21-260 Spring 2008

Homework #12 Solutions

Section 10.5

(7) Here $L = 1$, and $f(x) = \sin 2\pi x - \sin 5\pi x$, which can be written as $f(x) = \sin\left(\frac{2\pi x}{1}\right) - \sin\left(\frac{5\pi x}{1}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$ with $b_2 = 1$, $b_5 = -1$, and $b_n = 0$ for all other n.

So f may be considered to already have a Fourier sine series representation. The solution to the BVP is therefore

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{1}\right) \exp\left(-\frac{\alpha^2 n^2 \pi^2}{1} t\right)$$

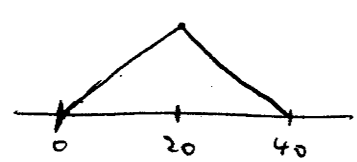
(where $\alpha^2 = 100$ here), with $c_2 = 1$, $c_5 = -1$, and $c_n = 0$ for all other n. So the solution is

$$u(x,t) = \sin(2\pi x)\exp(-400\pi^2 t) - \sin(5\pi x)\exp(-2500\pi^2 t)$$

(10) The BVP to solve is

$$u_{xx} = u_t \quad \text{for } x \in [0,40] \text{ and } t \geq 0$$
$$u(0,t) = 0$$
$$u(40,t) = 0$$
$$u(x,0) = f(x)$$

with $f(x) = \begin{cases} x & \text{for } x \in [0,20] \\ 40 - x & \text{for } x \in [20,40] \end{cases}$

The graph of f looks like this: 

So f is extendable to an odd function on $[-40, 40]$, or even on $(-\infty, \infty)$, in such a way that the extension is continuous (although that fact is not needed for the solution procedure).

So now we wish to find a Fourier sine series representing f, with coefficients $b_n = \frac{1}{40}\int_{-40}^{40} f(x) \sin\left(\frac{n\pi x}{40}\right) dx$,

and here f denotes the

$$\text{extension } f(x) = \begin{cases} -40 - x & \text{for } -40 \le x \le -20 \\ x & \text{for } -20 \le x \le 20 \\ 40 - x & \text{for } 20 \le x \le 40 \end{cases}$$

To simplify the computations, let's note that if g is an even function, then $\int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx$. Since f is odd, and each function $\sin\left(\frac{n\pi x}{40}\right)$ is also odd, the product of the two is even. Therefore, $b_n = \frac{1}{20}\int_0^{40} f(x) \sin\left(\frac{n\pi x}{40}\right) dx =$

$$\frac{1}{20}\int_0^{20} x \sin\left(\frac{n\pi x}{40}\right) dx + \frac{1}{20}\int_{20}^{40} (40 - x) \sin\left(\frac{n\pi x}{40}\right) dx$$

$$= \frac{1}{20}\left[\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right) - \frac{40}{n\pi} x \cos\left(\frac{n\pi x}{40}\right)\right]_0^{20} + \left[-\frac{80}{n\pi}\cos\left(\frac{n\pi x}{40}\right)\right]_{20}^{40}$$

$$- \frac{1}{20}\left[\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right) - \frac{40}{n\pi} x \cos\left(\frac{n\pi x}{40}\right)\right]_{20}^{40}$$

$$= \frac{1}{20}\left(\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{40}{n\pi}\cdot 20\cos\left(\frac{n\pi}{2}\right)\right) + \left(-\frac{80}{n\pi}\cos(n\pi) + \frac{80}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$- \frac{1}{20}\left(-\frac{40}{n\pi}\cdot 40\cos(n\pi) - \frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{40}{n\pi}\cdot 20\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{1}{10}\cdot\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{80}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{80}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$- \frac{80}{n\pi}\cos(n\pi) + \frac{80}{n\pi}\cos(n\pi)$$

$$\Longrightarrow \quad b_n = \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

So $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{40}\right)$, and the solution to the BVP is

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{40}\right)\exp\left(-\frac{n^2\pi^2}{1600}t\right)$, and since

$b_n = 0$ whenever n is even, let's write each odd n as

$n = 2K+1$. Then note that for K odd, $\sin\left(\frac{n\pi}{2}\right) = 1 = (-1)^{K+1}$.

