On the Finite Dimensional Characteristic Functions of the Brownian Rough Path

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Abstract

The Brownian rough path is the canonical lifting of Brownian motion to the free nilpotent Lie group of order 2. Equivalently it is a process taking values in the algebra of Lie polynomials of degree 2, which is described explicitly by the Brownian motion coupled with its area process. The aim of this article is to compute the finite dimensional characteristic functions of the Brownian rough path in $\mathbb{R}^d$ and obtain an explicit formula for $d = 2$.

Keywords: Brownian rough paths; Finite dimensional characteristic functions; Riccati system

1 Introduction

The rough path theory, developed by T. Lyons in his seminal work [11] in 1998, has led to significant applications in probability. In particular, it gives meaning to differential equations driven by a large class of stochastic processes from a pathwise point of view. The fundamental result in rough path theory, known as the universal limit theorem, asserts that the solution is continuous with respect to the driving process under the rough path topology. This result is particularly useful in the study of various problems in stochastic analysis.

The bridge connecting rough path theory and probability is the understanding of the rough path nature of various stochastic processes, and the fundamental example is of course the Brownian motion. Since Brownian sample paths have finite $(2 + \varepsilon)$-variation, the starting point is to prove the fact that Brownian motion has a canonical lifting to the free nilpotent Lie group G^2 of order 2 as geometric 2-rough paths. Such lifting, which was first studied by E. M. Sipiläinen [15] (see also [1], [12]), is now known as the Brownian rough path. The

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existence of the Brownian rough path enables us to establish a pathwise theory of SDEs driven by Brownian motion as a direct consequence of rough path theory.

A equivalent way of looking at the Brownian rough path is through its pullback to the Lie algebra $\mathfrak{g}^2$ of G^2 by the exponential diffeomorphism, which takes a simpler and more familiar form. More precisely, the Brownian rough path in $\mathfrak{g}^2$ is just the original Brownian motion coupled with its area process.

Lévy's stochastic area process

$$L_t = \frac{1}{2}\int_0^t W_s^1 dW_s^2 - W_s^2 dW_s^1$$

associated with a two dimensional Brownian motion $W = \{(W_t^1, W_t^2) : t \geqslant 0\}$ was first introduced by P. Lévy [10] in 1940. By using stochastic Fourier expansions, Lévy derived the conditional characteristic function of L_t with respect to W_t for fixed $t > 0$ as

$$\mathbb{E}[\exp(i\lambda L_t)|W_t = x] = \frac{t\lambda}{2\sinh(t\lambda/2)} \exp\left(\frac{|x|^2}{2t}\left(1 - \frac{t\lambda}{2}\coth\frac{t\lambda}{2}\right)\right).$$

It follows that the characteristic function of L_t is given by

$$\mathbb{E}[\exp(i\lambda L_t)] = \frac{1}{\cosh(\lambda t/2)}, \tag{1.1}$$

and the joint characteristic function of the coupled process $\{(W_t, L_t) : t \geqslant 0\}$ can be computed explicitly. The formula (1.1) is usually known as Lévy's area formula. After Lévy's original work, this formula has been derived by using different methods, see for instance [2], [4], [5] and [9].

The aim of the present article is to compute the finite dimensional characteristic functions (i.e. sampling any finite collection of times) of the Brownian rough path as a process in $\mathfrak{g}^2$ by further exploiting the idea of using Girsanov's transformation by K. Helmes and A. Schwane [4] for the one dimensional case. It turns out that the problem reduces to the solution of a recursive system of symmetric matrix Riccati equations and a system of independent first order linear matrix ODEs. In particular, we obtain an explicit formula for the planar Brownian rough path, extending the classical result (1.1) of Lévy for the one dimensional case. We expect that the higher dimensional case can also be solved explicitly by applying a standard matrix transformation so that it reduces to the two dimensional case. However, for technical simplicity we are not going to pursue this generality. From classical probability theory, the finite dimensional characteristic functions determine the law of the Brownian rough path according to Kolmogorov's extension theorem.

It should be pointed out that although the Brownian rough path is a Markov process, it is hard to study the finite dimensional case directly from the one dimensional result and the Markov property. This is because an explicit formula for the transition probability density is not available.

Before stating our main result in precise, it worths mentioning that the characteristic function for general quadratic Wiener functions has been intensively studied by the Japanese school from a functional analytic point of view. In particular, T. Hida [5] related the problem to the computation of a regularized Fredholm determinant

$$(\det{}_2(I - 2i\lambda B))^{-1/2},$$

where B is certain symmetric Hilbert-Schmidt operator associated with the quadratic Wiener functional. Along this approach, a large amount of works have been done to investigate the operator B and the regularized Fredholm determinant. See for instance [3], [6] and [7]. However, we are not going to adopt this general approach.

2 Main Results

In this section we present the main results of the present article.

Let $T^2(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2}$, and let $G^2(\mathbb{R}^d) \subset T^2(\mathbb{R}^d)$ be the free nilpotent Lie group of order 2 defined by

$$G^2(\mathbb{R}^d) = \left\{ 1 + \sum_{j=1}^d x^j e_j + \sum_{j,k=1}^d x^{jk} e_j \otimes e_k : \ x^j x^k = x^{jk} + x^{kj} \text{ for all } j,k \right\},$$

where $\{e_1, \cdots, e_d\}$ is the canonical basis of $\mathbb{R}^d$. The Lie algebra of $G^2(\mathbb{R}^d)$ is denoted by $\mathfrak{g}^2(\mathbb{R}^d)$. Equivalently, $\mathfrak{g}^2(\mathbb{R}^d)$ is the free nilpotent Lie algebra of order 2 defined by

$$\mathfrak{g}^2(\mathbb{R}^d) = \mathbb{R}^d \bigoplus [\mathbb{R}^d, \mathbb{R}^d] \subset T^2(\mathbb{R}^d),$$

where $[a,b] = a \otimes b - b \otimes a$ for $a, b \in T^2(\mathbb{R}^d)$. In particular, $\mathfrak{g}^2(\mathbb{R}^d)$ is a $(d^2+d)/2$-dimensional Lie algebra with basis $\{e_j, [e_k, e_l] : \ 1 \leqslant j \leqslant d, \ 1 \leqslant k < l \leqslant d\}$.

Suppose that $\{W_t : \ t \geqslant 0\}$ is a d-dimensional Brownian motion ($d \geqslant 2$). It is a well-known result (see for instance [1], [12]) that W_t has a canonical lifting to a process $\mathbf{W}_t$ on the group $G^2(\mathbb{R}^2)$ starting at the unit. This process is known as the *Brownian rough path.* The existence of such canonical lifting is the fundamental starting point to develop the pathwise stochastic calculus for Brownian motion in the framework of rough path theory.

