Simple Piecewise Geodesic Interpolation of Simple and Jordan Curves with Applications

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Abstract

We explicitly construct simple, piecewise minimizing geodesic, arbitrarily fine interpolation of simple and Jordan curves on a Riemannian manifold. In particular, a finite sequence of partition points can be specified in advance to be included in our construction. Then we present two applications of our main results: the generalized Green's theorem and the uniqueness of signature for planar Jordan curves with finite p-variation for $1 \leqslant p < 2$.

Keywords Piecewise Geodesic Interpolation · Simple Curves · Jordan Curves · Generalized Green's Theorem · Uniqueness of Signature
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1 Introduction

The classical proofs of many properties of Jordan curves (e.g. the Jordan curve theorem) or functions on Jordan curves (e.g. Cauchy's theorem) begin with the consideration of polygonal Jordan curves. As part of the proof of the Jordan curve theorem in [12], it was shown that for every planar Jordan curve, there is a polygonal Jordan curve that approximates the original Jordan curve arbitrarily well. Here we shall prove a stronger and more general fact that given a Jordan curve on a connected Riemannian manifold M and n points on the curve, there exists a simple, piecewise minimizing geodesic, arbitrarily fine interpolation which contains these n points as interpolation points. The proof relies on another main result of this paper for non-closed simple curves. Such case was first treated by Werness [13], in which the author used an inductive but non-constructive method. Here we provide another proof of this result, which

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has the advantage of being explicit and constructive, and hence numerically computable.

We would like to emphasize that our approximation, unlike that of [12] which is a direct consequence of our result, does not rely on the flatness of the Euclidean metric, and respects the parametrization of the curve, i.e. it is an interpolation rather than merely an approximation in the uniform norm. The latter is particularly important for applications in the context of rough path theory, where we approximate continuous paths by bounded variation ones in the p-variation metric. Such idea is fundamental to the study of the roughness of continuous paths, and particularly of sample paths of continuous stochastic processes (see [4], [6]).

We also give two applications of our main result.

Taking advantage of the fact that the p-variation of a piecewise linear interpolation of a path is bounded by the p-variation of the path itself, our approximation theorem gives immediately Green's theorem for planar Jordan curves with finite p-variation, where $1 \leqslant p < 2$. To our best knowledge, in the rough path literature, the only other attempt in extending Green's theorem to non-rectifiable curves appeared in [14], where Green's theorem was proved for the boundaries of α-Hölder domains for $\frac{1}{3} < \alpha < 1$. Our result is a partial generalization of Yam's. Yam's result requires the curve to be α-Hölder continuous under the conformal parametrization whereas our result only requires the curve to be α-Hölder continuous under some parametrization.

A long-standing open problem in rough path theory is to what extent a path can be determined from its iterated integrals of any order. This is usually known as the uniqueness of signature problem. Hambly and Lyons [8] proved that the iterated integrals of a rectifiable curve vanish if and only if the path is tree-like, based on a similar type of approximation result for tree-like paths. Using our approximation result, we prove the uniqueness of signature for planar Jordan curves with finite p-variations, where $1 \leqslant p < 2$. The case of non-closed simple curves was treated in [2]. To our best knowledge this is the strongest uniqueness of signature result so far for non-rectifiable curves.

Throughout the rest of this paper, all curves are assumed to be continuous.

2 Simple Piecewise Geodesic Interpolation of Simple and Jordan Curves

In this section, we are going to prove our main results about simple piecewise geodesic approximation of simple and Jordan curves in Riemannian manifolds. Although the most interesting and nontrivial case lies in the Euclidean plane, we formulate our problems in a Riemannian geometric setting of arbitrary dimension since the proofs do not rely on Euclidean geometry (that is, the "flatness" of Euclidean metric) at all.

Throughout this section, let M be a d-dimensional connected Riemannian manifold ($d \geqslant 2$).

The following lemma, which is an easy fact from Riemannian geometry, is fundamental for us to formulate our main results.

Lemma 2.1. *For any compact set $K \subset M$, there exists some $\varepsilon = \varepsilon_K > 0$, such that for any $x, y \in K$ with*

$$d(x, y) < \varepsilon,$$

there exists a unique minimizing geodesic in M joining x and y, where $d(\cdot, \cdot)$ denotes the Riemannian distance function.

Proof. For any $x \in K$, choose δ_x small enough such that $B(x, \delta_x)$ is a geodesically convex normal ball (see [5], p. 76, Proposition 4.2). By compactness, we have a finite covering of K :

$$K \subset \bigcup_{i=1}^{k} B\left(x_i, \frac{\delta_{x_i}}{2}\right),$$

where $x_1, \cdots, x_k \in K$. Let $\varepsilon = \frac{1}{2} \min\{\delta_{x_1}, \cdots, \delta_{x_k}\}$. It follows that for any $x, y \in K$ with $d(x, y) < \varepsilon$, there exists some $1 \leqslant i \leqslant k$, such that $x, y \in B(x_i, \delta_{x_i})$. Therefore, by geodesic convexity we know that x and y can be joined by a unique minimizing geodesic in M which lies in $B(x_i, \delta_{x_i})$. □

Now we are in position to state our main results.

The first main result is a simple piecewise geodesic approximation theorem for non-closed simple curves in M.

Theorem 2.1. *Let γ be a non-closed simple curve in M. For all $\varepsilon > 0$, there exists a finite partition*

$$\mathcal{P}_\varepsilon : \ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of $[0, 1]$, such that

(1) the mesh size of the partition $\|\mathcal{P}_\varepsilon\| = \max_{i=1,\cdots,n}(t_i - t_{i-1}) < \varepsilon$;

(2) for any $i = 1, \cdots, n$, $\gamma_{t_{i-1}}$ and γ_{t_i} can be joined by a unique minimizing geodesic in M, and the piecewise geodesic interpolation (more precisely, piecewise minimizing geodesic interpolation, and the same thereafter) $\gamma^{\mathcal{P}_\varepsilon}$ of γ over the partition points in $\mathcal{P}_\varepsilon$ is a simple curve.

The proof of Theorem 2.1 relies on the following crucial lemma, which depends heavily on properties of minimizing geodesics. In the Euclidean case, we illustrate the lemma in Figure 1, which says that if the lengths of the straight line segments $\overline{xy}$ and $\overline{zw}$ are both less than or equal to r, then at least one of the four line segments $\overline{xz}, \overline{xw}, \overline{yz}, \overline{yw}$ has length strictly less than r.

Lemma 2.2. *Let $x, y, z, w \in M$ and $\alpha : [0, 1] \to M$ (respectively, $\beta : [0, 1] \to M$) be a minimizing geodesic joining x and y (respectively, z and w). Assume that $\alpha([0, 1]) \bigcap \beta([0, 1]) \neq \emptyset$ and for some $r > 0$, $d(x, y) \leqslant r$, $d(z, w) \leqslant r$. Then at least one of $d(x, z), d(y, z), d(x, w), d(y, w)$ is strictly less than r.*

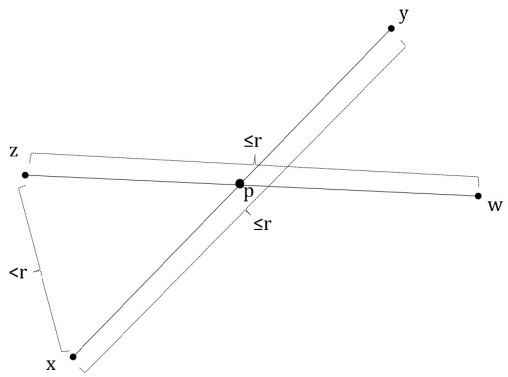

Figure 1: This figure illustrates the relative positions of the points in Lemma 2.2 in the Euclidean case. The lengths of the line segments $\overline{xy}$ and $\overline{zw}$ are less than or equal to r. Here the length of $\overline{zx}$ is strictly less than r.