But whenever K is even, $\sin\left(\frac{n\pi}{2}\right) = -1 = (-1)^{K+1}$. Hence we can

write $u(x,t) = \sum_{K=1}^{\infty} \frac{160(-1)^{K+1}}{(2K+1)^2\pi^2}\sin\left(\frac{(2K+1)\pi x}{40}\right)\exp\left(-\frac{(2K+1)^2\pi^2}{1600}t\right)$

(12) This time the BVP is
$$\begin{aligned} &u_{xx} = u_t \quad \text{for } x \in [0,40],\ t \geq 0 \\ &u(0,t) = 0 \\ &u(40,t) = 0 \\ &u(x,0) = x \leftarrow f(x) \end{aligned}$$

So f is extendable to $[-40, 40]$ simply by the formula $f(x) = x$, and the result is an odd function. Let's find a Fourier sine series for f. We compute $b_n = \frac{1}{40}\int_{-40}^{40} x \sin\left(\frac{n\pi x}{40}\right) dx$

$$= \frac{1}{40}\left[\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right) - \frac{40}{n\pi} x \cos\left(\frac{n\pi x}{40}\right)\right]_{-40}^{40}$$

$$= \frac{1}{40}\left(-\frac{40^2}{n\pi}\cos(n\pi) - \frac{40^2}{n\pi}\cos(-n\pi)\right).$$ Now $\cos(-z) = \cos z$ for any argument z, so $b_n = -\frac{80}{n\pi}\cos(n\pi) = -\frac{80}{n\pi}(-1)^n$, or

$$b_n = \frac{80(-1)^{n+1}}{n\pi}.$$

So $f(x) = \sum_{n=1}^{\infty} \frac{80(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{40}\right)$, and therefore the solution to the BVP is $u(x,t) = \sum_{n=1}^{\infty}\frac{80(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{40}\right)\exp\left(-\frac{n^2\pi^2}{1600}t\right)$

Section 10.6

9 (a) The BVP to solve is

$$(*)\begin{cases} \alpha^2 u_{xx} = u_t & \text{for } x \in [0,20],\ t \geq 0 \\ u(0,t) = 0 \\ u(20,t) = 60 \\ u(x,0) = 25 \end{cases}$$

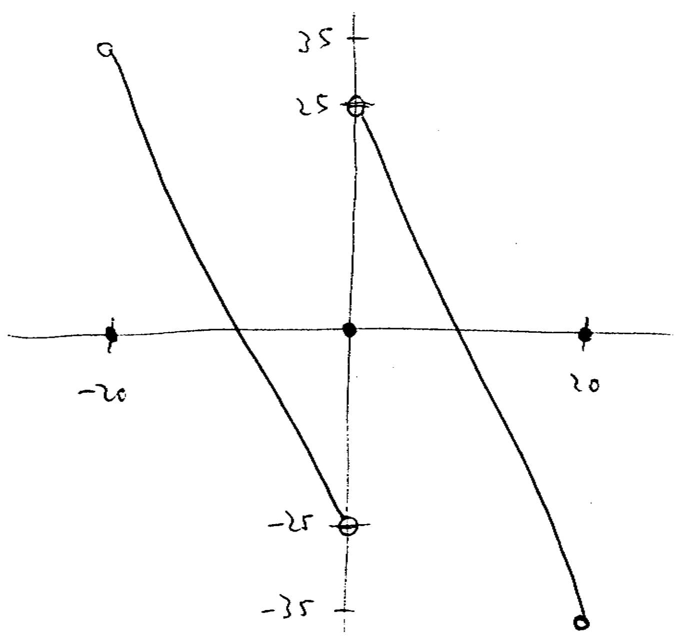
Unless I'm missing something, we are not told α^2.

Introduce $v(x)$, the linear function which is 0 at $x = 0$ and takes value 60 at $x = 20$. So that's just $v(x) = 3x$.

