21-260 Spring 2008

Homework #11 Solutions

Section 10.1

(4) The general solution to $y'' + y = 0$ is $y(t) = A\cos t + B\sin t$,

So if we want $y(L) = 0$, then we must have $A\cos L + B\sin L = 0$

Now we consider three cases:

(i) $\cos L = 0$

(ii) $\cos L \neq 0$ but $\sin L = 0$

(iii) $\cos L \neq 0$ and $\sin L \neq 0$

In case (i), we have $B\sin L = 0$. This means $B = 0$, because if $\cos L = 0$, then $\sin L$ cannot also be zero. So ... if $B=0$, then $y(t) = A\cos t$. But then the other boundary condition $y'(0) = 1$ cannot be met, for $y'(t) = -A\sin t \Rightarrow y'(0) = 0$ no matter what A is.

So there is no solution to the BVP if $\cos L = 0$.

In case (ii), $y(t) = A\cos L = 0 \Longrightarrow A = 0$. Therefore, $y(t) = B\sin t$. Then $y'(t) = B\cos t$, so $y'(0) = B \Rightarrow B = 1$. So we get exactly one solution $y(t) = \sin t$.

In case (iii), we can write $A\cos L + B\sin L = 0$ as $B = -A\cot L$. So $y(t) = A\cos t - A\cot L\sin t \Longrightarrow y'(t) = -A\sin t - A\cot L\cos t$ $\Rightarrow y'(0) = -A\cot L = 1 \Rightarrow A = -\tan L$.

Then $B = -A\cot L = -(-\tan L)\cot L = 1$

So we get the unique solution $y(t) = -\tan L \cos t + \sin t$.

(8) First we look for a particular solution to $y'' + 4y = \sin x$.

Assume $y_p(x) = A\cos x + B\sin x$.

Then $y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$$\Rightarrow y_p'' + 4y_p = -A\cos x - B\sin x + 4A\cos x + 4B\sin x$$

$$= 3A\cos x + 3B\sin x = \sin x$$

$$\Rightarrow 3B = 1$$

$$\Rightarrow B = \tfrac{1}{3}, \text{ and } A = 0.$$

So $y_p(x) = \frac{1}{3}\sin x$.

Then the general solution to $y'' + 4y = 0$ is $y(x) = A\cos 2x + B\sin 2x$,

so the gen. sol'n to $y'' + 4y = \sin x$ is

$$y(x) = \tfrac{1}{3}\sin x + A\cos 2x + B\sin 2x$$

Now we see if the boundary conditions can be satisfied by any member(s) of this family.

$$y(0) = \tfrac{1}{3}\sin 0 + A\cos 0 + B\sin 0 = A, \text{ so } A \text{ must be } 0.$$

Then, if $y(x) = \frac{1}{3}\sin x + B\sin 2x$, $y(\pi) = \frac{1}{3}\sin\pi + B\sin 2\pi = 0$.

So the condition $y(\pi) = 0$ is automatically satisfied. Therefore we have infinitely many solutions to the BVP, all of the form

$y(x) = \frac{1}{3} \sin x + B \sin 2x$

(16) First let's consider $\lambda = 0$. In that case the diff. eq. is $y'' = 0$, and the solutions are all linear functions. So if we have $y'(t) = 0$ for 2 values of t, then y is a constant function. So any function $y \equiv A$ works, and since we have nontrivial solutions, $\lambda = 0$ is an eigenvalue, with corresponding eigenfunctions $y \equiv A$, $A \neq 0$.
(or just take $y \equiv 1$ as a "representative eigenfunction".)

