0)$ and $\widehat{x}(T)$ lie in different components of $\widetilde{x}([0,T]) \setminus \{\widetilde{x}_t\}$. □

From now on, we shall equip $T\left(\left(\mathbb{R}^d\right)\right)$with the norm $|g| = \max_n |g|_{(\mathbb{R}^d)^{\otimes n}}$.

Corollary 18. *Let $x \in WG\Omega_p\left(\mathbb{R}^d\right)$. There exists a $\tilde{x} \in WG\Omega_p$ such that $\tilde{x}$ is a tree-reduction of x.*

Proof. Let $\hat{X} : [0,\tau] \to T\left(\left(\mathbb{R}^d\right)\right)$ be the simple path obtained by applying Lemma 15 then Lemma 17 to $t \to S(x)_{0,t}$. As the p variation of a path is invariant under reparametrisation,

$$\left|\pi_n\left(\hat{X}_{0,\cdot}\right)\right|_{p-var} = \left|\pi_n\left(\tilde{X}_{0,\cdot}\right)\right|_{p-var} \leqslant \left|\pi_n\left(S(x)_{0,\cdot}\right)\right|_{p-var},$$

where $\tilde{X}$ is the path obtained by applying Lemma 15 to $t \to S(x)_{0,t}$. Therefore, by construction $\pi_{\lfloor p \rfloor}\left(\hat{X}_{0,\cdot}\right)$ lies in $G^{\lfloor p \rfloor}\left(\mathbb{R}^d\right)$. Therefore by uniqueness of lift (Theorem 2.2.1 [12]), we have $S\left(\pi_{\lfloor p \rfloor}\left(\hat{X}_{0,\cdot}\right)\right) = \hat{X}_{0,\cdot}$. □

Now we prove the uniqueness of tree-reduced paths.

Lemma 19. *Let $x, y \in WG\Omega_p(\mathbb{R}^d)$ with $S(x)_{0,T} = S(y)_{0,T}$. Then for any $N \in \mathbb{N}$ and any C_c^K-one form ψ on $\mathbb{R}^{(d^N)}$ with $K > \lfloor p \rfloor$, we have*

$$\int_0^T \psi(dS_N(x)_{0,u}) = \int_0^T \psi(dS_N(y)_{0,u}). \tag{4}$$

Proof. Write $\psi = \sum_{|I| \leqslant N} \psi_I dX^I$, where X^I is the coefficient of $e_{i_1} \otimes \ldots \otimes e_{i_n}$ in $S_N(x)_{0,T}$ if $I = (i_1, \ldots, i_n)$. If those ψ_I are polynomials, (4) follows immediately from the shuffle product formula. In general, since ψ_I are compactly supported, according to [1] they can be approximated by polynomials under the C^K-norm. The result then follows from the continuity of rough path integrals with respect to the integrating one forms under the C^K-norm provided $K > \lfloor p \rfloor$ ([7], Theorem 10.50). □

Lemma 20. *Let $x \in WG\Omega_p(\mathbb{R}^d)$ and $t \to S(x)_{0,t}$ be simple. Then for any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that $S_N(x)_{0,s} \neq S_N(x)_{0,t}$ for every $N \geqslant N(\varepsilon)$ and $(s,t) \in \Delta$ with $|t - s| \geqslant \varepsilon$.*

Proof. Let $\Delta_\varepsilon = \{(s,t) \in \Delta :\ t - s \geqslant \varepsilon\}$. For each $(s,t) \in \Delta_\varepsilon$, since $X_s \neq X_t$, there exists some $N_{s,t} \in \mathbb{N}$ such that

$$S_{N_{s,t}}(x)_{0,s} \neq S_{N_{s,t}}(x)_{0,t}. \tag{5}$$

By continuity, (5) holds in a neighborhood of (s,t). The result then follows easily from a compactness argument on Δ_ε. □

Proposition 21. *Let $x, y \in WG\Omega_p(\mathbb{R}^d)$ be such that $S(x)_{0,1} = S(y)_{0,1}$ and $S(x)_{0,\cdot}$, $S(y)_{0,\cdot}$ are both simple. Then x and y differ by a reparametrization.*

Proof. It suffices to show that $\mathbf{X}_\cdot := S(x)_{0,\cdot}$ and $\mathbf{Y}_\cdot := S(y)_{0,\cdot}$ have the same image.