On the other hand, since the exponential map $\exp : \mathfrak{g}^2(\mathbb{R}^d) \to G^2(\mathbb{R}^d)$ is a diffeomorphism, it is equivalent to regard the Brownian rough path as a process $\mathbf{w}_t = \log \mathbf{W}_t$ in the Lie algebra $\mathfrak{g}^2(\mathbb{R}^d)$ starting at the origin. Here the exponential and logarithm are defined in terms of formal Taylor expansions in $T^2(\mathbb{R}^d)$. By direct calculation it is easy to see that

$$\mathbf{w}_t = \sum_{j=1}^d W_t^j e_j + \frac{1}{2} \sum_{1 \leqslant j < k \leqslant d} \left(\int_0^t W_s^j dW_s^k - W_s^k dW_s^j \right) [e_j, e_k].$$

The second term is just the area process, which we denote by L_t^{jk}.

Our aim is to compute the finite dimensional characteristic functions of $\mathbf{w}_t$ as a $(d^2+d)/2$-dimensional stochastic process in $\mathfrak{g}^2(\mathbb{R}^d)$. More precisely, we want to compute the expectation

$$\mathbb{E}\left[\exp\left(\sum_{l=1}^{n} i\langle\gamma_l, W_{t_l}\rangle + \sum_{l=1}^{n}\sum_{1\leqslant j<k\leqslant d} i\Gamma_l^{jk} L_{t_l}^{jk}\right)\right], \tag{2.1}$$

where $n \geqslant 1$, $0 < t_1 < \cdots < t_n$ and $\gamma_l \in \mathbb{R}^d$, $\Gamma_l^{jk} \in \mathbb{R}$ for $1 \leqslant l \leqslant n$, $1 \leqslant j < k \leqslant d$. From classical probability theory, this collection of expectations determines the finite dimensional distributions of the process $\mathbf{w}_t$, and hence determines the law of $\mathbf{w}_t$ according to Kolmogorov's extension theorem.

We are going to formulate (2.1) in a slightly different but equivalent way which is more convenient for us.

Given a $d \times d$ matrix A over $\mathbb{R}$, define the generalized Lévy's area process L^A with respect to A by

$$L_t^A = \int_0^t \langle AW_s, dW_s\rangle,\ t \geqslant 0.$$

The characteristic function of the coupled random vector (W_t, L_t^A) for fixed $t > 0$ was first computed in [4].

Now fix $0 < t_1 < \cdots < t_n$ and $d \times d$ matrices $A_1, \cdots, A_n$ over $\mathbb{R}^d$. We are going to compute the function

$$F(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^{n} i\langle\gamma_l, W_{t_l}\rangle + \sum_{l=1}^{n} i\Lambda_l L_{t_l}^{A_l}\right)\right], \tag{2.2}$$

where $\gamma_l \in \mathbb{R}^d, \Lambda_l \in \mathbb{R}$ $(1 \leqslant l \leqslant n)$. It is straight forward to see that (2.2) is equivalent to (2.1) by setting $\Lambda_l = 1$ and A_l to be the skew-symmetric matrix defined by $(A_l)_{jk} = -\Gamma_l^{jk}$ for $1 \leqslant j < k \leqslant d$.

Our first main result is the following. A^* denotes the transpose of a matrix A, and $\mathrm{Tr}(A)$ denotes its trace. Note that we do not require $A_1, \cdots, A_n$ to be skew-symmetric matrices.

Theorem 2.1. *The function $F(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$ is given by the following formula:*

$$\begin{aligned} F(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) \quad = \quad & \prod_{l=1}^{n} \exp\left(\frac{1}{2}\int_0^{t_l} \mathrm{Tr}(K_{i\Lambda_l,\cdots,i\Lambda_n}(s))ds\right. \\ & \left. -\frac{1}{2}\int_{t_{l-1}}^{t_l} \left|H_{i\Lambda_l,\cdots,i\Lambda_n}^{*-1}(s) H_{i\Lambda_l,\cdots,i\Lambda_n}^{*}(t_l)\mu_l\right|^2\right) \end{aligned}$$

Here $\{K_{i\Lambda_l,\cdots,i\Lambda_n}(t) :\ t \in [0, t_l], 1 \leqslant l \leqslant n\}$ is defined to be the solution to the *recursive system of n symmetric matrix Riccati equations (from $l=n$ to $l=1$):*

$$\begin{aligned}\frac{d}{dt}K_{i\Lambda_l,\cdots,i\Lambda_n}(t) &= C_{i\Lambda_l,\cdots,i\Lambda_n}(t) - K_{i\Lambda_l,\cdots,i\Lambda_n}(t)\left(\sum_{r=l+1}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(t)\right. \\ &\quad \left.+\sum_{r=l}^{n}(i\Lambda_r A_r)\right) - \left(\sum_{r=l+1}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(t)\right. \\ &\quad \left.+\sum_{r=l}^{n}(i\Lambda_r A_r^*)\right) K_{i\Lambda_l,\cdots,i\Lambda_n}(t) - K^2_{i\Lambda_l,\cdots,i\Lambda_n}(t),\ t\in[0,t_l],\end{aligned}$$

with $K_{i\Lambda_l,\cdots,i\Lambda_n}(t_l)=0,$ where

$$\begin{aligned}C_{i\Lambda_l,\cdots,i\Lambda_n}(t) &= \Lambda_l^2 A_l^* A_l - i\Lambda_l\left(\left(\sum_{r=l+1}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(t) + \sum_{r=l+1}^{n}(i\Lambda_r A_r^*)\right)A_l\right. \\ &\quad \left.+A_l^*\left(\sum_{r=l+1}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(t) + \sum_{r=l+1}^{n}(i\Lambda_r A_r)\right)\right),\ t\in[0,t_l].\end{aligned}$$

Moreover, $\{H_{i\Lambda_l,\cdots,i\Lambda_n}(t): t\in[0,t_l], 1\leqslant l\leqslant n\}$ is the solution to the system of n independent linear matrix ODEs:

$$\frac{d}{dt}H_{i\Lambda_l,\cdots,i\Lambda_n}(t) = \left(\sum_{r=l}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(t) + \sum_{r=l}^{n}(i\Lambda_r A_r)\right) H_{i\Lambda_l,\cdots,i\Lambda_n}(t), t\in[0,t_l],$$

with $H_{i\Lambda_l,\cdots,i\Lambda_n}(0)=\mathrm{Id}$ for $l=1,\cdots,n$. Finally $\{\mu_l: \ 1\leqslant l\leqslant n\}$ is defined recursively by $\mu_n=\gamma_n$,

$$\mu_l = \gamma_l + H^{*-1}_{i\Lambda_{l+1},\cdots,i\Lambda_n}(t_l) H^*_{i\Lambda_{l+1},\cdots,i\Lambda_n}(t_{l+1})\mu_{l+1},\ l=n-1,\cdots,1.$$

Remark 2.1. From the general theory of matrix Riccati equations (see for instance [8], [13], [14]), the Riccati system in Theorem 2.1 has a unique solution. Moreover, although each Λ_l can be absorbed into the matrix A_l, we keep Λ_l in the function F for technical purpose as we first consider the case when Λ_l are small and then apply a standard extension argument in complex analysis.

Our second main result is to solve the differential equations in Theorem 2.1 explicitly in the two dimensional case.