In case $\lambda < 0$, the general solution to $y'' + \lambda y = 0$ (or $y'' = -\lambda y = |\lambda| y$) is $y(t) = Ae^{\sqrt{-\lambda}t} + Be^{-\sqrt{-\lambda}t}$; so $y'(t) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}t} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}t}$, and if $y'(0) = 0$, then $A\sqrt{-\lambda} - B\sqrt{-\lambda} = 0$, or $\sqrt{-\lambda}(A - B) = 0$. Since $\sqrt{-\lambda} > 0$, this means $A = B$. So now we have

$$y'(t) = A\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}t} - e^{-\sqrt{-\lambda}t}\right),$$

and if $y'(\pi) = 0$, then $A\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}\right) = 0$. Again, $\sqrt{-\lambda} > 0$, so we either have $A = 0$, which would mean $A = B = 0$ $\Rightarrow y \equiv 0$ ... so that's no good ... or we have $e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{\lambda}\pi} = 0 \Rightarrow e^{\sqrt{-\lambda}\pi} = e^{-\sqrt{-\lambda}\pi}$, which means $\sqrt{-\lambda}\,\pi = -\sqrt{-\lambda}\,\pi$ (since the exponential function is a one-to-one function.) But this can only be true if $\lambda = 0$, so that's a contradiction.

So we only get the trivial solution $y \equiv 0$ if $\lambda \leq 0$.

Okey doke, now assume $\lambda > 0$. Then $y'' + \lambda y = 0$ has general solution $y(t) = A\cos(\sqrt{\lambda}\,t) + B\sin(\sqrt{\lambda}\,t)$, and so

$$y'(t) = -\sqrt{\lambda} A \sin(\sqrt{\lambda}\,t) + \sqrt{\lambda}\, B \cos(\sqrt{\lambda}\,t),$$

and $y'(0) = 0 \Rightarrow \sqrt{\lambda}\, B = 0. \Rightarrow B = 0$. So now we know $y'(t) = -\sqrt{\lambda}\, A \sin(\sqrt{\lambda}\,t)$, and $y'(\pi) = 0 \Rightarrow -\sqrt{\lambda}\, A \sin(\sqrt{\lambda}\,\pi) = 0$

Now this last condition does not force $A = 0$ (which would again yield only $y \equiv 0$ as a solution); if $A \neq 0$, then $\sin(\sqrt{\lambda}\,\pi) = 0$, which would mean that $\sqrt{\lambda}$ is an integer. Therefore λ is the square of a (positive) integer.

So $\lambda = n^2$ is an eigenvalue for any $n = 1, 2, 3, \ldots$, and then the corresponding eigenfunctions are $y(t) = A\cos(nt)$ for any $A \neq 0$.

So to summarize: The eigenvalues are $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 9$, ..., $\lambda_n = n^2$, and "representative" eigenfunctions are $y_0 \equiv 1$, $y_1(t) = \cos t$, $y_2(t) = \cos 2t$, $y_3(t) = \cos 3t$, ..., $y_n(t) = \cos nt$, ...

Section 10.2

(16) Here $f(x) = \begin{cases} x+1 & \text{for } -1 \le x < 0 \\ 1-x & \text{for } 0 \le x \le 1 \end{cases}$

(or $f(x) = 1 - |x|$ on $[-1, 1]$)

Since f is an even function, the Fourier series will just turn out to be a Fourier cosine series; in other words, all $b_N = 0$.

Now for $N \ge 1$, $a_N = \frac{1}{1} \int_{-1}^{1} f(x) \cos\left(\frac{N\pi x}{1}\right) dx$

($L = 1$ here)

$$= \int_{-1}^{1} f(x) \cos(N\pi x)\, dx = \int_{-1}^{0} (x+1) \cos(N\pi x)\, dx + \int_{0}^{1} (1-x) \cos(N\pi x)\, dx$$

$$= \int_{-1}^{0} x \cos(N\pi x)\, dx + \int_{-1}^{0} \cos(N\pi x)\, dx + \int_{0}^{1} \cos(N\pi x)\, dx$$

$$- \int_{0}^{1} x \cos(N\pi x)\, dx = \underbrace{\int_{-1}^{0} x \cos(N\pi x)\, dx}_{\text{I}} + \underbrace{\int_{-1}^{1} \cos(N\pi x)\, dx}_{\text{II}} - \underbrace{\int_{0}^{1} x \cos(N\pi x)\, dx}_{\text{III}}$$