Proof. Without loss of generality, we shall assume that all geodesics are parametrized at constant speed. Let $\alpha(u) = \beta(v) = p$ for some $u, v \in [0,1]$. Since α and β are minimizing geodesics, we know that

$$\begin{aligned} d(x,y) = d(x,p) + d(p,y) &\leqslant r, \\ d(z,w) = d(z,p) + d(p,w) &\leqslant r. \end{aligned}$$

Therefore, at least one of the following four cases happens:
(1) $d(x,p) \leqslant \frac{r}{2},\ d(z,p) \leqslant \frac{r}{2}$;
(2) $d(x,p) \leqslant \frac{r}{2},\ d(p,w) \leqslant \frac{r}{2}$;
(3) $d(p,y) \leqslant \frac{r}{2},\ d(z,p) \leqslant \frac{r}{2}$;
(4) $d(p,y) \leqslant \frac{r}{2};\ d(p,w) \leqslant \frac{r}{2}$.
First assume that (1) holds. It follows that

$$d(x,z) \leqslant d(x,p) + d(z,p) \leqslant r.$$

If $d(x,z) = r$, then

$$d(x,p) = d(z,p) = \frac{r}{2},$$

and hence (4) holds, which implies

$$d(y,w) \leqslant d(p,y) + d(p,w) \leqslant r.$$

If $d(y,w) = r$, then

$$d(p,y) = d(p,w) = \frac{r}{2}.$$

Consequently, we have $u = v = \frac{1}{2}$.

Now define

$$\widetilde{\alpha}(t) = \begin{cases} \alpha(t), & t \in \left[0, \frac{1}{2}\right]; \\ \beta(1-t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since

$$\text{Length}(\widetilde{\alpha}) = r = d(x, z),$$

$\widetilde{\alpha}$ is minimizing. Moreover, since any geodesic has constant speed, by definition we know that $\widetilde{\alpha}$ is parametrized proportionally to arc length. It follows from the first variation formula (see [5], p. 195, Proposition 2.4) that $\widetilde{\alpha}$ must be a geodesic. However, since $\widetilde{\alpha}|_{\left[0,\frac{1}{2}\right]} = \alpha|_{\left[0,\frac{1}{2}\right]}$, by the uniqueness of geodesics we have $\widetilde{\alpha} = \alpha$ and hence $y = z$. Similarly we have $x = w$.

The other cases can be treated in the same way, which completes the proof of the lemma. □

With the help of Lemma 2.2, we can now prove Theorem 2.1. The key idea is to construct a sequence of times $t_1, t_2, \ldots$, such that t_{i+1} is the *last* exit time of γ from a small geodesic ball around γ_{t_i} after time t_i. The uniform continuity of the inverse of the map $t \to \gamma_t$ will guarantee that t_i and t_{i+1} are close. We then need to argue that adjacent geodesic segments as well as non-adjacent geodesic segments in the approximation curve do not intersect. The latter uses Lemma 2.2. We illustrate the first step of the construction in Figure 2.

Proof of Theorem 2.1. Fix $\varepsilon > 0$. Since γ is a continuous and injective mapping from the compact space $[0, 1]$ to the Hausdorff space M, it is a homeomorphism from $[0, 1]$ to its image. By compactness and hence uniform continuity of γ^{-1} we know that there exists $\delta_\varepsilon > 0$ such that for any $s, t \in [0, 1]$,

$$d(\gamma_s, \gamma_t) < \delta_\varepsilon \implies |t - s| < \varepsilon.$$

We further assume that $\delta_\varepsilon < \varepsilon_{\gamma([0,1])}$, where $\varepsilon_{\gamma([0,1])}$ is the positive number in Lemma 2.1 depending on the compact set $\gamma([0, 1]) \subset M$. It follows from Lemma 2.1 that for any $s, t \in [0, 1]$ with $d(\gamma_s, \gamma_t) < \delta_\varepsilon$, γ_s and γ_t can be joined by a unique minimizing geodesic in M. Now define an increasing sequence of points $\{t_i\}_{i=0}^{\infty}$ in $[0, 1]$ inductively by setting $t_0 = 0$ and

$$t_i = \sup\left\{ t \in [t_{i-1}, 1] : \ \gamma_t \in \overline{B}\left(\gamma_{t_{i-1}}, \frac{\delta_\varepsilon}{2}\right)\right\}, \ i \geqslant 1.$$

We claim that there exists some $l \geqslant 1$, such that for all $i \geqslant l$, $t_i = 1$. In fact, if it is not the case, then for any $i \geqslant 1$, we have

$$t_{i-1} < t_i < 1 \text{ and } d\left(\gamma_{t_{i-1}}, \gamma_{t_i}\right) = \frac{\delta_\varepsilon}{2}.$$

On the other hand, by the uniform continuity of γ, there exists some $\eta_\varepsilon > 0$, such that for any $s, t \in [0, 1]$,

$$|t - s| < \eta_\varepsilon \implies d(\gamma_s, \gamma_t) < \frac{\delta_\varepsilon}{2}.$$

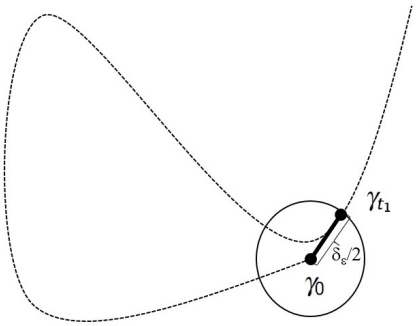

Figure 2: This figure illustrates the first step in the construction given in the proof of Theorem 2.1. The dotted line represents the simple curve γ. The solid geodesic segment joining the point γ_0 and γ_{t_1} represents the first step in the construction of the piecewise geodesic interpolation of γ. Note that we take t_1 to be the last exit time of γ from a $\frac{\delta_\varepsilon}{2}$-geodesic ball centered at γ_0.

Therefore, for any $i \geqslant 1$, $|t_i - t_{i-1}| \geqslant \eta_\varepsilon$, which is an obvious contradiction. Now set

$$l = \min\{i \geqslant 1 :\ t_i = 1\},$$

and define

$$\mathcal{P}_\varepsilon :\ 0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$$

to be a finite partition of $[0,1]$. Then it is easy to see that $\|\mathcal{P}_\varepsilon\| < \varepsilon$, where $\|\mathcal{P}_\varepsilon\|$ denote the mesh size of the partition $\mathcal{P}_\varepsilon$.

It remains to show that the piecewise geodesic interpolation $\gamma^{\mathcal{P}_\varepsilon}$ of γ over the points of $\mathcal{P}_\varepsilon$ is a simple curve.

