Axiomatizing AECs and Applications

Panglobal Algebra and Logic Seminar

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Overview

1. Main results
2. Abstract elementary classes (AECs)
3. Proof idea
4. Other results
5. Open questions

Main results

Definition

Let $L \subseteq L'$ be first-order languages, T be a first-order theory in L', Γ be a set of L'-types.

$$M' \in EC(T, \Gamma) \Leftrightarrow (M' \vDash T \text{ and } M' \text{ omits } \Gamma)$$

$$M \in PC(T, \Gamma, L) \Leftrightarrow (\exists M' \in EC(T, \Gamma) \text{ and } M = M' \restriction L)$$

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Theorem (Shelah)

Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$. *Then* K *is* $PC_{\lambda, 2^\lambda}$. *Namely, there are* $L' \supseteq \mathrm{L}(\mathbf{K})$, T *and* Γ *such that*

1. $K = PC(T, \Gamma, \mathrm{L}(\mathbf{K}))$;
2. $|T| \leq \lambda$ *and* $|\Gamma| \leq 2^\lambda$.

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Theorem (4.1,4.10)

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Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$*. Then* K *is* $PC_{\chi,\chi}$ *where* $\chi = \lambda + I_2(\lambda, \mathbf{K})$*.*
Moreover, if $\mathbf{K}$ *is categorical in* λ^+*, stable in* λ*, has* λ*-AP and* $I(\lambda, \mathbf{K}) \leq \lambda$*, then* $\mathbf{K}$ *is* $PC_{\lambda,\lambda}$*.*

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Moreover, if $\mathbf{K}$ *is categorical in* λ^+, *stable in* λ, *has* λ*-AP and* $I(\lambda, \mathbf{K}) \leq \lambda$, *then* $\mathbf{K}$ *is* $PC_{\lambda,\lambda}$.

Relevant result in the literature (using descriptive set theory):

Theorem (Shelah-Vasey)

Let $\mathbf{K}$ *be an AEC with* $\mathrm{LS}(\mathbf{K}) = \aleph_0$. *If* $\mathbf{K}$ *is stable in* $\aleph_0$, *has* $\aleph_0$*-AP and* $I(\aleph_0, \mathbf{K}) \leq \aleph_0$, *then* K *is* $PC_{\aleph_0,\aleph_0}$.

Let's also look at axiomatizations within the same language $\mathrm{L}(\mathbf{K})$.

Theorem (Shelah-Villaveces)

Let $\mathbf{K}$ *be an AEC,* $L = \mathrm{L}(\mathbf{K})$ *and* $\lambda = \mathrm{LS}(\mathbf{K})$. *Then* $\mathbf{K}$ *can be axiomatized by a sentence in* $L_{(2^{2^{\lambda^+}})^{+++},\lambda^+}$.

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Theorem (3.7)

Let $\mathbf{K}$ be an AEC, $L = \mathrm{L}(\mathbf{K})$, $\lambda = \mathrm{LS}(\mathbf{K})$ and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. Then $\mathbf{K}$ can be axiomatized by a sentence in $L_{\chi^+,\lambda^+}(\omega \cdot \omega)$ (game quantification of $\omega \cdot \omega$ steps).

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Definition

Let L be a finitary (first-order) language. An abstract elementary class $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in $L = \mathrm{L}(\mathbf{K})$ satisfies the following axioms:

1. K is a class of L-structures and $\leq_{\mathbf{K}}$ is a partial order on $K \times K$.
2. For $M_1, M_2 \in K$, $M_1 \leq_{\mathbf{K}} M_2$ implies $M_1 \subseteq M_2$ (as L-substructure).

Abstract elementary classes (AECs)

Definition (Continued)

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   a. If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.