Let $\{W_t: \ t\geqslant 0\}$ be a two dimensional Brownian motion, and recall that

$$L_t = \int_0^t W_s^1 dW_s^2 - W_s^2 dW_s^1,\ t\geqslant 0,$$

is the associated Lévy's area process. Define $\mathbf{w}_t = (W_t, L_t/2)$ to be the corresponding $\mathfrak{g}^2(\mathbb{R}^2)$-valued process ($\mathbf{w}_t$ is also known as the Brownian motion on the Heisenberg group). Then the finite dimensional characteristic functions of $\mathbf{w}_t$ is explicitly given by the following result.

Theorem 2.2. *Given $n \geqslant 1$ and $t_1 < \cdots < t_n$, let $G(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$ be the function defined by*

$$G(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^n i\langle \gamma_l, W_{t_l}\rangle + \sum_{l=1}^n i\Lambda_l L_{t_l}\right)\right]$$

for $\gamma_l \in \mathbb{R}^2, \Lambda_l \in \mathbb{R}$ $(1 \leqslant l \leqslant n)$. Then we have

$$G(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \prod_{l=1}^n \Bigg(\frac{c_l}{c_l \cosh(c_l(t_{l-1} - t_l)) + s_l \sinh(c_l(t_{l-1} - t_l))} \cdot \exp\left(-\frac{1}{2}\int_{t_{l-1}}^{t_l} \left|H_l^{*-1}(s) H_l^*(t_l)\mu_l\right|^2 ds\right)\Bigg).$$

Here $c_l = \sum_{r=l}^n \Lambda_r$ $(1 \leqslant l \leqslant n)$. $\{s_l : 1 \leqslant l \leqslant n\}$ is defined recursively by $s_n = 0$,

$$s_{l-1} = c_l \frac{c_l \sinh(c_l(t_{l-1} - t_l)) + s_l \cosh(c_l(t_{l-1} - t_l))}{c_l \cosh(c_l(t_{l-1} - t_l)) + s_l(t_l) \sinh(c_l(t_{l-1} - t_l))}, \ l = n, \cdots, 2.$$

$\{H_l(t) : t \in [0, t_l], 1 \leqslant l \leqslant n\}$ is defined by

$$H_l(t) = \exp\left(\begin{pmatrix} a_l(t) & -ic_l t \\ ic_l t & a_l(t) \end{pmatrix}\right), \ t \in [0, t_l],$$

with $a_l(t)$ given by

$$a_l(t) = \ln\left(\frac{c_l \cosh(c_l(t - t_l)) + s_l \sinh(c_l(t - t_l))}{c_l \cosh(c_l t_l) - s_l \sinh(c_l t_l)}\right), \ t \in [0, t_l].$$

Finally $\{\mu_l : 1 \leqslant l \leqslant n\}$ is defined in the same way as in Theorem 2.1 in terms of $\{\gamma_l, H_l\}$.

3 Proof of Theorem 2.1

In this section, we develop the proof of Theorem 2.1. The main idea is to apply change of measures and Girsanov's theorem recursively.

We always fix $n \geqslant 1$, $0 < t_1 < \cdots < t_n$, and $A_1, \cdots, A_n \in \mathrm{Mat}(d; \mathbb{R})$.

We first consider the real case (the moment generating function) and then apply a standard complexification argument. Recall that L_t^A is the generalized Lévy's area process associated with a matrix A. The following integrability result is a basic feature of quadratic Wiener functionals.

Lemma 3.1. *There exists $c > 0$, such that*

$$\sup\left\{\mathbb{E}\left[\exp\left(\sum_{l=1}^n \lambda_l L_{t_l}^{A_l}\right)\right] : \ \lambda_l \in (-c, c), 1 \leqslant l \leqslant n\right\} < \infty.$$

Proof. We only need to consider the case when $n = 1$. By Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda L_t^A\right)\right] &= \mathbb{E}\left[\exp\left(\lambda \int_0^t \langle AW_s, dW_s\rangle\right)\right] \\
&\leqslant \mathbb{E}^{1/2}\left[\exp\left(2\lambda \int_0^t \langle AW_s, dW_s\rangle - \frac{(2\lambda)^2}{2}\int_0^t |AW_s|^2 ds\right)\right] \\
&\quad \cdot \mathbb{E}^{1/2}\left[\exp\left(2\lambda^2 \int_0^t |AW_s|^2 ds\right)\right].
\end{aligned}$$

From Novikov's condition, it suffices to show that

$$\mathbb{E}\left[\exp\left(\lambda^2 \int_0^t |AW_s|^2 ds\right)\right] < \infty$$

when λ is small. By Jensen's inequality, it remains to see that

$$\sup_{s\in[0,t]} \mathbb{E}\left[\exp\left(\lambda^2 |AW_s|^2\right)\right] < \infty$$

when λ is small, which is an obvious fact from direct computation. □

Now define

$$h(\lambda_1, \cdots, \lambda_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^n \lambda_l L_{t_l}^{A_l}\right)\right], \ \lambda_1, \cdots, \lambda_n \in (-c, c),$$

where c is defined in Lemma 3.1. The following result gives a formula for h.

Proposition 3.1. *The function $h(\lambda_1, \cdots, \lambda_n)$ is given by*

$$h(\lambda_1, \cdots, \lambda_n) = \prod_{l=1}^n \exp\left(\frac{1}{2}\int_0^{t_l} \mathrm{Tr}(K_{\lambda_l, \cdots, \lambda_n}(s)) ds\right),$$

where $K_{\lambda_l, \cdots, \lambda_n}$ is defined as the solution to the Riccati system in Theorem 2.1 with the replacement $\lambda_l = i\Lambda_l$ ($1 \leqslant l \leqslant n$).

Proof. The proof is based on applying Girsanov's theorem recursively.

Starting from the interval $[0, t_n]$, by changing the original probability measure P, we have

$$\begin{aligned}
h(\lambda_1, \cdots, \lambda_n) &= \mathbb{E}\left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle + \lambda_n \int_0^{t_n} \langle A_n W_s, dW_s\rangle \right.\right. \\
&\quad \left.\left. - \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds + \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\right)\right] \\
&= \mathbb{E}_n\left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle + \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\right)\right],
\end{aligned}$$

where $\mathbb{E}_n$ denotes the expectation under the probability measure

$$d\mathbb{P}_n = \exp\left(\lambda_n \int_0^{t_n} \langle A_n W_s, dW_s\rangle - \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\right) d\mathbb{P}.$$

