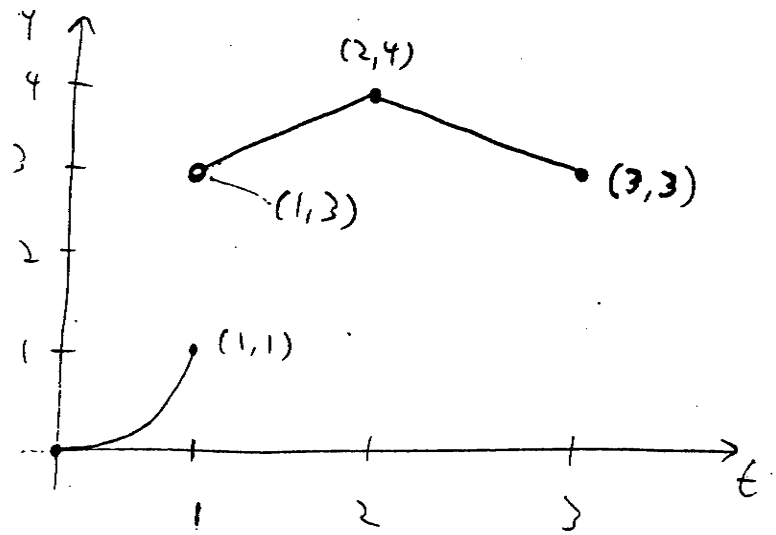
21-260 Spring 2008
Homework #7 Solutions

Section 6.1

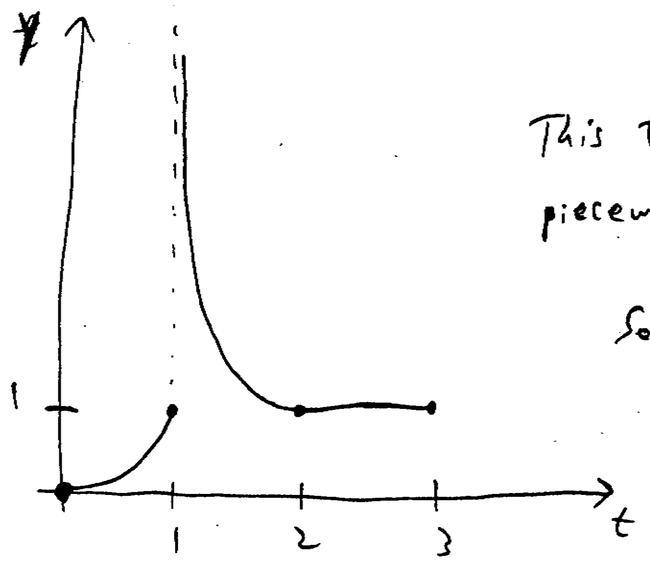
① The graph looks like this:



Since $\lim_{t \to 1^-} f(t) \neq \lim_{t \to 1^+} f(t)$, f is not continuous,

but f is piecewise continuous on $[0, 3]$.

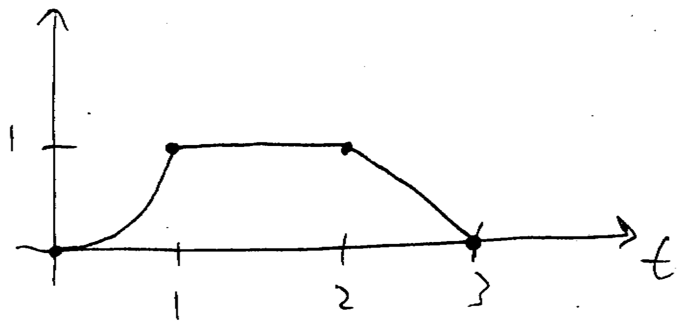
② Graph:



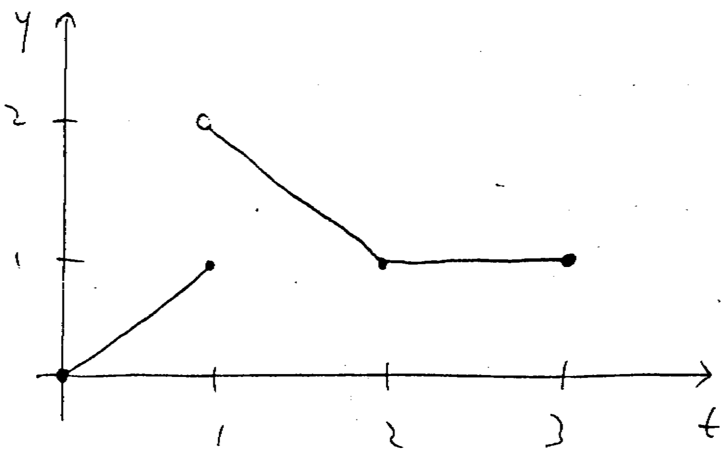
This function is not continuous nor even piecewise continuous, because $\lim_{t \to 1^+} f(t) = \infty$

So f has a "blow-up" discontinuity.