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   a. If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.
   b. Let $M_1, M_2, N_1, N_2 \in K$. If $f : M_1 \cong M_2$, $g : N_1 \cong N_2$, $g \supseteq f$ and $M_1 \leq_{\mathbf{K}} N_1$, then $M_2 \leq_{\mathbf{K}} N_2$.

$$\begin{tikzcd} N_1 \arrow[r, "g"] & N_2 \\ M_1 \arrow[u, "\leq_{\mathbf{K}}"] \arrow[r, "f"] & M_2 \arrow[u, dotted, "\leq_{\mathbf{K}}"] \end{tikzcd}$$

Abstract elementary classes (AECs)

Definition (Continued)

4. Coherence: Let $M_1, M_2, M_3 \in K$. If $M_1 \leq_{\mathbf{K}} M_3$, $M_2 \leq_{\mathbf{K}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \leq_{\mathbf{K}} M_2$.

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5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.

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5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\,\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.
6. Chain axioms: Let α be an ordinal and $\langle M_i : i < \alpha \rangle \subseteq K$ such that for $i < j < \alpha$, $M_i \leq_{\mathbf{K}} M_j$.
    1. Then $M = \bigcup_{i<\alpha} M_i$ is in K and for all $i < \alpha$, $M_i \leq_{\mathbf{K}} M$.
    2. Let $N \in K$. If in addition for all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

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$$I_2(\lambda, \mathbf{K}) = |\{(M, N)/_{\cong} : M \leq_{\mathbf{K}} N \text{ both in } K_\lambda\}|$$

where $(M_1, N_1) \cong (M_2, N_2)$ iff $M_1 \leq_{\mathbf{K}} N_1$, $M_2 \leq_{\mathbf{K}} N_2$ and there is $g : N_1 \cong N_2$ such that $g \restriction M_1 : M_1 \cong M_2$.

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Fact

$I_2(\lambda, \mathbf{K}) \leq 2^\lambda$.

In the main results, we had $\chi = \lambda + I_2(\lambda, \mathbf{K})$, and bypassed I_2 under more assumptions.

Proof idea

Adapt the original proof by Shelah!

Definition

Let I be an index set. A directed system $\langle M_i : i \in I \rangle \subseteq K$ indexed by I satisfies the following: for any $i, j \in I$, there is $k \in I$ such that $M_i \leq_{\mathbf{K}} M_k$ and $M_j \leq_{\mathbf{K}} M_k$.

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Given $M \in K$, we index the system by finite tuples of elements in M, ordered by inclusion.

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Fact

Let $\langle M_i : i \in I\rangle \subseteq K$ be a directed system. Then

1. $M = \bigcup_{i \in I} M_i \in K$;
2. For all $i \in I$, $M_i \leq_{\mathbf{K}} M$;
3. Let $N \in K$. If in addition for all $i \in I$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

Proof idea

Let M be an $\mathrm{L}(\mathbf{K})$-structure, $\lambda = \mathrm{LS}(\mathbf{K})$.
Strategy: to ensure that $M \in K$, we require M to be the union of a directed system in K.

1. Expand $\mathrm{L}(\mathbf{K})$ by adding λ-many functions. They map a finite tuple to a $\mathbf{K}$-structure containing it:
$$a \in |M|^{<\omega} \mapsto \{f_i(a) : i < \lambda\} = |M_a| \text{ with } M_a \in K.$$

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   - "$M_a \in K$" is by listing all the isomorphism types in K_λ, there are $I(\lambda, \mathbf{K})$-many.
2. Stipulate that $\langle M_a : a \in |M|^{<\omega}\rangle$ forms a directed system:
If $a \cup b \subseteq c$ then $M_a \leq_{\mathbf{K}} M_c$ and $M_b \leq_{\mathbf{K}} M_c$.

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If $a \cup b \subseteq c$ then $M_a \leq_{\mathbf{K}} M_c$ and $M_b \leq_{\mathbf{K}} M_c$.
   - "$\leq_{\mathbf{K}}$" is by listing all the isomorphism types of pairs $N_0 \leq_{\mathbf{K}} N_1$ in K_λ, there are $I_2(\lambda, \mathbf{K})$-many.

Proof idea

We have axiomatized **K** in an $L'_{\chi^+,\omega}$-sentence σ where L' contains extra functions than L(**K**) and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. To convert this to a PC-class, we use the following fact:

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Chang's presentation theorem

Let θ be an infinite cardinal, L be a language of size $\leq \theta$, T be an $L_{\theta^+,\omega}$-theory contained in a fragment of size $\leq \theta$, then the models of T is a $PC_{\theta,\theta}$-class.