Note that $\{W_t : t \in [0, t_n]\}$ is not a Brownian motion under the probability measure $\mathbb{P}_n$. However, by Girsanov's theorem, under $\mathbb{P}_n$, we know that the process

$$W_t^{(n)} = W_t - \lambda_n \int_0^t A_n W_s ds,\ t \in [0, t_n],$$

is a Brownian motion, and the original process $\{W_t : t \in [0, t_n]\}$ satisfies the following SDE:

$$dW_t = \lambda_n A_n W_t dt + dW_t^{(n)}, \quad t \in [0, t_n].$$

In order to eliminate the Lebesgue integral over $[0, t_n]$, let $C_n(t) = -\lambda_n^2 A_n^* A_n$ ($t \in [0, t_n]$) and introduce the following matrix Riccati equation:

$$\frac{d}{dt} K_{\lambda_n}(t) = C_n(t) - \lambda_n \left(K_{\lambda_n}(t) A_n + A_n^* K_{\lambda_n}(t)\right) - K_{\lambda_n}^2(t), \tag{3.1}$$

with $K_{\lambda_n}(t_n) = 0$. By the symmetry of (3.1), we know that the solution $\{K_n(t) : t \in [0, t_n]\}$ is symmetric. Therefore, we have

$$\begin{aligned}
& h(\lambda_1, \cdots, \lambda_n) \\
= \quad & \mathbb{E}_n\left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle - \frac{1}{2}\int_0^{t_n} W_s^* C_n(s) W_s ds\right)\right] \\
= \quad & \mathbb{E}_n\left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle - \frac{1}{2}\int_0^{t_n} W_s^* \frac{d}{ds} K_{\lambda_n}(s) W_s ds\right.\right. \\
& -\frac{1}{2}\int_0^{t_n} W_s^* \left(K_{\lambda_n}(s)(\lambda_n A_n) + \lambda_n A_n^* K_{\lambda_n}(s)\right) W_s ds \\
& \left.\left. -\frac{1}{2}\int_0^{t_n} |K_{\lambda_n}(s) W_s|^2 ds\right)\right] \\
= \quad & \mathbb{E}_n\left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle - \frac{1}{2}\int_0^{t_n} W_s^* \frac{d}{ds} K_{\lambda_n}(s) W_s ds\right.\right. \\
& \left.\left. - \int_0^{t_n} \langle K_{\lambda_n}(s) W_s, \lambda_n A_n W_s\rangle ds - \frac{1}{2}\int_0^{t_n} |K_{\lambda_n}(s) W_s|^2 ds\right)\right].
\end{aligned}$$

By change of measure again, we have

$$
\begin{aligned}
& h(\lambda_1, \cdots, \lambda_n) \\
= \quad & \widetilde{\mathbb{E}}_n \left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_{\lambda_n}(s) W_s ds \right.\right. \\
& \left.\left. - \int_0^{t_n} \langle K_{\lambda_n}(s) W_s, \lambda_n A_n W_s \rangle ds - \int_0^{t_n} \langle K_{\lambda_n}(s) W_s, dW_s^{(n)} \rangle \right)\right] \\
= \quad & \widetilde{\mathbb{E}}_n \left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_{\lambda_n}(s) W_s ds \right.\right. \\
& \left.\left. - \int_0^{t_n} \langle K_{\lambda_n}(s) W_s, dW_s \rangle \right)\right],
\end{aligned}
$$

where $\widetilde{\mathbb{E}}_n$ denotes the expectation under the probability measure

$$
d\widetilde{\mathbb{P}}_n = \exp\left(\int_0^{t_n} \langle K_{\lambda_n}(s) W_s, dW_s^{(n)} \rangle - \frac{1}{2} \int_0^{t_n} |K_n(s) W_s|^2 ds \right) d\mathbb{P}_n.
$$

Again from Girsanov's theorem, under $\widetilde{\mathbb{P}}_n$, the process

$$
\widetilde{W}_t^{(n)} = W_t^{(n)} - \int_0^t K_{\lambda_n}(s) W_s ds, \ t \in [0, t_n],
$$

is a Brownian motion, and the original process $\{W_t : \ t \in [0, t_n]\}$ now satisfies the following SDE:

$$
dW_t = (K_{\lambda_n}(t) + \lambda_n A_n) W_t dt + d\widetilde{W}_t^{(n)}, \ t \in [0, t_n].
$$

It should be pointed out that under $\widetilde{\mathbb{P}}_n$, the semi-martingale $\{W_t : \ t \in [0, t_n]\}$ has the same quadratic variation process as that of the Brownian motion.

Now define a function

$$
\varphi(t, w) = w^* K_{\lambda_n}(t) w, \ t \in [0, t_n], w \in \mathbb{R}^d.
$$

By applying Itô's formula to the process $\{F(t, W_t) : t \in [0, t_n]\}$, we have

$$
\int_0^{t_n} W_s^* \frac{d}{ds} K_{\lambda_n}(s) W_s ds + 2 \int_0^{t_n} \langle K_{\lambda_n}(s) W_s, dW_s \rangle + \int_0^{t_n} \mathrm{Tr}(K(s)) ds = 0.
$$

Therefore, we arrive at

$$
\begin{aligned}
& h(\lambda_1, \cdots, \lambda_n) \\
= \quad & \exp\left(\frac{1}{2} \int_0^{t_n} \mathrm{Tr}(K_{\lambda_n}(s)) ds \right) \cdot \widetilde{\mathbb{E}}_n \left[\exp\left(\sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s \rangle \right) \right].
\end{aligned}
$$

Now we apply a similar argument over the interval $[0, t_{n-1}]$. Here the main difference is that $\{W_t : t \in [0, t_{n-1}]\}$ is no longer a Brownian motion under the probability measure $\widetilde{\mathbb{P}}_n$. However, again by a change of measure, we have

$$\begin{aligned}
&h(\lambda_1, \cdots, \lambda_n) \exp\left(-\frac{1}{2}\int_0^{t_n} \mathrm{Tr}(K_{\lambda_n}(s))ds\right) \\
&= \widetilde{\mathbb{E}}_n \left[\exp\left(\sum_{l=1}^{n-2} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle + \lambda_{n-1}\int_0^{t_{n-1}} \langle A_{n-1}W_s, dW_s\rangle\right)\right] \\
&= \widetilde{\mathbb{E}}_n \left[\exp\left(\sum_{l=1}^{n-2} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle\right.\right. \\
&\quad + \lambda_{n-1}\int_0^{t_{n-1}} \langle A_{n-1}W_s, (K_{\lambda_n}(s) + \lambda_n A_n)W_s\rangle ds \\
&\quad + \lambda_{n-1}\int_0^{t_{n-1}} \langle A_{n-1}W_s, d\widetilde{W}_s^{(n)}\rangle - \frac{\lambda_{n-1}^2}{2}\int_0^{t_{n-1}} |A_{n-1}W_s|^2 ds \\
&\quad \left.\left.+\frac{\lambda_{n-1}^2}{2}\int_0^{t_{n-1}} |A_{n-1}W_s|^2 ds\right)\right] \\
&= \mathbb{E}_{n-1} \left[\exp\left(\sum_{l=1}^{n-2} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle\right.\right. \\
&\quad + \lambda_{n-1}\int_0^{t_{n-1}} \langle A_{n-1}W_s, (K_{\lambda_n}(s) + \lambda_n A_n)W_s\rangle ds \\
&\quad \left.\left.+\frac{\lambda_{n-1}^2}{2}\int_0^{t_{n-1}} |A_{n-1}W_s|^2 ds\right)\right].
\end{aligned}$$

Here $\mathbb{E}_{n-1}$ is the expectation under the probability measure

$$\begin{aligned}
d\mathbb{P}_{n-1} \quad = \quad & \exp\left(\lambda_{n-1}\int_0^{t_{n-1}} \langle A_{n-1}W_s, d\widetilde{W}_s^{(n)}\rangle\right. \\
& \left.-\frac{\lambda_{n-1}^2}{2}\int_0^{t_{n-1}} |A_{n-1}W_s|^2 ds\right) d\widetilde{\mathbb{P}}_n.
\end{aligned}$$