To see this, first notice that for adjacent intervals $[t_{i-1}, t_i]$, $[t_i, t_{i+1}]$, we have

$$\gamma^{\mathcal{P}_\varepsilon}|_{[t_{i-1},t_i]} \bigcap \gamma^{\mathcal{P}_\varepsilon}|_{[t_i,t_{i+1}]} = \{\gamma_{t_i}\}.$$

In fact, if it is not the case, then there exist $s_1 \in [t_{i-1}, t_i)$ and $s_2 \in (t_i, t_{i+1}]$ such that

$$\gamma_{s_1}^{\mathcal{P}_\varepsilon} = \gamma_{s_2}^{\mathcal{P}_\varepsilon} \neq \gamma_{t_i}.$$

If $i < l-1$, then by applying Lemma 2.1 with $x = \gamma_{t_i}$ and $y = \gamma_{s_1}^{\mathcal{P}_\varepsilon} = \gamma_{s_2}^{\mathcal{P}_\varepsilon}$, $\gamma^{\mathcal{P}_\varepsilon}|_{[s_1,t_i]}$ is a reparametrization of the reversal of $\gamma^{\mathcal{P}_\varepsilon}|_{[t_i,s_2]}$, which we denote as $\overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}|_{[t_i,s_2]}$. In particular, $\gamma^{\mathcal{P}_\varepsilon}|_{[t_i,t_{i+1}]}$ and $\overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}|_{[t_{i-1},t_i]}$ are geodesics that starts at the same position with the same initial velocity. By the uniqueness of geodesic, either $\gamma^{\mathcal{P}_\varepsilon}([t_i, t_{i+1}]) \subseteq \overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}([t_{i-1}, t_i])$ or $\overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}([t_{i-1}, t_i]) \subseteq \gamma^{\mathcal{P}_\varepsilon}([t_i, t_{i+1}])$. In particular we have either $\gamma^{\mathcal{P}_\varepsilon}|_{[t_{i-1},t_i]}$ passes through $\gamma_{t_{i+1}}$ or $\gamma^{\mathcal{P}_\varepsilon}|_{[t_i,t_{i+1}]}$ passes through $\gamma_{t_{i-1}}$. As $\overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}|_{[t_{i-1},t_i]}$ and $\gamma^{\mathcal{P}_\varepsilon}|_{[t_i,t_{i+1}]}$ are minimizing geodesics and we have $d(\gamma_{t_i}, \gamma_{t_{i-1}}) = d(\gamma_{t_i}, \gamma_{t_{i+1}})$, we conclude that $\gamma_{t_{i-1}} = \gamma_{t_{i+1}}$ which contradicts that γ is simple. Figure 3 illustrates this argument.

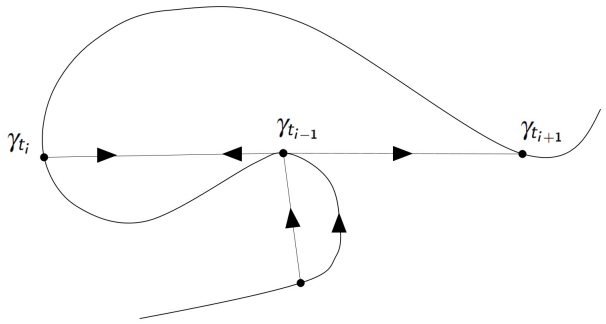

Figure 3: This figure illustrates the argument in the proof of Theorem 2.1 that two adjacent line segment of the approximation curve we constructed cannot intersect. The straight line represents the geodesic segments in the piecewise geodesic interpolation of γ. $\gamma_{t_{i-1}}, \gamma_{t_i}, \gamma_{t_{i+1}}$ are subdivision points of the curve. If the two adjacent line segments do intersect, as in the figure below, then $\gamma_{t_{i+1}}$ would be closer to $\gamma_{t_{i-1}}$ than to γ_{t_i} which would contradict our construction.

If $i = l-1$, then arguing as in the case $i < l-1$, we have either $\gamma^{\mathcal{P}_\varepsilon}([t_i, t_{i+1}]) \subseteq \overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}([t_{i-1}, t_i])$ or $\overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}([t_{i-1}, t_i]) \subseteq \gamma^{\mathcal{P}_\varepsilon}([t_i, t_{i+1}])$. However, as $i = l-1$, we have $d\left(\gamma_{t_i}, \gamma_{t_{i+1}}\right) \leqslant \frac{\delta_\varepsilon}{2} = d\left(\gamma_{t_{i-1}}, \gamma_{t_i}\right)$ and hence $\gamma^{\mathcal{P}_\varepsilon}([t_i, t_{i+1}]) \subseteq \overleftarrow{\gamma^{\mathcal{P}_\varepsilon}}([t_{i-1}, t_i])$. In particular, $\gamma^{\mathcal{P}_\varepsilon}|_{[t_{i-1}, t_i]}$ passes through $\gamma_{t_{i+1}}$. Therefore, $d\left(\gamma_{t_{i-1}}, \gamma_{t_{i+1}}\right) \leqslant d\left(\gamma_{t_{i-1}}, \gamma_{t_i}\right)$ which contradicts the construction of $\{t_i\}_{i=0}^{l}$.

On the other hand, if $[t_{i-1}, t_i]$ and $[t_{j-1}, t_j]$ $(i < j)$ are non-adjacent intervals and

$$\gamma^{\mathcal{P}_\varepsilon}|_{[t_{i-1}, t_i]} \bigcap \gamma^{\mathcal{P}_\varepsilon}|_{[t_{j-1}, t_j]} \neq \emptyset,$$

then by Lemma 2.2 we know that at least one of

$$d\left(\gamma_{t_{i-1}}, \gamma_{t_{j-1}}\right), d\left(\gamma_{t_i}, \gamma_{t_{j-1}}\right), d\left(\gamma_{t_{i-1}}, \gamma_{t_j}\right), d\left(\gamma_{t_i}, \gamma_{t_j}\right)$$

is strictly less than $\frac{\delta_\varepsilon}{2}$. However, this again contradicts the construction of $\{t_i\}_{i=0}^{l}$.

Now the proof is complete. □

The same technique of proof will allow us to prove our second main result, which is concerned with simple piecewise geodesic approximations of Jordan curves. This result significantly strengthens Theorem 2.1.

Theorem 2.2. *Let $\gamma : [0,1] \to M$ be a Jordan curve. Assume that $0 < \tau_1 < \cdots < \tau_k < 1$ are k fixed points in $[0,1]$. Then for any $\varepsilon > 0$, there exists a finite partition*

$$\mathcal{P}_\varepsilon : \ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of $[0,1]$, such that

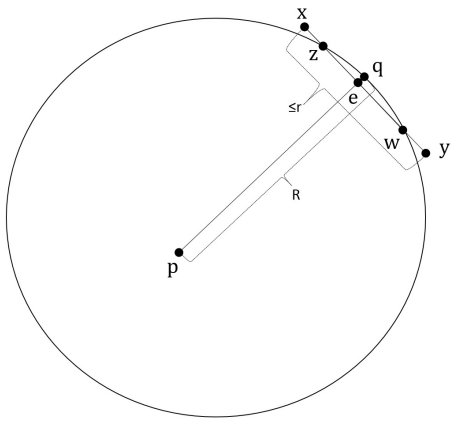

Figure 4: This figure illustrates the relative positions of the points involved in Lemma 2.3.

(1) $\tau_1, \cdots, \tau_k$ *are partition points of* $\mathcal{P}_\varepsilon$*;*
(2) $\|\mathcal{P}_\varepsilon\| < \varepsilon$*;*
(3) for $i = 1, \cdots, n$, $\gamma_{t_{i-1}}$ *and* γ_{t_i} *can be joined by a unique minimizing geodesic in* M*, and the piecewise geodesic interpolation* $\gamma^{\mathcal{P}_\varepsilon}$ *of* γ *over the partition points in* $\mathcal{P}_\varepsilon$ *is a Jordan curve.*

The proof of Theorem 2.2 relies on the following geometric fact. It is illustrated by Figure 4.

