21-260 Spring 2008

Homework #10 Solutions

Section 7.6

(3) Here $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$, so to determine the eigenvalues, we

consider $\det(A - rI) = \det\begin{bmatrix} 2-r & -5 \\ 1 & -2-r \end{bmatrix} = (2-r)(-2-r) + 5$

$= (r-2)(r+2) + 5 = r^2 - 4 + 5 = r^2 + 1 = 0$ for $r = \pm i$.

So if we write the eigenvalues as $\alpha \pm i\beta$, then in this case we have $\alpha = 0$ and $\beta = 1$

To find an eigenvector ξ corresponding to eigenvalue i, we consider $A - iI = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \sim \begin{bmatrix} 2-i & -5 \\ 2-i & (-2-i)(2-i) \end{bmatrix}$

multiply row 2 by $2-i$

And we note $(-2-i)(2-i) = (i+2)(i-2) = i^2 - 4 = -1 - 4 = -5$.

So $A - iI$ is row-equivalent to $\begin{bmatrix} 2-i & -5 \\ 2-i & -5 \end{bmatrix}$ and the algebraic system $(A - iI)\xi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore just reduces to the single equation $(2-i)\xi_1 - 5\xi_2 = 0$. The nonzero vector $\xi = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$ satisfies this equation, so this is a suitable eigenvector.

Writing $\xi = u + iv = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and using the formula from lecture for the general solution to $x' = Ax$, we have

$$x(t) = c_1 \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t \right) + c_2 \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t \right)$$

or $$x(t) = c_1 \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 \begin{bmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{bmatrix}$$

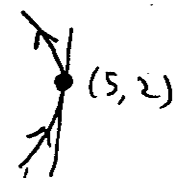
Solutions are periodic, so the trajectories are closed curves in the phase plane, possibly ellipses or something of this sort. Whether they are oriented clockwise or counterclockwise is not entirely clear, but we can check one (nontrivial) trajectory to find out.

(I say this because $c_1 = c_2 = 0$ yields the zero solution, i.e., $x_1 \equiv 0$ and $x_2 \equiv 0$, so the entire trajectory of this solution consists of just the point $(0,0)$.)

Let's set $c_1 = 1$, $c_2 = 0$ and consider $x(t) = \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix}$

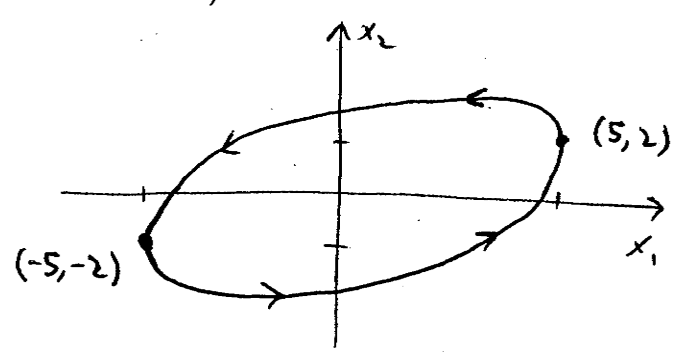
We have $x(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, and then $x'(0) = A x(0) =$

$\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix}$. So this says that x_2 is initially increasing, while x_1 is neither increasing nor decreasing at the instant $t=0$. We deduce that the trajectory has a vertical tangent line at the point $(5, 2)$, and since the origin is enclosed inside the trajectory, something like this must be occurring:

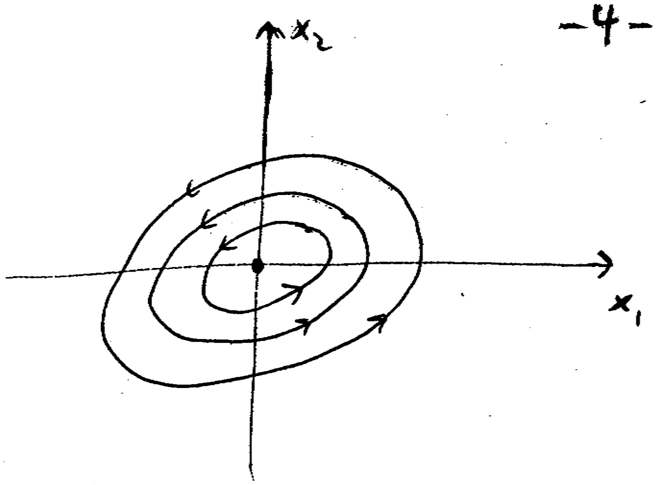


(... although this part of the curve would correspond to $t < 0$)