Moreover, under $\mathbb{P}_{n-1}$, the process

$$W_t^{(n-1)} = \widetilde{W}_t^{(n)} - \lambda_{n-1}\int_0^t A_{n-1}W_s ds, \ t \in [0, t_{n-1}],$$

is a Brownian motion, and $\{W_t : t \in [0, t_{n-1}]\}$ satisfies the following SDE:

$$dW_t = (K_{\lambda_n}(t) + \lambda_n A_n + \lambda_{n-1}A_{n-1})W_t dt + dW_t^{(n-1)}, \quad t \in [0, t_{n-1}].$$

Now let

$$\begin{aligned}
C_{n-1}(t) \quad = \quad & -\lambda_{n-1}^2 A_{n-1}^* A_{n-1} - \lambda_{n-1}\left((K_{\lambda_n}(t) + \lambda_n A_n^*)A_{n-1}\right. \\
& \left.+A_{n-1}^*(K_{\lambda_n}(t) + \lambda_n A_n)\right), \ t \in [0, t_{n-1}],
\end{aligned}$$

and introduce the following matrix Riccati equation:

$$\begin{aligned}\frac{d}{dt}K_{\lambda_{n-1},\lambda_n}(t) &= C_{n-1}(t) - K_{\lambda_{n-1},\lambda_n}(t)\left(K_{\lambda_n}(t)+\lambda_n A_n+\lambda_{n-1}A_{n-1}\right)\\ &\quad -\left(K_{\lambda_n}(t)+\lambda_n A_n^*+\lambda_{n-1}A_{n-1}^*\right)K_{\lambda_{n-1},\lambda_n}(t)\\ &\quad -K^2_{\lambda_{n-1},\lambda_n}(t),\ \ t\in[0,t_{n-1}],\end{aligned}$$

with $K_{\lambda_{n-1},\lambda_n}(t_{n-1})=0$. It follows that

$$\begin{aligned}&h(\lambda_1,\cdots,\lambda_n)\exp\left(-\frac{1}{2}\int_0^{t_n}\mathrm{Tr}(K_{\lambda_n}(s))ds\right)\\ &=\mathbb{E}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2}\lambda_l\int_0^{t_l}\langle A_lW_s,dW_s\rangle-\frac{1}{2}\int_0^{t_{n-1}}W_s^*C_{n-1}(s)W_sds\right)\right]\\ &=\mathbb{E}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2}\lambda_l\int_0^{t_l}\langle A_lW_s,dW_s\rangle-\frac{1}{2}\int_0^{t_{n-1}}W_s^*\frac{d}{ds}K_{\lambda_{n-1},\lambda_n}(s)W_sds\right.\right.\\ &\quad -\int_0^{t_{n-1}}\langle K_{\lambda_{n-1},\lambda_n}(s)W_s,(K_{\lambda_n}(s)+\lambda_nA_n+\lambda_{n-1}A_{n-1})W_s\rangle ds\\ &\quad\left.\left.-\frac{1}{2}\int_0^{t_{n-1}}|K_{\lambda_{n-1},\lambda_n}(s)W_s|^2ds\right)\right]\\ &=\widetilde{\mathbb{E}}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2}\lambda_l\int_0^{t_l}\langle A_lW_s,dW_s\rangle-\frac{1}{2}\int_0^{t_{n-1}}W_s^*\frac{d}{ds}K_{\lambda_{n-1},\lambda_n}(s)W_sds\right.\right.\\ &\quad -\int_0^{t_{n-1}}\langle K_{\lambda_{n-1},\lambda_n}(s)W_s,(K_{\lambda_n}(s)+\lambda_nA_n+\lambda_{n-1}A_{n-1})W_s\rangle ds\\ &\quad\left.\left.-\int_0^{t_{n-1}}\langle K_{\lambda_{n-1},\lambda_n}(s)W_s,dW_s^{(n-1)}\rangle\right)\right]\\ &=\widetilde{\mathbb{E}}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2}\lambda_l\int_0^{t_l}\langle A_lW_s,dW_s\rangle-\frac{1}{2}\int_0^{t_{n-1}}W_s^*\frac{d}{ds}K_{\lambda_{n-1},\lambda_n}(s)W_sds\right.\right.\\ &\quad\left.\left.-\int_0^{t_{n-1}}\langle K_{\lambda_{n-1},\lambda_n}(s)W_s,dW_s\rangle\right)\right].\end{aligned}$$

Here we have changed the probability measure from $\mathbb{P}_{n-1}$ to

$$\begin{aligned}d\widetilde{\mathbb{P}}_{n-1} &= \exp\left(\int_0^{t_{n-1}}\langle K_{\lambda_{n-1},\lambda_n}(s)W_s,dW_s^{(n-1)}\rangle\right.\\ &\quad\left.-\frac{1}{2}\int_0^{t_{n-1}}|K_{\lambda_{n-1},\lambda_n}(s)W_s|^2ds\right)d\mathbb{P}_{n-1}.\end{aligned}$$

Again by applying Itô's formula to the process $\{W_t^*K_{\lambda_{n-1},\lambda_n}(t)W_t : t\in[0,t_{n-1}]\}$ and noting that the quadratic variation process of $\{W_t :\ t\in[0,t_{n-1}]\}$ under

$\widetilde{\mathbb{P}}_{n-1}$ is the same as that of the Brownian motion, we obtain

$$\int_0^{t_{n-1}} W_s^* \frac{d}{ds} K_{\lambda_{n-1},\lambda_n}(s) W_s ds + 2\int_0^{t_{n-1}} \langle K_{\lambda_{n-1},\lambda_n}(s) W_s, dW_s\rangle$$
$$+ \int_0^{t_{n-1}} \mathrm{Tr}(K_{\lambda_{n-1},\lambda_n}(s))ds = 0.$$

Therefore, we arrive at

$$h(\lambda_1, \cdots, \lambda_n) \exp\left(-\frac{1}{2}\int_0^{t_n} \mathrm{Tr}(K_{\lambda_n}(s))ds\right)$$
$$= \exp\left(\frac{1}{2}\int_0^{t_{n-1}} \mathrm{Tr}(K_{\lambda_{n-1},\lambda_n}(s))ds\right) \cdot \widetilde{\mathbb{E}}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle\right)\right].$$

Now the proof follows from a standard induction argument. □

Having understood the case when $\gamma_1 = \cdots = \gamma_n = 0$, we are now able to compute the function

$$f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^n i\langle\gamma_l, W_{t_l}\rangle + \sum_{l=1}^n \lambda_l L_{t_l}^{A_l}\right)\right]$$

for $\gamma_1, \cdots, \gamma_n \in \mathbb{R}$ and $\lambda_1, \cdots, \lambda_n \in (-c, c)$.

Note from the proof of Proposition 3.1 that each time after a change of measure, the distribution of W_t is changed. To compute the function f, it is necessary to keep track of the process W_t under each transformation. Here main difficulty is that if we transform over the interval $[0, t_k]$, the new distribution of W_t over $[t_k, t_n]$ is hard to compute. The key observation to get around this issue is that under each new probability measure, the diffusion form (the SDE) of W_t is invariant.