Lemma 2.3. *Let* $B(p,R)$ *be a geodesically convex normal ball centered at* $p \in M$*, and let* $q \in \partial B(p,R)$*. Assume that* $x, y \in \overline{B}(p,R)^c$ *and there exists a minimizing geodesic* $\alpha : [0,1] \to M$ *joining* x *and* y*. If* $\alpha([0,1]) \bigcap \overline{pq} \neq \emptyset$ *and* $d(x,y) \leqslant r$ *for some* $0 < r < R$*, where* $\overline{pq}$ *denotes the image of the unique minimizing geodesic in* M *joining* p *and* q*, then*

$$d(x,q) < r, \ d(y,q) < r.$$

Proof. The conclusion is obvious if $q \in \alpha([0,1])$. Otherwise, let $t \in (0,1)$ be the unique time such that $e := \alpha(t) \in B(p,R)$ is the intersection point of $\alpha([0,1])$ and $\overline{pq}$. By using the fact that $B(p,R)$ is a geodesically convex normal ball, it is easy to show that there exists a unique $u \in (0,t)$ and a unique $v \in (t,1)$, such that $z := \alpha(u)$ and $w := \alpha(v)$ lie on $\partial B(p,R)$. Observe that e and q are distinct, for their equality would contradict the fact that $r < R$. Now it follows

from properties of minimizing geodesics that

$$\begin{aligned} d(x,q) &\leqslant d(x,e)+d(e,q) \\ &= d(x,e)+d(p,q)-d(p,e) \\ &= d(x,e)+d(p,w)-d(p,e) \\ &\leqslant d(x,e)+d(e,w) \\ &= d(x,w) \\ &< d(x,y) \\ &\leqslant r. \end{aligned}$$

Similarly, we have $d(y,q)<r$. □

Now we can prove Theorem 2.2. Our proof is constructive and the idea is as follows. Recall that the times $\tau_1,\ldots,\tau_k$ should to included in our partition. Firstly, We find small disjoint geodesic balls around the points $\gamma_{\tau_i},\ldots,\gamma_{\tau_k},\gamma_1$. Secondly, we connect each point γ_{τ_i} by two radial minimizing geodesics to the point where γ *first* enters the geodesic ball around γ_{τ_i} before time τ_i and to the point where γ *last* exists the geodesic ball. Finally, we construct simple piecewise geodesic interpolation for each piece of simple curves outside those geodesic balls inductively, by using the algorithm in Theorem 2.1. To make sure that those approximation curves do not intersect the geodesic segments inside those geodesic balls, we need to use Lemma 2.3. Figure 5 illustrates the idea when $k=2$.

Proof of Theorem 2.2. Take an arbitrary $\tau\in(0,\tau_1)$. Since γ is a Jordan curve, we know that $\gamma_\tau,\gamma_{\tau_1},\cdots,\gamma_{\tau_k},\gamma_{\tau_{k+1}}\in M$ are all distinct, where we set $\tau_{k+1}=1$. By the Hausdorff property, there exists some $\delta>0$ such that the closed metric balls $\overline{B}(\gamma_{\tau_1},\delta),\cdots,\overline{B}(\gamma_{\tau_{k+1}},\delta)$ are all disjoint and $\gamma_\tau\notin\bigcup_{i=1}^{k+1}\overline{B}(\gamma_{\tau_i},\delta)$.

For the moment, by periodic extension and restriction we regard γ as defined on $[\tau,\tau+1]$ with starting and end points being $\gamma(\tau)$.

Now fix $\varepsilon>0$. Without loss of generality we assume that

$$\varepsilon<\min\left\{\tau,\tau_1-\tau,\frac{\tau_2-\tau_1}{2},\cdots,\frac{\tau_{k+1}-\tau_k}{2}\right\}.$$

First of all, by the uniform continuity of $\gamma|_{[\tau,\tau_i]}^{-1}$ and $\gamma|_{[\tau_i,\tau+1]}^{-1}$, there exists some $\delta_\varepsilon>0$, such that for all $i=1,\cdots,k+1$, any $s,t\in[\tau,\tau_i]$ or $s,t\in[\tau_i,\tau+1]$,

$$d(\gamma_s,\gamma_t)<\delta_\varepsilon \implies |t-s|<\varepsilon.$$

Now set $U_i=B(\gamma_{\tau_i},\delta_\varepsilon)$. Here we assume that δ_ε is small enough so that each U_i is a geodesically convex normal ball and Lemma 2.1 holds for those γ_s,γ_t with $d(\gamma_s,\gamma_t)<2\delta_\varepsilon$. Define

$$\begin{aligned} u_i &= \inf\left\{t\in[\tau,\tau_i]:\ \gamma_t\in\overline{U_i}\right\}, \\ v_i &= \sup\left\{t\in[\tau_i,\tau+1]:\ \gamma_t\in\overline{U_i}\right\}. \end{aligned}$$

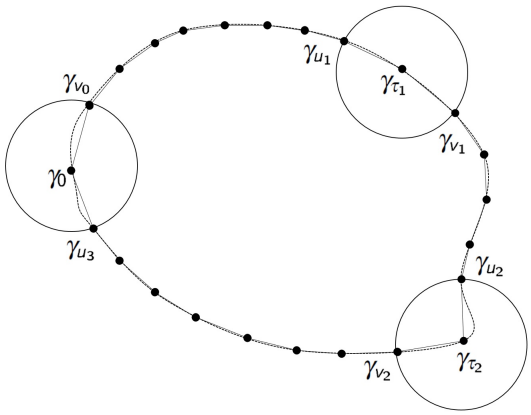

Figure 5: This figure illustrates the idea of proving of Theorem 2.2 when $k = 2$. The dotted line represents the curve γ. The solid line represents the piecewise geodesic interpolation of γ.

To return to the original time interval $[0, 1]$, let $v_0 = v_{k+1} - 1$. We have $|v_0| < \varepsilon$, $|\tau_i - u_i| < \varepsilon$, $|v_i - \tau_i| < \varepsilon$ and

$$0 < v_0 < u_1 < \tau_1 < v_1 < \cdots < u_k < \tau_k < v_k < u_{k+1} < 1,$$

and

$$\gamma_{u_i} \neq \gamma_{v_i},\ d\left(\gamma_{\tau_i}, \gamma_{u_i}\right) = d\left(\gamma_{\tau_i}, \gamma_{v_i}\right) = \delta_\varepsilon.$$

Moreover, we have

$$\gamma|_{(v_0,u_1)\cup(v_1,u_2)\cup\cdots\cup(v_{k-1},u_k)\cup(v_k,u_{k+1})} \bigcap \left(\bigcup_{i=1}^{k+1} \overline{U_i}\right) = \emptyset.$$

We will take $v_0, u_1, \tau_1, v_1, \cdots, u_k, \tau_k, v_k, u_{k+1}$ as part of the partition points in $\mathcal{P}_\varepsilon$. In particular, v_0 will be the first point, u_{k+1} will be the last point (except 0 and 1), and u_i, τ_i, v_i will be successive points in $\mathcal{P}_\varepsilon$, so the piecewise geodesic interpolation of γ over those small intervals is a finite sequence of radial geodesics of the balls centered at γ_{τ_i} with radius δ_ε for $i = 1, \cdots, k+1$.

For the next step, notice that $\gamma|_{[v_0,u_1]}, \gamma|_{[v_1,u_2]}, \cdots, \gamma|_{[v_k,u_{k+1}]}$ are $k+1$ non-closed simple curves with disjoint images. We are going to use the constructive procedure in the proof of Theorem 2.1 to define a simple piecewise geodesic approximation of each $\gamma|_{[v_{i-1},u_i]}$ $(i = 1, \cdots, k+1)$ with partition size smaller than ε inductively, such that the resulting piecewise geodesic closed curve over $[0, 1]$ is simple. That will complete the proof of the theorem.