OK, so $\text{I} = \left[\frac{x}{N\pi} \sin(N\pi x) + \frac{1}{N^2\pi^2} \cos(N\pi x) \right]_{-1}^{0}$

$$= \frac{1}{N^2\pi^2} - \frac{1}{N^2\pi^2} \cos(-N\pi)$$

Now, $\cos(-N\pi) = \cos(N\pi) = (-1)^N$

So $I = \dfrac{1-(-1)^N}{N^2\pi^2} = \begin{cases} 0 & \text{if } N \text{ is even} \\ \dfrac{2}{N^2\pi^2} & \text{if } N \text{ is odd} \end{cases}$

Now, $II = \left[\dfrac{1}{N\pi}\sin(N\pi x)\right]_{-1}^{1} = 0$

And $III = \left[-\dfrac{x}{N\pi}\sin(N\pi x) - \dfrac{1}{N^2\pi^2}\cos(N\pi x)\right]_0^1$

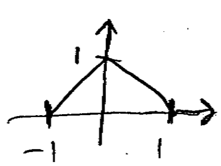
$= -\dfrac{1}{N^2\pi^2}\cos(N\pi) + \dfrac{1}{N^2\pi^2} = \dfrac{-(-1)^N+1}{N^2\pi^2} = \begin{cases} 0 & \text{if } N \text{ even} \\ \dfrac{2}{N^2\pi^2} & \text{if } N \text{ odd} \end{cases}$

So for $N \geq 1$, $a_N = \begin{cases} 0 & \text{if } N \text{ even} \\ \dfrac{4}{N^2\pi^2} & \text{if } N \text{ odd} \end{cases}$

Then $a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} x+1\,dx + \int_0^1 1-x\,dx$

$= \left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \left[x - \tfrac{1}{2}x^2\right]_0^1 = -\left(\tfrac{1}{2}-1\right) + \left(1-\tfrac{1}{2}\right)$

$= 1 \implies a_0 = 1$. (or just note that the graph of f

looks like this: 

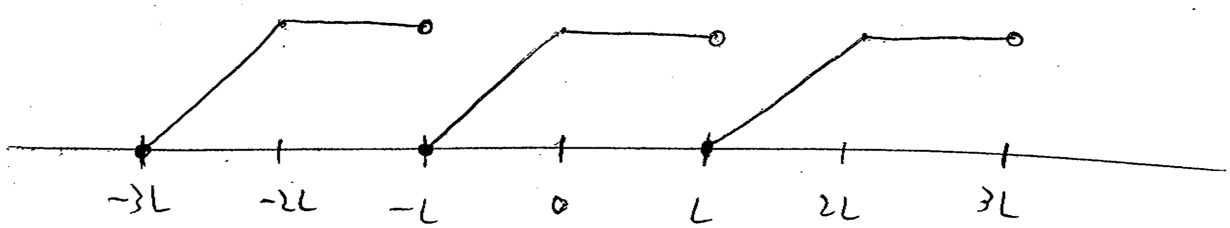
So even a Pitt student could figure

out that $\int_{-1}^{1} f(x)\,dx$ is 1.)

So ... the Fourier cosine series for f is

$$\frac{1}{2} + \sum_{N \text{ odd}} \frac{4}{N^2\pi^2} \cos(N\pi x) = \frac{1}{2} + \sum_{K=0}^{\infty} \frac{4}{(2K+1)^2\pi^2} \cos[(2K+1)\pi x]$$

17 For three periods (of length $2L$) this thing looks like:



So this function is neither even nor odd.