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We have axiomatized **K** in an $L'_{\chi^+,\omega}$-sentence σ where L' contains extra functions than $L(\mathbf{K})$ and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. To convert this to a PC-class, we use the following fact:

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Taking $T = \{\sigma\}$, we obtain **K** as a $PC_{\chi,\chi}$-class.

Proof idea

Theorem (4.10)

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in λ^+, stable in λ, has λ-AP and $I(\lambda, \mathbf{K}) \leq \lambda$, then $\mathbf{K}$ is $PC_{\lambda,\lambda}$.

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$I(\lambda, \mathbf{K}) \leq \lambda$ allows us to list at most λ individual models. Unfortunately, we do not know how to obtain a better bound of I_2.

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$I(\lambda, \mathbf{K}) \leq \lambda$ allows us to list at most λ individual models. Unfortunately, we do not know how to obtain a better bound of I_2.
We bypass this by building limit models. Recall that N is (λ, ω)-limit over M_0 if $N = \bigcup_{i<\omega} M_i$ where $M_{i+1} \in K_\lambda$ is universal over M_i.

$$
\begin{tikzcd}
M_b \arrow[rr, dotted, "\cong"] & & M_c \\
& M_a \arrow[ul, "{(\lambda,\omega)}"] \arrow[ur, "{(\lambda,\omega)}"'] &
\end{tikzcd}
$$

In other words, if $I(\lambda, \mathbf{K}) = \lambda$ and $\leq_{\mathbf{K}}$ is replaced by "(λ, ω)-limit", then $I_2(\lambda, \mathbf{K}) = \lambda$!

Other results

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- any set of T-types Γ of size $\leq \kappa$,
- if there exists $M \in EC(T, \Gamma)$ with $(Q^M, <^M)$ of order type $\geq \delta$,
- then there is $N \in EC(T, \Gamma)$ with $(Q^N, <^N)$ ill-founded.

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Definition

Let $\mathbf{K}$ be an AEC. $K^< = \{\langle |M|, |N| \rangle \mid N <_{\mathbf{K}} M\}$.

Other results

Theorem (Shelah)

Let $\mathbf{K}$ *be an AEC,* $\theta \geq \mathrm{LS}(\mathbf{K})$*. Suppose the following hold:*

1. $K, K^{<}$ *are both* $PC_{\theta,\theta}$*;*
2. $\mathbf{K}$ *is categorical in both* θ *and* θ^{+}*;*
3. $\delta(\theta, 1) = \theta^{+}$ *(true for* $\aleph_0$ *and strong limits of countable cofinality).*

Then $K_{\theta^{++}} \neq \emptyset$*.*

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(1) is true for $\theta \geq 2^\lambda$ *where* $\lambda = \mathrm{LS}(\mathbf{K})$.

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Corollary (4.8)

(1) is true for $\theta \geq \lambda + I_2(\lambda, \mathbf{K})$*. Moreover, under* λ*-AP and stability in* λ*, (2) already implies (1) for* $\theta = \lambda$*.*

Other results

Let $\mu \geq \aleph_0$. Define PC^μ as before except that the underlying languages L, L' are $(< \mu)$-ary.

K is	**K** is axiomatizable in	**K** is
An AEC	$L_{\chi^+,\lambda^+}(\omega \cdot \omega)$	$PC_{\chi,\chi}$
A μ-AEC	$L_{\chi^+,\lambda^+}(\mu \cdot \mu)$	$PC^\mu_{\chi,\chi}$

Open questions

Question

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$.

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- Can we relate the infinitary logics $L_{\alpha,\beta}$ and $L_{\gamma,\epsilon}(\delta)$?
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- Can we relate the infinitary logics $L_{\alpha,\beta}$ and $L_{\gamma,\epsilon}(\delta)$?
  - This might remove the game quantification in our axiomatization.
- Does the Hanf number exist for μ-AECs?
  - Hanf number is the threshold cardinal for arbitrarily large models. In AECs, the Hanf number is $\beth_{(2^\lambda)^+}$.