If $\mathbf{X}_\tau \notin \mathbf{Y}[0,1]$, then there exists $\varepsilon > 0$ such that $|\mathbf{X}_\tau - \mathbf{Y}_\sigma| > \varepsilon$ for all $\sigma \in [0,1]$. Let s and t be such that $|\mathbf{X}_{\tau'} - \mathbf{Y}_\sigma| \geqslant \varepsilon$ for all $s \leqslant \tau' \leqslant t$ and all σ. Take N_1 such that $\frac{\omega(0,1)^{n/p}}{(n/p)!} < \varepsilon/2$ for all $n \geqslant N_1$, where ω is a control for both x and y, then by Theroem 2.2.1 in [12],

$$\sup_{\sigma \in [0,1]} \left| S_n(x)_{0,\tau'} - S_n(y)_{0,\sigma} \right| \geqslant \varepsilon$$

for all $n \geqslant N_1$, $s \leqslant \tau' \leqslant t$ and all σ.

As $\mathbf{X}_s \neq \mathbf{X}_t$ so there exists a functional $f \in \left(\mathbb{R}^d\right)^{\otimes n*}$ such that $f\circ\mathbf{X}_s \neq f\circ\mathbf{X}_t$ for some n. Suppose without loss of generality that $f \circ \mathbf{X}_s < f \circ \mathbf{X}_{0,t}$. Then let R_1,R_2 be such that

$$f \circ \mathbf{X}_{0,s} < R_1 < R_2 < f \circ \mathbf{X}_{0,t}.$$

.

Let s_2 and t_2 be defined by

$$\begin{aligned} s_2 &= \inf\left\{u \geqslant s : f \circ \mathbf{X}_u \geqslant R_1\right\} \\ t_2 &= \sup\left\{u \leqslant t : f \circ \mathbf{X}_u \leqslant R_2\right\}. \end{aligned}$$

Then $f \circ \mathbf{X}_{s_2} = R_1,\ f \circ \mathbf{X}_{t_2} = R_2$ and for all $u \in (s, s_2)$ and $u \in (t_2, t)$, we have either $f \circ \mathbf{X}_u < R_1$ or $f \circ \mathbf{X}_u > R_2$.

As $\mathbf{X}.$ is simple, by Lemma 20 there exists N_2 such that for all $n \geqslant N_2$, $S_n\left(x\right)_{u,v} \neq 1$ for all $|u - v| \geqslant \max\left(s_2 - s, t - t_2\right)$.

Take $N_2 \geqslant |I| \vee N_1$. Then

$$S_{N_2}\left(x\right)|_{[s_2,t_2]} \cap S_{N_2}\left(x\right)|_{[0,s]\cup[t,1]} = \emptyset.$$

Let $U \subset V$ be an open sets in $T^{N_2}\left(\mathbb{R}^d\right)$ such that

$$S_{N_2}\left(x\right)|_{[s_2,t_2]} \subset U,\ S_{N_2}\left(x\right)|_{[0,s]\cup[t,1]} \cup S_{N_2}\left(y\right)|_{[0,1]} \subset V^c.$$

Let ψ denote a bump function in $T^{N_2}\left(\mathbb{R}^d\right)$ with respect to (U, V), so that $\psi\left(z\right) = 1$ for $z \in U$ and $\psi\left(z\right) = 0$ for $z \in V^c$. Let

$$W = \left\{x^{N_2} \in T^{N_2}\left(\mathbb{R}^d\right):\ R_1 \leqslant f \circ x^{N_2} \leqslant R_2\right\}$$

, and 1_W be the indicator function on W.

Then define ϕ on $T^{N_2}\left(\mathbb{R}^d\right)$ by

$$\phi\left(x^{N_2}\right) = \begin{cases} \left(f \circ x^{N_2} - f \circ x^{N_2}_{s_2}\right)^{2k}\left(f \circ x^{N_2} - x^{N_2}_{t_2}\right)^{2k}, & x^{N_2} \in W, \\ 0, & x^{N_2} \in W^c. \end{cases}$$

where k is chosen to be arbitrarily large to satisfy the regularity assumptions in Lemma 19.