Let $\gamma^{(0)}$ be the Jordan curve such that

$$\gamma^{(0)} = \gamma,\ \text{on}\ [v_0, u_1] \cup [v_1, u_2] \cup \cdots \cup [v_{k-1}, u_k] \cup [v_k, u_{k+1}],$$

and it is the minimizing geodesic (radial segment of the corresponding normal ball) on each small interval of

$$[0, v_0], [u_1, \tau_1], [\tau_1, v_1], \cdots, [u_k, \tau_k], [\tau_k, v_k], [u_{k+1}, 1].$$

By the construction in the proof of Theorem 2.1, we may find a partition

$$\mathcal{P}^{(1)}_{[v_0,u_1]}: \ v_0 = w^{(1)}_0 < w^{(1)}_1 < \cdots < w^{(1)}_{l_1-1} < w^{(1)}_{l_1} = u_1$$

so that $\left\|\mathcal{P}^{(1)}_{[v_0,u_1]}\right\| < \varepsilon$, the geodesic interpolation $\gamma^{\mathcal{P}^{(1)}_{[v_0,u_1]}}$ of $\gamma|_{[v_0,u_1]}$ over the partition points in $\mathcal{P}^{(1)}_{[v_0,u_1]}$ is simple and

$$\begin{aligned} d\left(\gamma_{w^{(1)}_{i-1}}, \gamma_{w^{(1)}_i}\right) &= \delta^{(1)}_\varepsilon, \ i = 1, \cdots l_1 - 1, \\ d\left(\gamma_{w^{(1)}_{l_1-1}}, \gamma_{u_1}\right) &\leqslant \delta^{(1)}_\varepsilon, \end{aligned}$$

for some $\delta^{(1)}_\varepsilon > 0$.

Moreover, we may choose $\delta^{(1)}_\varepsilon$small enough so that $\operatorname{dist}\left(\gamma^{\mathcal{P}^{(1)}_{[v_0u_1]}}, \gamma^{(0)}|_{[\tau_1,1]}\right) > 0$ and $\delta^{(1)}_\varepsilon < \delta_\varepsilon$.

Now we will show that

$$\gamma^{\mathcal{P}^{(1)}_{[v_0,u_1]}} \bigcap \gamma^{(0)}|_{[0,v_0)\cup(u_1,\tau_1]} = \emptyset.$$

In fact, if $\gamma^{\mathcal{P}^{(1)}_{[v_0,u_1]}} \bigcap \gamma^{(0)}|_{[0,v_0)} \neq \emptyset$, then from the construction of $\left\{w^{(1)}_i\right\}$, there exists some $i \geqslant 2$, such that $\gamma_{w^{(1)}_{i-1}}, \gamma_{w^{(1)}_i} \in \overline{U_{k+1}}^c$ and

$$\overline{\gamma_{w^{(1)}_{i-1}} \gamma_{w^{(1)}_i}} \bigcap \gamma^{(0)}|_{[0,v_0)} \neq \emptyset,$$

where $\overline{\gamma_{w^{(1)}_{i-1}} \gamma_{w^{(1)}_i}}$ denotes the image of the unique minimizing geodesic joining $\gamma_{w^{(1)}_{i-1}}$ and $\gamma_{w^{(1)}_i}$. However, since $d\left(\gamma_{w^{(1)}_{i-1}}, \gamma_{w^{(1)}_i}\right) \leqslant \delta^{(1)}_\varepsilon < \delta_\varepsilon$, we know from Lemma 2.3 that

$$d\left(\gamma_{v_0}, \gamma_{w^{(1)}_{i-1}}\right) < \delta^{(1)}_\varepsilon, \ d\left(\gamma_{v_0}, \gamma_{w^{(1)}_i}\right) < \delta^{(1)}_\varepsilon,$$

which is an obvious contradiction to the construction of $\left\{w^{(1)}_i\right\}^{l_1}_{i=0}$. On the other hand, if $\gamma^{\mathcal{P}^{(1)}_{[v_0,u_1]}} \bigcap \gamma^{(0)}|_{(u_1,\tau_1]} \neq \emptyset$, then there exists some $i \leqslant l_1 - 1$, such that $\overline{\gamma_{w^{(1)}_{i-1}} \gamma_{w^{(1)}_i}} \bigcap \gamma^{(0)}|_{(u_1,\tau_1]} \neq \emptyset$.

Since $\gamma_{w^{(1)}_{i-1}}, \gamma_{w^{(1)}_i} \in \overline{U_1}^c$ and $d\left(\gamma_{w^{(1)}_{i-1}}, \gamma_{w^{(1)}_i}\right) = \delta^{(1)}_\varepsilon < \delta_\varepsilon$, we know again from Lemma 2.3 that

$$d\left(\gamma_{u_1}, \gamma_{w^{(1)}_{i-1}}\right) < \delta^{(1)}_\varepsilon, \ d\left(\gamma_{u_1}, \gamma_{w^{(1)}_i}\right) < \delta^{(1)}_\varepsilon.$$

But this is also a contradiction to the construction of $\left\{w_i^{(1)}\right\}_{i=0}^{l_1}$.

Therefore, the closed curve $\gamma^{(1)}$ over $[0,1]$ defined by

$$\gamma_t^{(1)} = \begin{cases} \gamma_t^{\mathcal{P}^{(1)}_{[v_0,u_1]}}, & t \in [v_0, u_1]\,; \\ \gamma_t^{(0)}, & t \in [0,1] \setminus [v_0, u_1]\,, \end{cases}$$

is a Jordan curve.

Now consider $\gamma|_{[v_1,u_2]}$. The previous argument can be carried through easily with respect to the Jordan curve $\gamma^{(1)}$, and we obtain a finite partition

$$\mathcal{P}^{(2)}_{[v_1,u_2]}:\ v_1 = w_0^{(2)} < w_1^{(2)} < \cdots < w_{l_2-1}^{(2)} < w_{l_2}^{(2)} = u_2,$$

such that $\left\|\mathcal{P}^{(2)}_{[v_1,u_2]}\right\| < \varepsilon$, and the closed curve $\gamma^{(2)}$ over $[0,1]$ defined by

$$\gamma_t^{(2)} = \begin{cases} \gamma_t^{\mathcal{P}^{(2)}_{[v_1,u_2]}}, & t \in [v_1, u_2]\,; \\ \gamma_t^{(1)}, & t \in [0,1] \setminus [v_1, u_2]\,, \end{cases}$$

is a Jordan curve, where $\gamma^{\mathcal{P}^{(2)}_{[v_1,u_2]}}$ is the geodesic interpolation of $\gamma|_{[v_1,u_2]}$ over the partition points in $\mathcal{P}^{(2)}_{[v_1,u_2]}$. By induction, we are able to construct simple piecewise geodesic approximation of each piece of γ outside $\cup_{i=1}^{k+1}\overline{U_i}$ and finally obtain a finite partition $\mathcal{P}_\varepsilon$ of $[0,1]$ with partition points

$$\{0\} \bigcup \left(\bigcup_{i=1}^{k+1} \left\{ v_{i-1}, w_1^{(i)}, \cdots, w_{l_i-1}^{(i)}, u_i, \tau_i \right\}, \right)$$

such that $\|\mathcal{P}_\varepsilon\| < \varepsilon$, and the geodesic interpolation $\gamma^{\mathcal{P}_\varepsilon}$ (which is $\gamma^{(k+1)}$ by induction) of γ over the points of $\mathcal{P}_\varepsilon$ is a Jordan curve.

Now the proof is complete. □

Remark 2.1. By slight modification of the proof, it is not hard to see that Theorem 2.2 also holds for non-closed simple curves. In this case, it strengthens the result of Theorem 2.1.

Remark 2.2. It is possible to generalize our main results to infinite dimensional spaces with suitable geodesic properties. For technical simplicity we are not going to present the details.

3 Applications

In this section, we shall demonstrate two applications of Theorem 2.2. Here we assume that $M = \mathbb{R}^2$.

3.1 Green's theorem for Jordan curves with finite p-variation $(1 \leqslant p < 2)$

We will prove a generalized Green's theorem for planar Jordan curves with finite p-variation, where $1 \leqslant p < 2$. First we shall briefly recall basic facts about Young's integration.