For $N \geq 1$, $a_N = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{N\pi x}{L}\right) dx$

$$= \frac{1}{L}\left(\int_{-L}^{0} (x+L) \cos\left(\frac{N\pi x}{L}\right) dx + \int_{0}^{L} L \cos\left(\frac{N\pi x}{L}\right) dx \right)$$

$$= \frac{1}{L}\left(\underbrace{\int_{-L}^{0} x \cos\left(\frac{N\pi x}{L}\right) dx}_{\mathrm{I}} + \underbrace{\int_{-L}^{L} L \cos\left(\frac{N\pi x}{L}\right) dx}_{\mathrm{II}} \right) \qquad (*)$$

Now $$I = \left[\frac{Lx}{N\pi} \sin\left(\frac{N\pi x}{L}\right) + \frac{L^2}{N^2\pi^2} \cos\left(\frac{N\pi x}{L}\right) \right]_{-L}^{0}$$

$$= \frac{L^2}{N^2\pi^2} - \frac{L^2}{N^2\pi^2}\cos(-N\pi) = \frac{L^2(1-(-1)^N)}{N^2\pi^2} = \begin{cases} 0, & N \text{ even} \\ \frac{2L^2}{N^2\pi^2}, & N \text{ odd} \end{cases}$$

Then $$II = \left[\frac{L^2}{N\pi} \sin\left(\frac{N\pi x}{L}\right) \right]_{-L}^{L} = 0$$

So from (*) we get $$a_N = \begin{cases} 0 & \text{if } N \text{ is even} \\ \frac{2L}{N^2\pi^2} & \text{if } N \text{ is odd} \end{cases}$$

for $N \geq 1$. Then $$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\underbrace{\left(\frac{3}{2}L^2\right)}_{\text{from the graph}}$$

$$\Longrightarrow a_0 = \frac{3}{2}L.$$

Now $$b_N = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{N\pi x}{L}\right) dx$$

$$= \frac{1}{L}\left(\int_{-L}^{0} (x+L)\sin\left(\frac{N\pi x}{L}\right) dx + \int_{0}^{L} L \sin\left(\frac{N\pi x}{L}\right) dx \right)$$

$$= \frac{1}{L}\left(\underbrace{\int_{-L}^{0} x \sin\left(\frac{N\pi x}{L}\right) dx}_{I} + \underbrace{\int_{-L}^{L} L \sin\left(\frac{N\pi x}{L}\right) dx}_{II} \right) \qquad (**)$$

Now $I = \left[-\frac{Lx}{N\pi} \cos\left(\frac{N\pi x}{L}\right) + \frac{L^2}{N^2\pi^2} \sin\left(\frac{N\pi x}{L}\right) \right]_{-L}^{0}$

$= -\left(-\frac{L(-L)}{N\pi} \cos(-N\pi) \right) = -\frac{L^2}{N\pi} \cos(N\pi)$

$= \frac{-L^2(-1)^N}{N\pi}$, or $\frac{L^2(-1)^{N+1}}{N\pi} = \begin{cases} \frac{-L^2}{N\pi} & \text{if } N \text{ is even} \\ \frac{L^2}{N\pi} & \text{if } N \text{ is odd} \end{cases}$

Then $II = \left[-\frac{L^2}{N\pi} \cos\left(\frac{N\pi x}{L}\right) \right]_{-L}^{L} =$

$-\frac{L^2}{N\pi} \cos(N\pi) + \frac{L^2}{N\pi} \underbrace{\cos(-N\pi)}_{\text{same as } \cos(N\pi)} = 0$

So from $(**)$, $b_N = \frac{L(-1)^{N+1}}{N\pi}$

So the Fourier series for f is

$\frac{3L}{4} + \underbrace{\sum_{N \text{ odd}} \frac{2L}{N^2\pi^2} \cos\left(\frac{N\pi x}{L}\right)}$ $+ \sum_{N=1}^{\infty} \frac{L(-1)^{N+1}}{N\pi} \sin\left(\frac{N\pi x}{L}\right)$, and

this can be alternately written as $\sum_{K=1}^{\infty} \frac{2L}{(2K-1)^2\pi^2} \cos\left(\frac{(2K-1)\pi x}{L}\right)$,

and then we can just change the name of this index from K to N, and

Combine the two series to obtain

$$\frac{3L}{4} + \sum_{N=1}^{\infty}\left[\frac{2L}{(2N-1)^2\pi^2}\cos\left(\frac{(2N-1)\pi x}{L}\right) + \frac{L(-1)^{N+1}}{N\pi}\sin\left(\frac{N\pi x}{L}\right)\right]$$