Since sine and cosine have period 2π, the solution $x(t)$ also has this period, and so the trajectory loops around from $(5, 2)$ at $t=0$, counterclockwise into all four quadrants, coming around to arrive at $(5, 2)$ again at $t = 2\pi$. You can check that at $t = \pi$ we hit the point $(-5, -2)$ with another vertical tangent line. (You might think about where the horizontal tangent lines are.) So the trajectory looks roughly like this:



And for a "full" phase portrait, we can give a picture like this:



(4) $\det(A - rI) = \det\begin{bmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{bmatrix} = (2-r)(-1-r) + \frac{9}{2}$

$= (r-2)(r+1) + \frac{9}{2} = r^2 - r - 2 + \frac{9}{2} = r^2 - r + \frac{5}{2} = 0$ when

$$r = \frac{1 \pm \sqrt{1 - 4\left(\frac{5}{2}\right)}}{2} = \frac{1 \pm \sqrt{-9}}{2} = \frac{1}{2} \pm \frac{3}{2} i$$

So these are the eigenvalues; let's find an eigenvector corresponding to $r = \frac{1}{2} + \frac{3}{2} i$. $A - rI = \begin{bmatrix} 2 - \frac{1}{2} - \frac{3}{2} i & -\frac{5}{2} \\ \frac{9}{5} & -1 - \frac{1}{2} - \frac{3}{2} i \end{bmatrix}$

$$= \begin{bmatrix} \frac{3}{2} - \frac{3}{2} i & -\frac{5}{2} \\ \frac{9}{5} & -\frac{3}{2} - \frac{3}{2} i \end{bmatrix} \sim \begin{bmatrix} 3 - 3i & -5 \\ \frac{18}{5} & -3 - 3i \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 - 3i & -5 \\ \frac{18}{5}(3-3i) & \underbrace{(-3-3i)(3-3i)} \end{bmatrix} \sim \begin{bmatrix} 3 - 3i & -5 \\ 3 - 3i & -5 \end{bmatrix}$$

$(3i+3)(3i-3) = 9i^2 - 9 = -18$

OK, so we need $(3-3i)\xi_1 - 5\xi_2 = 0$, so $\xi = \begin{bmatrix} 5 \\ 3-3i \end{bmatrix}$ will work.

Then $\xi = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ -3 \end{bmatrix} = u + iv$, so the general solution is

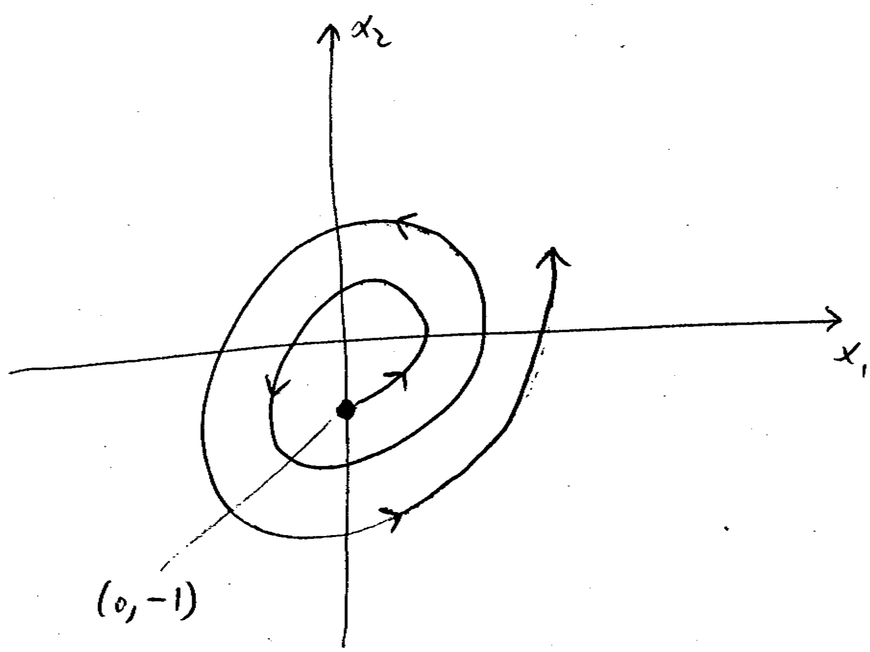
$$x(t) = e^{\frac{1}{2}t}\left[c_1\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\cos\left(\tfrac{3}{2}t\right) - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\sin\left(\tfrac{3}{2}t\right)\right) + c_2\left(\begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos\left(\tfrac{3}{2}t\right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\sin\left(\tfrac{3}{2}t\right)\right)\right]$$