Proposition 3.2. *The function $f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)$ is given by*

$$f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \prod_{l=1}^n \exp\left(\frac{1}{2}\int_0^{t_l} \mathrm{Tr}(K_{\lambda_l,\cdots,\lambda_n}(s))ds - \frac{1}{2}\int_{t_{l-1}}^{t_l} |H^{*-1}_{\lambda_l,\cdots,\lambda_n}(s) H^*_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l|^2 ds\right),$$

where $K_{\lambda_l,\cdots,\lambda_n}, H_{\lambda_l,\cdots,\lambda_n}, \mu_l$ are defined in the same way as in Theorem 2.1 with the replacement $\lambda_l = i\Lambda_l$ $(1 \leqslant l \leqslant n)$.

Proof. By using the same notation as in Proposition 3.1, we have

$$f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \widetilde{\mathbb{E}}_n\left[\exp\left(\sum_{l=1}^n i\langle\gamma_l, W_{t_l}\rangle\right) \cdot \Delta_n\right],$$

where

$$\Delta_n = \exp\left(\frac{1}{2}\int_0^{t_n} \mathrm{Tr}(K_{\lambda_n}(s))ds + \sum_{l=1}^{n-1} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle\right).$$

Under $\widetilde{\mathbb{P}}_n$, the process $\{W_t : t \in [0, t_n]\}$ is a diffusion of the form

$$dW_t = (K_{\lambda_n}(t) + \lambda_n A_n)W_t dt + d\widetilde{W}_t^{(n)},\ t \in [0, t_n]. \tag{3.2}$$

Let $\{H_{\lambda_n}(t) : t \in [0, t_n]\}$ be the solution to the following linear matrix ODE:

$$\frac{d}{dt} H_{\lambda_n}(t) = (K_{\lambda_n}(t) + \lambda_n A_n)H_{\lambda_n}(t),\ t \in [0, t_n],$$

with $H_{\lambda_n}(0) = \mathrm{Id}$. Note that $H_{\lambda_n}(t)$ is invertible. By solving (3.2) explicitly, we have

$$W_t = H_{\lambda_n}(t)\int_0^t H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)},\ t \in [0, t_n].$$

Therefore,

$$\begin{aligned}
&f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)\\
&= \widetilde{\mathbb{E}}_n\left[\exp\left(\sum_{l=1}^{n-1} i\langle\gamma_l, W_{t_l}\rangle + i\langle\gamma_n, H_{\lambda_n}(t_n)\int_0^{t_n} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}\rangle\right)\cdot\Delta_n\right]\\
&= \widetilde{\mathbb{E}}_n\left[\exp\left(\sum_{l=1}^{n-1} i\langle\gamma_l, W_{t_l}\rangle + i\langle\gamma_n, H_{\lambda_n}(t_n)\int_0^{t_{n-1}} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}\rangle\right.\right.\\
&\quad\left.\left. + i\langle\gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}\rangle\right)\cdot\Delta_n\right].
\end{aligned}$$

Observe that the stochastic integral $\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}$ is independent of the rest terms as the integrand is deterministic, we have

$$\begin{aligned}
&f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)\\
&= \widetilde{\mathbb{E}}_n\left[\exp\left(i\langle\gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}\rangle\right)\right]\\
&\quad\cdot\widetilde{\mathbb{E}}_n\left[\exp\left(\sum_{l=1}^{n-1} i\langle\gamma_l, W_{t_l}\rangle + i\langle H_{\lambda_n}^{*-1}(t_{n-1})H_{\lambda_n}^*(t_n)\gamma_n, W_{t_{n-1}}\rangle\right)\cdot\Delta_n\right]\\
&= R_n\cdot\widetilde{\mathbb{E}}_n\left[\exp\left(\sum_{l=1}^{n-2} i\langle\gamma_l, W_{t_l}\rangle + \langle\mu_{n-1}, W_{t_{n-1}}\rangle\right)\cdot\Delta_n\right],
\end{aligned}$$

where

$$R_n = \widetilde{\mathbb{E}}_n\left[\exp\left(i\langle\gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)d\widetilde{W}_s^{(n)}\rangle\right)\right]$$

and

$$\mu_{n-1} = \gamma_{n-1} + H^{*-1}_{\lambda_n}(t_{n-1}) H^*_{\lambda_n}(t_n)\gamma_n.$$

Similarly, by applying the change of measures as in the proof of Proposition 3.1, we obtain that

$$\begin{aligned} & f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) \\ = {} & R_n \cdot \widetilde{\mathbb{E}}_{n-1}\left[\exp\left(\sum_{l=1}^{n-2} i\langle \gamma_l, W_{t_l}\rangle + \langle \mu_{n-1}, W_{t_{n-1}}\rangle\right) \cdot \Delta_{n-1}\right], \end{aligned}$$

where

$$\begin{aligned} \Delta_{n-1} &= \exp\left(\frac{1}{2}\int_0^{t_{n-1}} \mathrm{Tr}(K_{\lambda_{n-1},\lambda_n}(s))ds + \frac{1}{2}\int_0^{t_n} \mathrm{Tr}(K_{\lambda_n}(s))ds \right. \\ &\quad \left. + \sum_{l=1}^{n-2} \lambda_l \int_0^{t_l} \langle A_l W_s, dW_s\rangle\right). \end{aligned}$$

Let $\{H_{\lambda_{n-1},\lambda_n}(t) : t \in [0, t_{n-1}]\}$ be the solution to the equation

$$\begin{aligned} \frac{d}{dt} H_{\lambda_{n-1},\lambda_n}(t) &= (K_{\lambda_n}(t) + K_{\lambda_{n-1},\lambda_n}(t) + \lambda_n A_n + \lambda_{n-1}A_{n-1})H_{\lambda_{n-1},\lambda_n}(t), \\ & \quad t \in [0, t_{n-1}], \end{aligned}$$

with $H_{\lambda_{n-1},\lambda_n}(0) = \mathrm{Id}$, we then obtain that

$$\begin{aligned} & f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) \\ = {} & R_n \cdot R_{n-1} \cdot \widetilde{\mathbb{E}}_{n-1}\left[\exp\left(\sum_{l=1}^{n-3} i\langle \gamma_l, W_{t_l}\rangle + i\langle \mu_{n-2}, W_{t_{n-2}}\rangle\right) \cdot \Delta_{n-1}\right], \end{aligned}$$

where

$$R_{n-1} = \widetilde{\mathbb{E}}_{n-1}\left[\exp\left(i\langle \mu_{n-1}, H_{\lambda_{n-1},\lambda_n}(t_{n-1}) \int_{t_{n-2}}^{t_{n-1}} H^{-1}_{\lambda_{n-1},\lambda_n}(s) d\widetilde{W}^{(n-1)}_s\rangle\right)\right]$$

and

$$\mu_{n-2} = \gamma_{n-2} + H^{*-1}_{\lambda_{n-1},\lambda_n}(t_{n-2}) H^*_{\lambda_{n-1},\lambda_n}(t_{n-1})\mu_{n-1}.$$