Definition 3.1. Let (X, d) be a metric space. We say a function $\gamma : [0,1] \to X$ has finite p-variation if

$$\|\gamma\|_p := \left(\sup_{\mathcal{P}} \sum_{t_0 < t_1 < \ldots < t_n} d\left(\gamma_{t_i}, \gamma_{t_{i+1}}\right)^p \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all finite partitions of $[0,1]$.

Theorem 3.1. *(L.C. Young, [15], see also [11], Theorem 1.16) Let $p, q \geqslant 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$. Let $\gamma, \tilde{\gamma} : [0,1] \to \mathbb{R}^d$ be two continuous paths with finite p-variation and q-variation respectively. Then the following limit exists:*

$$\int_0^1 \gamma \otimes \mathrm{d}\tilde{\gamma} := lim_{|\mathcal{P}| \to 0} \sum_{\mathcal{P}:\ 0 = t_0 < \ldots < t_n = 1} \gamma_{t_i} \otimes \left(\tilde{\gamma}_{t_{i+1}} - \tilde{\gamma}_{t_i}\right).$$

Moreover, the function $\int_0^{\cdot} \gamma \otimes \mathrm{d}\tilde{\gamma}$ has finite q-variation and

$$\left\| \int_0^{\cdot} \gamma \otimes \mathrm{d}\tilde{\gamma} \right\|_q \leqslant 2\zeta\left(\frac{1}{p} + \frac{1}{q}\right) \|\gamma\|_p \|\tilde{\gamma}\|_q,$$

where $\zeta(\cdot)$ is the classical Riemann zeta function.

The following lemma demonstrates the importance of piecewise linear approximation for the p-variation metric.

Lemma 3.1. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a path with finite p-variation, where $p \geqslant 1$. Let $f : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ be a Lipschitz function with Lipschitz constant C. Then*

1. *([11], Lemma 1.12 and Lemma 1.18) $\|f(\gamma)\|_p \leqslant C \|\gamma\|_p$.*
2. *([11], Proposition 1.14 and Remark 1.19) For all $q > p$,*

$$\left\| f(\gamma) - f\left(\gamma^{\mathcal{P}}\right) \right\|_q \to 0$$

as $\|\mathcal{P}\| \to 0$.

We now prove Green's theorem for non-smooth Jordan curves.

Theorem 3.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be functions with continuous first order derivatives, and let $\gamma : [0,1] \to \mathbb{R}^2$ be a positively oriented Jordan curve with finite p-variation, where $1 \leqslant p < 2$. Let $x., y.$ denote the first and second coordinate components of $\gamma.$ respectively. Then*

$$\int_0^1 \left(f(\gamma_s)\,\mathrm{d}y_s - g(\gamma_s)\,\mathrm{d}x_s\right) = \int_{\mathrm{Int}(\gamma)} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x\mathrm{d}y,$$

where the integral on the L.H.S. is understood as the Young's integral, and $\mathrm{Int}(\gamma)$ *denotes the interior of* γ.

Proof. Fix $\varepsilon > 0$. According to Theorem 2.2, let $\mathcal{P}_\varepsilon$ be a finite partition of $[0,1]$ such that $\|\mathcal{P}_\varepsilon\| < \varepsilon$, and the piecewise linear interpolation $\gamma^{\mathcal{P}_\varepsilon}$ of γ over the partition points in $\mathcal{P}_\varepsilon$ is a Jordan curve. Let $x^{\mathcal{P}_\varepsilon}, y^{\mathcal{P}_\varepsilon}$ be the first and second components of $\gamma^{\mathcal{P}_\varepsilon}$ respectively. It follows from the classical Green's theorem for piecewise smooth Jordan curve that

$$\int_0^1 \left(f\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon} - g\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) \mathrm{d}x_s^{\mathcal{P}_\varepsilon}\right) = \int_{\mathrm{Int}(\gamma^{\mathcal{P}_\varepsilon})} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

For any $q \in (p, 2)$, we know that

$$\begin{aligned}
&\left|\int_0^1 \left(f\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon} - \int_0^1 f\left(\gamma_s\right) \mathrm{d}y_s\right)\right| \\
=&\left|\int_0^1 \left(f\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) - f\left(\gamma_s\right)\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon} + \int_0^1 f\left(\gamma_s\right) \mathrm{d}\left(y_s^{\mathcal{P}_\varepsilon} - y_s\right)\right| \\
\leqslant&\left|\int_0^1 \left(f\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) - f\left(\gamma_s\right)\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon}\right| + \left|\int_0^1 f\left(\gamma_s\right) \mathrm{d}\left(y_s^{\mathcal{P}_\varepsilon} - y_s\right)\right| \\
\leqslant&2\zeta\left(\frac{2}{q}\right)\left(\left\|f\left(\gamma_\cdot^{\mathcal{P}_\varepsilon}\right) - f\left(\gamma_\cdot\right)\right\|_q \|\gamma\|_q + \|f\left(\gamma_\cdot\right)\|_q \left\|\gamma_\cdot^{\mathcal{P}_\varepsilon} - \gamma_\cdot\right\|_q\right),
\end{aligned}$$

where the final inequality follows from Theorem 3.1 and Lemma 3.1. Therefore, by Lemma 3.1,

$$\int_0^1 f\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon} \to \int_0^1 f\left(\gamma_s\right) \mathrm{d}y_s$$

as $\varepsilon \to 0$. Similarly,

$$\int_0^1 g\left(\gamma_s^{\mathcal{P}_\varepsilon}\right) \mathrm{d}y_s^{\mathcal{P}_\varepsilon} \to \int_0^1 g\left(\gamma_s\right) \mathrm{d}y_s$$

as $\varepsilon \to 0$.

On the other hand, as γ has finite p variation, it has a $\frac{1}{p}$-Hölder parametrization. Therefore, γ has Hausdorff dimension less than 2. In particular, this means that $\gamma([0,1])$ has zero Lebesgue measure. By applying the bounded convergence theorem to the integrand

$$\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathbf{1}_{\mathrm{Int}(\gamma^{\mathcal{P}_\varepsilon})},$$

we have

$$\int_{\mathrm{Int}(\gamma^{\mathcal{P}_\varepsilon})} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x\mathrm{d}y \to \int_{\mathrm{Int}(\gamma)} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$.

Now the proof is complete. □

Remark 3.1. For rectifiable paths, there is a version of Green's theorem which works for non-simple closed curves, involving the winding number of a path. An interesting inequality in this respect is the Banchoff-Pohl inequality (see [1]), which generalizes the isoperimetric inequality and asserts that the winding number of a rectifiable curve is square-integrable. The reason for the "simple closed" condition in our version of Green's theorem is because in general the winding number of a non-simple non-rectifiable curve is not integrable. The fact that we can approximate the rough Jordan curves by piecewise linear interpolations which are still Jordan means that the winding number of each approximation is an indicator function, which is bounded by the indicator function of a neighborhood of $\mathrm{Int}(\gamma)$.

Remark 3.2. A direct consequence of Theorem 3.2 is Cauchy's theorem for Jordan curves with finite p-variation where $1 \leqslant p < 2$, according to the Cauchy-Riemann equation for holomorphic functions.

3.2 Uniqueness of signature for Jordan curves with finite p-variation $(1 \leqslant p < 2)$

The sequence of iterated integrals (formally known as the signature) of paths plays a key role in rough path theory. A central open problem in this area is to determine a path from its iterated integrals. In the case of bounded variation paths, Hambly and Lyons [8] proved that two paths of bounded variation can have the same sequence of iterated integrals if and only if they can be obtained from each other by a "tree-like deformation". In [2], it was proved that for planar simple curves with finite p-variation, the sequence of iterated integrals of the path determines the path up to reparametrization. In the context of stochastic processes, it was proved in [9] that the sequence of iterated Stratonovich's integrals of Brownian motion determines the Brownian sample paths almost surely. This result was extended to diffusion processes in [7].