Section 10.5

(1) If $u(x,t) = X(x)\,T(t)$, then $u_{xx} = X''T$, and $u_t = XT'$,

so $x u_{xx} + u_t = 0 \Rightarrow x X''T + XT' \equiv 0$, and indeed

we can separate variables and get $\dfrac{xX''}{X} = -\dfrac{T'}{T}$. So each

side of this equation is a constant

function, i.e., $\dfrac{xX''}{X} \equiv \sigma$ and also

$-\dfrac{T'}{T} \equiv \sigma$ for some number σ.

So this leads to the pair $\left\{\begin{array}{c} xX'' - \sigma X = 0 \\ T' + \sigma T = 0 \end{array}\right\}$ of ODEs.

(3) If $u(x,t) = X(x)\,T(t)$, then $u_{xx} = X''T$, $u_{xt} = X'T'$,

and $u_t = XT'$. So $u_{xx} + u_{xt} + u_t = X''T + X'T' + XT'$

$= X''T + T'(X' - X) \equiv 0 \Rightarrow X''T = T'(X - X') \Rightarrow$

$\dfrac{X''}{X - X'} = \dfrac{T'}{T}$ for all x and t. So $\dfrac{X''}{X - X'}$ is a constant function ; let's say $\dfrac{X''}{X-X'} \equiv \beta$. Then $\dfrac{T'}{T} \equiv \beta$ as well.

So this gives the following pair of ODEs :

$$X'' + \beta X' - \beta X = 0$$
$$T' - \beta T = 0$$

(22) Assuming $u(x,y,t) = X(x)\,Y(y)\,T(t)$, we have $u_{xx} = X''YT$ and $u_{yy} = XY''T$ and $u_t = XYT'$.

So $\alpha^2(u_{xx} + u_{yy}) = u_t$

$$\Rightarrow \alpha^2 X''YT + \alpha^2 XY''T = XYT'$$

Now ... obviously an equation does not have three sides, so how do we separate variables? Well, first separate one from the other two:

$$\frac{\alpha^2 X''Y + \alpha^2 XY''}{XY} = \frac{T'}{T}$$

This equation must hold for all triplets (x,y,t). So if you fix x and y, you just get some number on the left ... call it δ. Now vary t, and you find $\dfrac{T'(t)}{T(t)} = \delta$ for all t. So that means $\dfrac{T'}{T}$ is a constant function, yea, $\dfrac{T'}{T} \equiv \delta$.

So therefore $$\frac{\alpha^2 X''(x)\, Y(y) + \alpha^2 X(x)\, Y''(y)}{X(x)\, Y(y)} = \delta$$ for all x and y.

Now $$\frac{\alpha^2 X'' Y + \alpha^2 X Y''}{XY} = \frac{\alpha^2 X''}{X} + \frac{\alpha^2 Y''}{Y} = \delta$$

$$\Longrightarrow \quad \frac{\alpha^2 X''}{X} = \delta - \frac{\alpha^2 Y''}{Y},$$ and each side of this equation must be a constant function, with common value γ, say.

So $$\frac{\alpha^2 X''}{X} \equiv \gamma \quad \text{and} \quad \delta - \frac{\alpha^2 Y''}{Y} \equiv \gamma$$

So we get these three equations:

$$\alpha^2 X'' - \gamma X = 0$$

$$\alpha^2 Y'' + (\gamma - \delta) Y = 0$$

$$T' - \delta T = 0$$