It follows from a simple induction argument that

$$f(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \prod_{l=1}^{n} \left(R_l \cdot \exp\left(\frac{1}{2}\int_0^{t_l} \mathrm{Tr}(K_{\lambda_l,\cdots,\lambda_n}(s))ds\right)\right).$$

Here for $1 \leqslant l \leqslant n$,

$$R_l = \widetilde{\mathbb{E}}_l\left[\exp\left(i\langle \mu_l, H_{\lambda_l,\cdots,\lambda_n}(t_l) \int_{t_{l-1}}^{t_l} H^{-1}_{\lambda_l,\cdots,\lambda_n}(s) d\widetilde{W}^{(l)}_s\rangle\right)\right],$$

and $\{H_{\lambda_l,\cdots,\lambda_n}(t) : t \in [0,t_l]\}$ is the solution to the equation

$$\frac{d}{dt}H_{\lambda_l,\cdots,\lambda_n}(t) = \left(\sum_{r=l}^{n} K_{\lambda_l,\cdots,\lambda_n}(t) + \sum_{r=l}^{n}(\lambda_l A_l)\right) H_{\lambda_l,\cdots,\lambda_n}(t),\ t \in [0,t_l],$$

with $H_{\lambda_l,\cdots,\lambda_n}(0) = \mathrm{Id}$. Moreover, $\{\mu_l\}$ is defined recursively by $\mu_n = \gamma_n$ and

$$\mu_l = \gamma_l + H^{*-1}_{\lambda_{l+1},\cdots,\lambda_n}(t_l)H^{*}_{\lambda_{l+1},\cdots,\lambda_n}(t_{l+1})\mu_{l+1},\ l = n-1,\cdots,1.$$

Now it remains to compute R_l explicitly. But this is easy since the random variables involved are Gaussian under the corresponding probability measures. More precisely, we have

$$\begin{aligned} R_l &= \widetilde{\mathbb{E}}_l\left[\exp\left(i\langle H^{*}_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l, \int_{t_{l-1}}^{t_l} H^{-1}_{\lambda_l,\cdots,\lambda_n}(s)d\widetilde{W}^{(n)}_s\right)\right] \\ &= \widetilde{\mathbb{E}}_l\left[\exp\left(i\int_{t_{l-1}}^{t_l}\langle H^{*-1}_{\lambda_l,\cdots,\lambda_n}(s)H^{*}_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l, d\widetilde{W}^{(n)}_s\rangle\right)\right] \\ &= \exp\left(-\frac{1}{2}\int_{t_{l-1}}^{t_l}|H^{*-1}_{\lambda_l,\cdots,\lambda_n}(s)H^{*}_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l|^2 ds\right). \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} & f(\gamma_1,\cdots,\gamma_n;\lambda_1,\cdots,\lambda_n) \\ = & \prod_{l=1}^{n}\exp\left(\frac{1}{2}\int_0^{t_l}\mathrm{Tr}(K_{\lambda_l,\cdots,\lambda_n}(s))ds - \frac{1}{2}\int_{t_{l-1}}^{t_l}|H^{*-1}_{\lambda_l,\cdots,\lambda_n}(s)H^{*}_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l|^2 ds\right). \end{aligned}$$

□

From the previous discussion, we can see that the computation of f reduces to the solution of a recursive system of symmetric matrix Riccati equations and the solution of a system of independent first order linear matrix ODEs. To complete the proof of Theorem 2.1, it remains to apply a standard complexification argument.

Lemma 3.2. *Fix $\gamma_1,\cdots,\gamma_n \in \mathbb{R}$. When c is small enough, the function*

$$\phi(z_1,\cdots,z_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^{n} i\langle\gamma_l, W_{t_l}\rangle + \sum_{l=1}^{n} z_l L^{A_l}_{t_l}\right)\right]$$

is holomorphic in the domain $D_c = \{(z_1,\cdots,z_n) \in \mathbb{C}^n : \mathrm{Re}(z_l) \in (-c,c), 1 \leqslant l \leqslant n\}$ of $\mathbb{C}^n$. Moreover, the function

$$\begin{aligned} \psi(\lambda_1,\cdots,\lambda_n) &= \prod_{l=1}^{n}\exp\left(\frac{1}{2}\int_0^{t_l}\mathrm{Tr}(K_{\lambda_l,\cdots,\lambda_n}(s))ds \right.\\ &\qquad \left. -\frac{1}{2}\int_{t_{l-1}}^{t_l}|H^{*-1}_{\lambda_l,\cdots,\lambda_n}(s)H^{*}_{\lambda_l,\cdots,\lambda_n}(t_l)\mu_l|^2 ds\right) \end{aligned}$$

defined on $\mathbb{R}^n$ can be extended holomorphically to$\mathbb{C}^n$. Such extension is unique, and when restricted to D_c, we have

$$\psi(z_1, \cdots, z_n) = \phi(z_1, \cdots, z_n).$$

Proof. By Lemma 3.1, when c is small, we know that $\phi(z_1, \cdots, z_n)$ is well defined on D_c. The continuity of $\phi(z_1, \cdots, z_n)$ follows easily from uniform integrability. Moreover, since the function

$$(z_1, \cdots, z_n) \mapsto \exp\left(\sum_{l=1}^{n} i\langle \gamma_l, W_{t_l}\rangle + \sum_{l=1}^{n} z_l L_{t_l}^{A_l}\right)$$

is holomorphic on $\mathbb{C}^n$ for every ω, by Fubini's theorem and Morera's theorem, we conclude that $\phi(z_1, \cdots, z_n)$ is holomorphic on D_c.

On the other hand, it is apparent that the recursive system of matrix Riccati equations and the system of independent matrix ODEs defined in Proposition 3.2 depend analytically on $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and extend naturally to the case when $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Therefore, when $(\lambda_1, \cdots, \lambda_n)$ is replaced by $(z_1, \cdots, z_n) \in \mathbb{C}^n$, the two systems determine solutions depending holomorphically on $z_1, \cdots, z_n$. In other words, $\psi(\lambda_1, \cdots, \lambda_n)$ possesses a holomorphic extension to $\mathbb{C}^n$. The uniqueness of such extension is a direct consequence of the identity theorem.

Finally, from Proposition 3.2 we know that ϕ and ψ coincide on the set $\{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_l \in (-c, c), 1 \leqslant l \leqslant n\}$. Therefore, again by the identity theorem, they must coincide on D_c. ☐

Now the result of Theorem 2.1 follows from the fact that

$$\{(i\Lambda_1, \cdots, i\Lambda_n) : \Lambda_l \in \mathbb{R}, 1 \leqslant l \leqslant n\} \subset D_c.$$

4 The Two Dimensional Brownian Rough Path

In this section, we are going to solve the differential equations involved in Theorem 2.1 explicitly for the two dimensional case. In particular this leads to an explicit formula (Theorem 2.2) for the finite dimensional characteristic functions of the planar Brownian rough path, extending the classical result (1.1) of Lévy.