Writing the solution to (*) as $u(x,t) = w(x,t) + v(x)$, the function w solves the following auxiliary BVP:

$$(**) \quad \begin{cases} \alpha^2 w_{xx} = w_t & \text{for } x \in [0,20],\ t \geq 0 \\ w(0,t) = 0 \\ w(20,t) = 0 \\ w(0,x) = 25 - 3x & \leftarrow \text{call this } g(x) \end{cases}$$

We solve this BVP (with zero boundary conditions) by finding a Fourier sine series for g. Re-defining g to be 0 at $x=0$ and $x=20$, and extending g to $[-20,20]$ as an odd function, we get this:



Then $b_n = \frac{2}{20}\int_0^{20} g(x)\sin\left(\frac{n\pi x}{20}\right)dx = \frac{1}{10}\int_0^{20}(25-3x)\sin\left(\frac{n\pi x}{20}\right)dx$

$$= \frac{1}{10}\left[-\frac{25\cdot 20}{n\pi}\cos\left(\frac{n\pi x}{20}\right)\right]_0^{20} - \frac{3}{10}\left[\frac{20^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{20}\right) - \frac{20}{n\pi}x\cos\left(\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left(-\frac{500}{n\pi}\cos(n\pi) + \frac{500}{n\pi}\right) - \frac{3}{10}\left(-\frac{400}{n\pi}\cos(n\pi)\right) = \frac{70}{n\pi}\cos(n\pi) + \frac{50}{n\pi}$$

So $b_n = \dfrac{70(-1)^n + 50}{n\pi} = \begin{cases} \dfrac{-20}{n\pi} & \text{when } n \text{ is odd} \\ \dfrac{120}{n\pi} & \text{when } n \text{ is even} \end{cases}$

So the solution to $(**)$ is

$$w(x,t) = \sum_{n=1}^{\infty} \left(\frac{70(-1)^n + 50}{n\pi} \right) \sin\left(\frac{n\pi x}{20}\right) \exp\left(-\frac{\alpha^2 n^2 \pi^2}{400} t\right)$$

and then the solution $u(x,t)$ to $(*)$ is $3x + w(x,t)$.

$\boxed{\text{Section 10.7}}$

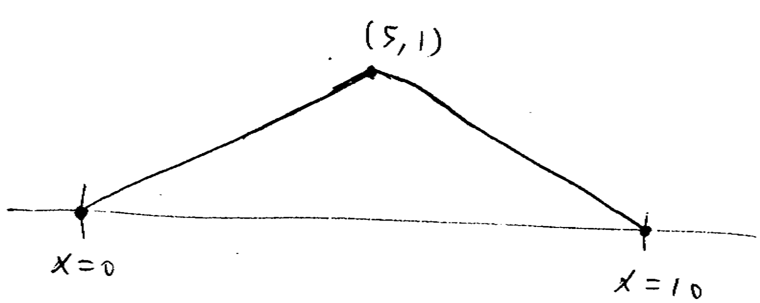
1 (a) The BVP to solve is
$$\begin{aligned} u_{xx} &= u_{tt} \quad \text{for } 0 \le x \le 10,\ t \ge 0 \\ u(0,t) &= 0 \\ u(10,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= 0 \end{aligned}$$

The solution u is of the form $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) \cos\left(\frac{n\pi t}{10}\right)$,

where b_n are the Fourier sine series coefficients for f. That is,

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$, and here $f(x) = \begin{cases} \frac{x}{5} & \text{for } 0 \le x \le 5 \\ 2 - \frac{x}{5} & \text{for } 5 \le x \le 10 \end{cases}$

The graph of f looks like this:



So we are modeling a string which is pulled up a distance of one unit (from the axis of equilibrium) in the center, and then released.

The FSS for f has coefficients $b_n = \frac{2}{10}\int_0^{10} f(x)\sin\left(\frac{n\pi x}{10}\right)dx$

$$= \frac{1}{5}\int_0^5 \frac{x}{5}\sin\left(\frac{n\pi x}{10}\right)dx + \frac{1}{5}\int_5^{10}\left(2-\frac{x}{5}\right)\sin\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{1}{25}\int_0^5 x\sin\left(\frac{n\pi x}{10}\right)dx + \frac{2}{5}\int_5^{10}\sin\left(\frac{n\pi x}{10}\right)dx - \frac{1}{25}\int_5^{10} x\sin\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{1}{25}\left[\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right) - \frac{10x}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\right]_0^5 + \left[-\frac{4}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\right]_5^{10}$$

$$- \frac{1}{25}\left[\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right) - \frac{10x}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\right]_5^{10}$$

$$= \frac{1}{25}\left(\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{50}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) + \left(-\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$- \frac{1}{25}\left(-\frac{100}{n\pi}\cos(n\pi) - \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{50}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

OK, so the solution is $u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{n\pi t}{10}\right)$

(or you can re-index the series to get rid of the zero terms corresponding to even values of n.)