Here we are going to prove the uniqueness of signature for planar Jordan curves with finite p-variation, where $1 \leqslant p < 2$ is fixed throughout the rest of this section.

We shall follow [10] and embed the sequence of iterated integrals of a path into the tensor algebra, which gives us a very nice algebraic structure to work with.

Assume that $\left(\mathbb{R}^d\right)^{\otimes n}$ is the tensor produce space equipped with the Euclidean norm by identifying it with $\mathbb{R}^{d^n}$. Let

$$T\left(\mathbb{R}^d\right) = \oplus_{n=0}^{\infty} \left(\mathbb{R}^d\right)^{\otimes n},$$

and let $\pi_n : T\left(\mathbb{R}^d\right) \to \left(\mathbb{R}^d\right)^{\otimes n}$ denote the projection map. Assume that $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is the standard basis of $\mathbb{R}^d$, and $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_d^*\}$ is the corresponding dual basis of $\mathbb{R}^{d*}$. We embed $T\left(\left(\mathbb{R}^d\right)^*\right)$ into $T\left(\mathbb{R}^d\right)^*$ by extending the

relation

$$\mathbf{e}_{i_1}^* \otimes \ldots \otimes \mathbf{e}_{i_n}^* \left(\mathbf{e}_{j_1} \otimes \ldots \otimes \mathbf{e}_{j_k}\right) = \begin{cases} 1 & \text{if } n = k \text{ and } i_1 = j_1, \cdots, i_k = j_k, \\ 0 & \text{otherwise,} \end{cases}$$

linearly.

Definition 3.2. Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous path with finite p-variation, then the formal series of tensors

$$S(\gamma)_{0,1} := 1 + \sum_{i=1}^{\infty} \int_{0<s_1<\ldots<s_i<1} \mathrm{d}\gamma_{s_1} \otimes \ldots \otimes \mathrm{d}\gamma_{s_i},$$

defined in terms of Young's integrals, is called the signature of γ over $[0,1]$.

We will briefly recall three important properties of signature.

Proposition 3.1. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous path with finite p-variation. Then*

1. ([11], p. 32) Let $r : [0,1] \to [0,1]$ be a continuous increasing function, then

$$S(\gamma_\cdot)_{0,1} = S\left(\gamma_{r(\cdot)}\right)_{0,1}.$$

2. ([11], p. 32) For all $a \in \mathbb{R}^d$,

$$S(a + \gamma_\cdot)_{0,1} = S(\gamma_\cdot)_{0,1}.$$

3. ([11], Corollary 2.11) Let γ_n be a sequence of paths with finite p-variation and

$$\|\gamma - \gamma_n\|_p \to 0,$$

as $n \to \infty$, then for each $k \in \mathbb{N}$,

$$\left|\pi_k\left(S(\gamma_n)_{0,1}\right) - \pi_k\left(S(\gamma)\right)\right| \to 0$$

as $n \to \infty$.

It turns out that some terms in the signature of a curve can be reduced to a single line integral. This is the key idea to prove our uniqueness of signature result.

Proposition 3.2. *Let γ be a positively oriented Jordan curve with finite p-variation. Let $x_\cdot, y_\cdot$ be the first and second coordinate components of γ respectively. Then for any $k, n \geqslant 0$*

$$\begin{aligned} & \mathbf{e}_1^{*\otimes(k+1)} \otimes \mathbf{e}_2^{*\otimes(n+1)} \left(S(\gamma)_{0,1}\right) \\ = & \int_0^1 \int_0^{s_{n+k+2}} \cdots \int_0^{s_2} \mathrm{d}x_{s_1} \ldots \mathrm{d}x_{s_{k+1}} \mathrm{d}y_{s_{k+2}} \ldots \mathrm{d}y_{s_{n+k+2}} \\ = & \frac{1}{k!n!} \int_{\mathrm{Int}(\gamma)} (x - x_0)^k (y_1 - y)^n \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\int_0^1 \int_0^{s_{n+k+2}} \cdots \int_0^{s_2} \mathrm{d}x_{s_1}^{\mathcal{P}_\varepsilon} \ldots \mathrm{d}x_{s_{k+1}}^{\mathcal{P}_\varepsilon} \mathrm{d}y_{s_{k+2}}^{\mathcal{P}_\varepsilon} \ldots \mathrm{d}y_{s_{n+k+2}}^{\mathcal{P}_\varepsilon} \\
= \;& \frac{1}{(k+1)!n!} \int_0^1 \left(x_{s_{k+1}}^{\mathcal{P}_\varepsilon} - x_0^{\mathcal{P}_\varepsilon}\right)^{k+1} \left(y_1^{\mathcal{P}_\varepsilon} - y_{s_{k+1}}^{\mathcal{P}_\varepsilon}\right)^n \mathrm{d}y_{s_{k+1}}^{\mathcal{P}_\varepsilon} \\
= \;& \frac{1}{k!n!} \int_{\mathrm{Int}(\gamma^{\mathcal{P}_\varepsilon})} (x - x_0)^k (y_1 - y)^n \, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By Lemma 3.1,

$$\begin{aligned}
&\int_0^1 \int_0^{s_{n+k+2}} \cdots \int_0^{s_2} \mathrm{d}x_{s_1}^{\mathcal{P}_\varepsilon} \ldots \mathrm{d}x_{s_{k+1}}^{\mathcal{P}_\varepsilon} \mathrm{d}y_{s_{k+2}}^{\mathcal{P}_\varepsilon} \ldots \mathrm{d}y_{s_{n+k+2}}^{\mathcal{P}_\varepsilon} \\
\to \;& \int_0^1 \int_0^{s_{n+k+2}} \cdots \int_0^{s_2} \mathrm{d}x_{s_1} \ldots \mathrm{d}x_{s_{k+1}} \mathrm{d}y_{s_{k+2}} \ldots \mathrm{d}y_{s_{n+k+2}}
\end{aligned}$$

as $\varepsilon \to 0$.

As in the proof of Theorem 3.2,

$$\int_{\mathrm{Int}(\gamma^{\mathcal{P}_\varepsilon})} (x - x_0)^k (y_1 - y)^n \, \mathrm{d}x\mathrm{d}y \to \int_{\mathrm{Int}(\gamma)} (x - x_0)^k (y_1 - y)^n \, \mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$. Therefore, the result follows. □

Remark 3.3. The case of $k = 1, n = 0$ for Proposition 3.2 has been proved by Werness [13]. The main difficulty in extending to the general case involves mainly the interchange of iterated integrals.

The following lemma is the main reason why our result only works for Jordan curves.

Lemma 3.2. *Let $\gamma, \widetilde{\gamma} : [0,1] \to \mathbb{R}^2$ be two positively oriented Jordan curves such that $\gamma([0,1]) = \widetilde{\gamma}([0,1])$ and $\gamma_0 = \widetilde{\gamma}_0$. There exists a continuous increasing function $r : [0,1] \to [0,1]$ such that $\gamma_{r(t)} = \widetilde{\gamma}$. In other words, γ and $\widetilde{\gamma}$ are equal up to a reparametrization.*

Proof. As γ is a Jordan curve,$\gamma([0,1])\backslash\gamma_0$ and $\widetilde{\gamma}([0,1])\backslash\widetilde{\gamma}_0$ are both homeomorphic to $(0,1)$. Therefore, the function $r : (0,1) \to (0,1)$ defined by $r(t) = \gamma^{-1} \circ \widetilde{\gamma}(t)$ is a homeomorphism $(0,1) \to (0,1)$. Hence, it is strictly monotone. This implies that $\lim_{t\to 0} r(t)$ exists. Moreover, it is easy to see that $\lim_{t\to 0} r(t) \in \{0,1\}$.