We now show that $\int_0^1 \phi\left(x^{N_2}_v\right) dx^{N_2}_v \neq 0$. Note that as $x^{N_2}_v \notin V$,

$$\int_t^1 \phi\left(x^{N_2}_v\right) \mathrm{d}f \circ x^{N_2}_v = 0 = \int_0^s \phi\left(x^{N_2}_v\right) \mathrm{d}f \circ x^{N_2}_v.$$

As $x^{N_2}_v < R_1$ or $x^{N_2}_v > R_2$ for $v \in (s, s_2) \cup (t_2, t)$. We also have

$$\int_s^{s_2} \phi\left(x^{N_2}_v\right) \mathrm{d}f \circ x^{N_2}_v = 0 = \int_{t_2}^t \phi\left(x^{N_2}_v\right) \mathrm{d}x^{N_2}_v.$$

Therefore,

$$\int_0^1 \phi\left(x_v^{N_2}\right) \mathrm{d}f \circ x_v^{N_2} = \int_{s_2}^{t_2} \phi\left(x_v^{N_2}\right) \mathrm{d}f \circ x_v^{N_2} \neq 0.$$

On the other hand, $\int_0^1 \phi\left(y_v^{N_2}\right) \mathrm{d}f \circ y_v^{N_2} = 0$ as $y_v^{N_2} \notin V$ for all v. The proof completes by Lemma 19. □

We now prove Theorem 3.

Proof of Theorem 3. Let $x \in WG\Omega_p$ such that $S(x)_{0,T} = 1$. Fix $p' > p$. Define

$$h_t = \left|\tilde{x}^t\right|_{p-var}$$

where $\tilde{x}^t$ is the tree-reduction of $x|_{[0,t]}$. We now prove that if $S(x)_{0,1} = 1$, then h is a height function.

Obviously, $h \geqslant 0$, $h_0 = h_T = 0$. We now prove that h is continuous. Let $s < t$. Let $\tilde{x}^s$ and $\tilde{x}^t$ be the tree-reduction of $x|_{[0,s]}$ and $x|_{[0,t]}$ respectively. Let x' be such that

$$S(x')_{0,v} = \begin{cases} S(\tilde{x}^s)_{0,v}, & v \in [0,s], \\ S(x)_{0,v}, & v \in [s,t]. \end{cases}$$

Suppose $|\tilde{x}^t|_{p-var} \geqslant |\tilde{x}^s|_{p-var}$, then

$$\begin{aligned} & h_t - h_s \\ = & \left|\tilde{x}^t\right|_{p-var} - |\tilde{x}^s|_{p-var} \\ \leqslant & |x'|_{p-var} - |\tilde{x}^s|_{p-var} \text{ as } x' \text{ is a reduction of } \tilde{x} \\ = & |x'|_{p-var,[0,t]} - |x'|_{p-var,[0,s]} \end{aligned}$$

and the last expression goes to zero as s goes to t.

In the case $|\tilde{x}^s|_{p-var} \geqslant |\tilde{x}^t|_{p-var}$, continuity is obtained similarly by defining x' so that

$$S(x')_{0,v} = \begin{cases} S(\tilde{x}^t)_{0,v}, & v \in [0,t], \\ S(x)_{0,2t-v}, & v \in [t, 2t-s]. \end{cases}$$

We now need that if $h_t = h_s = \inf_{u\in[s,t]} h_u$, then $x_t = x_s$.

Suppose $S(x)_{s,t} \neq 1$. Define $\tilde{x}$ on $[0,s]$ to be tree-reduction of $x|_{[0,s]}$. Extend the definition of $\tilde{x}$ so that $\tilde{x}|_{[s,t]}$ is the tree-reduction of $x|_{[s,t]}$.

We will prove that $v \to S(\tilde{x})_{0,v}$ is simple. By definition, we already have $v \to S(\tilde{x})_{0,v}$ simple on $[0,s]$ and $[s,t]$. We just need to prove that

$$S(\tilde{x})|_{[0,s)} \cap S(\tilde{x})_{[s,t]} = \emptyset.$$

Suppose for contradiction that there is $s' < s$ and $t' \in (s,t)$ that $S(\tilde{x})_{0,s'} = S(\tilde{x})_{0,t'}$ .Then $h_{s'} = h_s$. But then

$$h_{s'} = |\tilde{x}|_{p-var,[0,s']} < |\tilde{x}|_{p-var,[0,s]} = h_s.$$

Hence $h_{t'} < h_s$, a contradiction. Hence $v \to S(\tilde{x})_{0,v}$ is a simple curve.

Therefore,

$$h_t = |\tilde{x}|_{p-var,[0,t]} > |\tilde{x}|_{p-var,[0,s]} = h_s.$$

This is again a contradiction. □

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