Recall that

$$G(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \mathbb{E}\left[\exp\left(\sum_{l=1}^{n} i\langle \gamma_l, W_{t_l}\rangle + \sum_{l=1}^{n} i\Lambda_l L_{t_l}\right)\right],$$

where W_t is a two dimensional Brownian motion and L_t is Lévy's area process. In terms of Theorem 2.1, in this case we have

$$A_1 = \cdots = A_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: A.$$

Now let $\{k_{i\Lambda_l,\cdots,i\Lambda_n}(t) : t \in [0, t_l]\}$ $(1 \leqslant l \leqslant n)$ be the solution to the system of real scalar Riccati equations defined recursively from $l = n$ to $l = 1$ by

$$\begin{aligned}\frac{d}{dt} k_{i\Lambda_l,\cdots,i\Lambda_n} &= \left(\Lambda_l^2 + 2\Lambda_l \sum_{r=l+1}^{n} \Lambda_r\right) - 2\left(\sum_{r=l+1}^{n} k_{i\Lambda_r,\cdots,i\Lambda_n}(t)\right) k_{i\Lambda_l,\cdots,i\Lambda_n}(t) \\ &\quad - k_{i\Lambda_l,\cdots,i\Lambda_n}^2(t),\ t \in [0, t_l], \end{aligned} \tag{4.1}$$

with $k_{i\Lambda_l,\cdots,i\Lambda_n}(t_l) = 0$. By direct computation it is easy to see that

$$K_{i\Lambda_l,\cdots,i\Lambda_n}(t) = \begin{pmatrix} k_{i\Lambda_l,\cdots,i\Lambda_n} & 0 \\ 0 & k_{i\Lambda_l,\cdots,i\Lambda_n} \end{pmatrix},\ t \in [0, t_l], 1 \leqslant l \leqslant n,$$

solves the matrix Riccati system defined in Theorem 2.1 for this case. In other words, by uniqueness the solution is given by real scalar functions times the identity matrix.

To solve the system (4.1), we add the equations from $r = n$ to $r = l$ to obtain that

$$\frac{d}{dt}\left(\sum_{r=l}^{n} k_{i\Lambda_r,\cdots,i\Lambda_n}(t)\right) = \left(\sum_{r=l}^{n} \Lambda_r\right)^2 - \left(\sum_{r=l}^{n} k_{i\Lambda_r,\cdots,i\Lambda_n}\right)^2,\ t \in [0, t_l].$$

Let

$$c_l = \sum_{r=l}^{n} \Lambda_r,\ s_l(t) = \sum_{r=l}^{n} k_{i\Lambda_r,\cdots,i\Lambda_n}(t),\ t \in [0, t_l], 1 \leqslant l \leqslant n.$$

Suppose that $c_l \neq 0$ for every l. From explicit integration we obtain

$$s_l(t) = c_l \frac{c_l \sinh(c_l(t - t_l)) + s_l(t_l)\cosh(c_l(t - t_l))}{c_l \cosh(c_l(t - t_l)) + s_l(t_l)\sinh(c_l(t - t_l))}. \tag{4.2}$$

Since $k_{i\Lambda_l,\cdots,i\Lambda_n}(t_l) = 0$, we know that $\{s_l(t_l)\}$ is defined recursively by $s_n(t_n) = 0$ and

$$s_{l-1}(t_{l-1}) = c_l \frac{c_l \sinh(c_l(t_{l-1} - t_l)) + s_l(t_l)\cosh(c_l(t_{l-1} - t_l))}{c_l \cosh(c_l(t_{l-1} - t_l)) + s_l(t_l)\sinh(c_l(t_{l-1} - t_l))}$$

for $l = n, \cdots, 2$. It follows from Theorem 2.1 that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sum_{l=1}^{n} i\Lambda_l L_{t_l}\right)\right] &= \exp\left(\sum_{l=1}^{n} \frac{1}{2}\int_0^{t_l} \mathrm{Tr}(K_{i\Lambda_l,\cdots,i\Lambda_n}(s))ds\right) \\ &= \exp\left(\sum_{l=1}^{n} \int_0^{t_l} k_l(s)ds\right) \\ &= \exp\left(\sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} s_l(u)du\right) \\ &= \prod_{l=1}^{n} \frac{c_l}{c_l \cosh(c_l(t_{l-1} - t_l)) + s_l(t_l)\sinh(c_l(t_{l-1} - t_l))}. \end{aligned} \tag{4.3}$$

If $c_l = 0$ for some l, again by explicit integration we have

$$s_l(t) = \frac{s_l(t_l)}{1 + s_l(t_l)(t - t_l)} = \frac{s_{l+1}(t_l)}{1 + s_{l+1}(t_l)(t - t_l)}.$$

It is easy to see that $s_l(t)$ is the limit of (4.2) when $c_l \to 0$. Moreover, the resulting term in the product (4.3) becomes

$$\frac{1}{1 + s_l(t_l)(t_{l-1} - t_l)},$$

which is also the limit as $c_l \to 0$. Therefore (4.2) and (4.3) represent the general case.

On the other hand, if we let $\{\Phi_l(t) : t \in [0, t_l], 1 \leqslant l \leqslant n\}$ be the coefficient matrices of the linear system for $H_{i\Lambda_l,\cdots,i\Lambda_n}$ defined in Theorem 2.1, then from the previous discussion we can see that

$$\Phi_l(s)\Phi_l(t) = \Phi_l(t)\Phi_l(s), \ \ s, t \in [0, t_l].$$

Therefore, the linear system can be solved explicitly, and we obtain that

$$H_{i\Lambda_l,\cdots,i\Lambda_n}(t) = \exp\left(\int_0^t \left(\sum_{r=l}^n K_{i\Lambda_r,\cdots,i\Lambda_n}(u) + \left(\sum_{r=l}^n i\Lambda_r\right) A\right) du\right), \ \ t \in [0, t_l].$$

From the previous discussion, we conclude that

$$H_{i\Lambda_l,\cdots,i\Lambda_n}(t) = \exp\left(\left(\begin{array}{cc} a_l(t) & -ic_l t \\ ic_l t & a_l(t) \end{array}\right)\right), \ \ t \in [0, t_l],$$

where

$$\begin{aligned} a_l(t) &= \int_0^t \left(\sum_{r=l}^n K_{i\Lambda_r,\cdots,i\Lambda_n}(u)\right) du \\ &= \int_0^t s_l(u) du \\ &= \ln\left(\frac{c_l \cosh(c_l(t - t_l)) + s_l(t_l)\sinh(c_l(t - t_l))}{c_l \cosh(c_l t_l) - s_l(t_l)\sinh(c_l t_l)}\right) \ \ t \in [0, t_l]. \quad (4.4) \end{aligned}$$

Note that (4.4) works for the case $c_l = 0$ as well.

Finally, according to Theorem 2.1, we arrive at

$$\begin{aligned} & G(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) \\ = & \prod_{l=1}^n \left(\frac{c_l}{c_l \cosh(c_l(t_{l-1} - t_l)) + s_l(t_l)\sinh(c_l(t_{l-1} - t_l))}\right. \\ & \left. \cdot \exp\left(-\frac{1}{2}\int_{t_{l-1}}^{t_l} \left|H_l^{*-1}(s)H_l^*(t_l)\mu_l\right|^2 ds\right)\right), \end{aligned}$$

where $\{\mu_l\}$ is defined in the same way as in Theorem 2.1. Now the proof of Theorem 2.2 is complete.

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