(3) This function is continuous on $[0,3]$, for at every point $a \in [0,3]$, we have $\lim_{t \to a} f(t) = f(a)$. Here's the graph:



(4) The graph looks like this:



f is piecewise continuous.

Section 6.2

(1) If $F(s) = \dfrac{3}{s^2+4}$, and we compare this with Item 5 in Table 6.2.1, we see that it isn't quite right whether we set $a = 3$ or $a = 2$. However, with $a = 2$ we

do have $\frac{2}{s^2+4}$, which is $\mathcal{L}\{\sin 2t\}$. But also,

$$\frac{3}{s^2+4} = \frac{3}{2}\cdot\frac{2}{s^2+4} = \frac{3}{2}\mathcal{L}\{\sin 2t\} = \mathcal{L}\{\tfrac{3}{2}\sin 2t\}$$ using

linearity. So $f(t) = \frac{3}{2}\sin 2t$ satisfies $F(s) = \frac{3}{s^2+4}$.

(2) $F(s) = \frac{4}{(s-1)^3}$ is a translation of the function $\frac{4}{s^3}$,

which is $2\cdot\frac{2}{s^3} = 2\,\mathcal{L}\{t^2\}$. This in turn is $\mathcal{L}\{2t^2\}$.

Therefore $F(s) = \mathcal{L}\{2t^2e^t\}$, using the property of the

Laplace transform that if $F = \mathcal{L}\{f\}$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

(So in this case we have $f(t) = 2t^2$ and $a=1$.)

(3) $$F(s) = \frac{2}{s^2+3s-4} = \underbrace{\frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}}$$

for some A and B. So multiplying this equation by $(s+4)(s-1)$,

we get $2 = A(s-1) + B(s+4)$

$$\Rightarrow\ 2 = (A+B)s + (-A+4B)$$

So we need $\left.\begin{aligned} A+B &= 0\\ -A+4B &= 2\end{aligned}\right\} \Rightarrow A = -\frac{2}{5},\ B = \frac{2}{5}$

So $F(s) = -\frac{2}{5} \cdot \frac{1}{s+4} + \frac{2}{5} \cdot \frac{1}{s-1}$

$= -\frac{2}{5} \cdot \frac{1}{s-(-4)} + \frac{2}{5} \cdot \frac{1}{s-1}$

$= -\frac{2}{5} \mathcal{L}\{e^{-4t}\} + \frac{2}{5} \mathcal{L}\{e^{t}\}$

$= \mathcal{L}\{-\frac{2}{5} e^{-4t} + \frac{2}{5} e^{t}\}.$

So $f(t) = \frac{2}{5}(e^{t} - e^{-4t})$

(7) This one's a little tricky because our first thought, to factor the denominator, doesn't work. That's because $s^2 - 2s + 2$ has no real roots. Note that the discriminant ($b^2 - 4ac$ in the quadratic formula) is $(-2)^2 - 4(1)(2) = 4 - 8 < 0$. So the key is to get the denominator into the form $(s-a)^2 + b^2$, with an eye toward Items 9 and 10 in Table 6.2.1. So in other words, we should <u>complete the square</u>, like so:

$s^2 - 2s + 2 = (s^2 - 2s + 1) + 1 = (s-1)^2 + 1$, and so

$F(s) = \frac{2s+1}{s^2 - 2s + 2} = \frac{2s+1}{(s-1)^2 + 1}$. Now this doesn't conform

to either Item 9 or Item 10, but note what happens if we manipulate the numerator so that the quantity $s-1$ is present:

$$F(s) = \frac{2s - 2 + 2 + 1}{(s-1)^2 + 1} = \frac{2(s-1) + 3}{(s-1)^2 + 1}.$$ Now split

this into two terms, and write $F(s) = \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}.$

Now we are in business, since

$$F(s) = 2 \cdot \frac{s-1}{(s-1)^2 + 1} + 3 \cdot \frac{1}{(s-1)^2+1}$$

$$= 2 \mathcal{L}\{e^t \cos t\} + 3 \mathcal{L}\{e^t \sin t\}$$

$$= \mathcal{L}\{2e^t \cos t + 3e^t \sin t\}.$$

So $f(t) = e^t (2\cos t + 3 \sin t)$