If $\lim_{t\to 0} r(t) = 0$, then r can be extended to a continuous increasing function on $[0,1]$. As $\gamma_{r(t)} = \widetilde{\gamma}_t$, we know that γ and $\widetilde{\gamma}$ equal up to reparametrization. If $\lim_{t\to 0} r(t) = 1$, then $\lim_{t\to 1} r(t) = 0$ and $r(t)$ is decreasing. This implies that $r(1-t)$ is an increasing continuous function. Therefore, γ and $\widetilde{\gamma}$ have opposite orientations, which is a contradiction. □

Now we are in position to state and prove our result on the uniqueness of signature for planar Jordan curves.

Theorem 3.3. *Let $\gamma, \tilde{\gamma} : [0,1] \to \mathbb{R}^2$ be a Jordan curves with finite p-variation. Then $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$ if and only if γ and $\tilde{\gamma}$ is a translation and a reparametrization of each other.*

Proof. Sufficiency follows from Proposition 3.1. We now consider the necessity part.

By applying a translation we may assume that $\gamma_1 = \gamma_0 = \tilde{\gamma}_0 = \tilde{\gamma}_1 = 0$.

As $\mathbf{e}_1^* \otimes \mathbf{e}_2^* \left(S(\gamma)_{0,1} \right) = \mathbf{e}_1^* \otimes \mathbf{e}_2^* \left(S(\tilde{\gamma})_{0,1} \right)$, by Proposition 3.2 we have

$$(-1)^{\varepsilon(\gamma)} \int_{\mathrm{Int}(\gamma)} \mathrm{d}x\mathrm{d}y = (-1)^{\varepsilon(\tilde{\gamma})} \int_{\mathrm{Int}(\tilde{\gamma})} \mathrm{d}x\mathrm{d}y,$$

where $\varepsilon(\gamma)$ is 0 if γ is positively oriented and 1 otherwise. As $\int_{\mathrm{Int}(\gamma)} \mathrm{d}x\mathrm{d}y$ and $\int_{\mathrm{Int}(\tilde{\gamma})} \mathrm{d}x\mathrm{d}y$ are both positive, we must have γ and $\tilde{\gamma}$ oriented in the same direction.

Without loss of generality, assume both γ and $\tilde{\gamma}$ are positively oriented. By Proposition 3.2 and that $S(\gamma)_{0,1} = S(\tilde{\gamma})_{0,1}$, we have

$$\int_{\mathrm{Int}(\gamma)} (x - x_0)^k (y_1 - y)^n \, \mathrm{d}x\mathrm{d}y = \int_{\mathrm{Int}(\tilde{\gamma})} (x - \tilde{x}_0)^k (\tilde{y}_1 - y)^n \, \mathrm{d}x\mathrm{d}y$$

for all $k, n \geq 0$. Therefore,

$$\int_{\mathrm{Int}(\gamma)} e^{i(\lambda_1 x + \lambda_2 y)} \mathrm{d}x\mathrm{d}y = \int_{\mathrm{Int}(\tilde{\gamma})} e^{i(\lambda_1 x + \lambda_2 y)} \mathrm{d}x\mathrm{d}y$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

Both $\mathbf{1}_{\mathrm{Int}(\gamma)}$ and $\mathbf{1}_{\mathrm{Int}(\tilde{\gamma})}$ are in L^1 and by the injectivity of the Fourier transform on L^1, we have

$$\mathbf{1}_{\mathrm{Int}(\gamma)}(x, y) = \mathbf{1}_{\mathrm{Int}(\tilde{\gamma})}(x, y)$$

for almost every $(x, y) \in \mathbb{R}^2$. In particular, this implies that both $\mathrm{Int}(\gamma) \setminus \overline{\mathrm{Int}(\tilde{\gamma})} \subset \mathrm{Int}(\gamma) \setminus \mathrm{Int}(\tilde{\gamma})$ and $\mathrm{Int}(\tilde{\gamma}) \setminus \overline{\mathrm{Int}(\gamma)} \subset \mathrm{Int}(\gamma) \setminus \overline{\mathrm{Int}(\tilde{\gamma})}$ are null sets in $\mathbb{R}^2$. However, since both $\mathrm{Int}(\tilde{\gamma}) \setminus \mathrm{Int}(\gamma)$ and $\mathrm{Int}(\gamma) \setminus \overline{\mathrm{Int}(\tilde{\gamma})}$ are open, they must be empty. Therefore,

$$\overline{\mathrm{Int}(\tilde{\gamma})} = \overline{\mathrm{Int}(\gamma)}.$$

By the Jordan curve theorem, we have

$$\overline{\mathbb{R}^2 \setminus \overline{\mathrm{Int}(\tilde{\gamma})}} = \mathbb{R}^2 \setminus \mathrm{Int}(\tilde{\gamma}).$$

Therefore, $\mathrm{Int}(\tilde{\gamma}) = \mathrm{Int}(\gamma)$ and so $\gamma([0,1]) = \tilde{\gamma}([0,1])$ and by Lemma 3.2, γ and $\tilde{\gamma}$ are equal up to reparametrization. □

Remark 3.4. The proof of Theorem 3.3 gives a very explicit way of computing the moments of the finite measure $\mathbf{1}_{\mathrm{Int}(\gamma)}(x, y) \, \mathrm{d}x\mathrm{d}y$ from the signature of γ. However, due to possible difficulty of numerically inverting the Fourier transform, this is still not an explicit reconstruction scheme for a Jordan curve from its signature. In fact, an explicit reconstruction scheme in general remains a significant open problem in rough path theory.

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References

[1] T.F. Banchoff and W.F. Pohl, A generalization of the isoperimetric inequality, J. Differential Geom., 6 (2), pp. 175-192.

[2] H. Boedihardjo, H. Ni and Z. Qian, Uniqueness of signature for simple curves, arXiv:1304.0755.

[3] S.S. Chern, W. Chen and K.S. Lam, *Lectures on differential geometry*, World Scientific, 1999.

[4] L. Coutin and Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions, Probab. Theory Related Fields, 122 (2002), pp. 108-140.

[5] M.P. Do Carmo, *Riemannian geometry*, Springer, 1992.

[6] P.K. Friz and N.B. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge University Press, 2010.

[7] X. Geng and Z. Qian, On the Uniqueness of Stratonovich's Signatures of Multidimensional Diffusion Paths, arXiv:1304.6985.

[8] B. Hambly and T. Lyons, Uniqueness for the signature of a path of bounded variation and the reduced path group, Ann. of Math., 171 (2010), pp. 109-167.

[9] Y. Le Jan and Z. Qian, Stratonovich's signatures of Brownian motion determine Brownian sample paths, Probab. Theory Related Fields, 157 (2012), pp. 209-223.

[10] T. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana, 14 (1998), pp. 215-310.

[11] T. Lyons, M. Caruana, and T. Levy. *Differential equations driven by rough paths*, Lecture Note in Mathematics, Vol. 1908, Springer, Berlin, 2007.

[12] H. Tverberg, A proof of the Jordan curve theorem, Bull. Lond. Math. Soc., 12 (1980), pp. 34-38.

[13] B. Werness, Regularity of Schramm-Loewner evolutions, annular crossings, and rough path theory, Electron. J. Probab., 17 (2012), pp. 1-21.

[14] P. Yam, *Analytical and topological aspects of signatures*, D.Phil. thesis, Oxford University, 2008.

[15] L. C. Young. An inequality of Hölder type connected with Stieltjes integration, Acta Math., 67 (1936), pp. 251-282.