$$= e^{\frac{1}{2}t}\left[c_1\begin{bmatrix} 5\cos\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{bmatrix} + c_2\begin{bmatrix} 5\sin\left(\frac{3}{2}t\right) \\ -3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{bmatrix}\right]$$

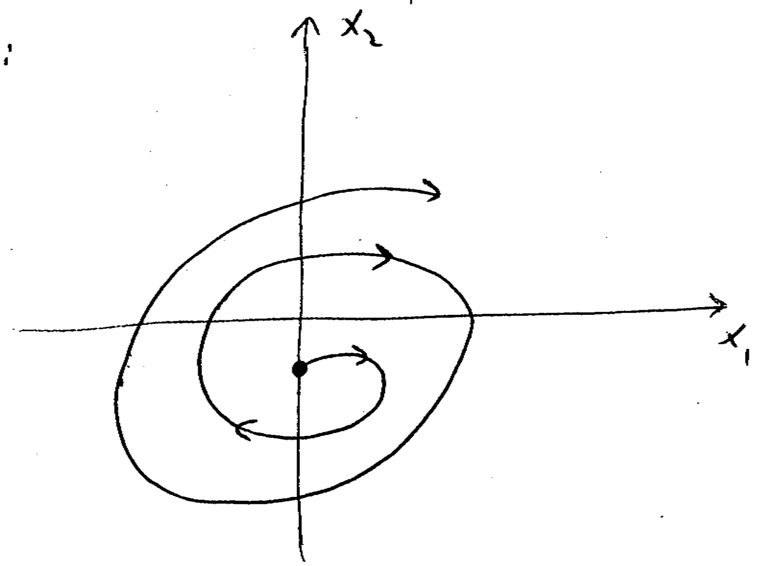
(Note $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$ here.) Since $\alpha > 0$, solutions spiral out away from the origin. Let's check a particular solution to determine the orientation of the spirals. I'll set $c_1 = 0$ and $c_2 = \frac{1}{3}$; this yields the solution $x(t) = e^{\frac{1}{2}t}\begin{bmatrix} \frac{5}{3}\sin\left(\frac{3}{2}t\right) \\ \sin\left(\frac{3}{2}t\right) - \cos\left(\frac{3}{2}t\right) \end{bmatrix}$. So $x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Then $x'(0) = A\,x(0) = \begin{bmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix}$

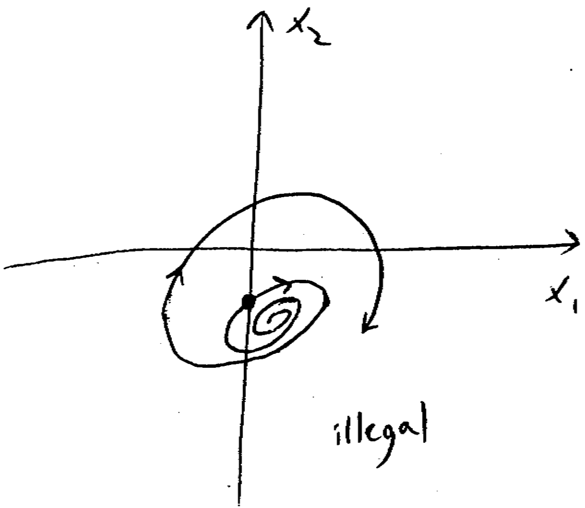
So $x_1', x_2' > 0$ initially, so the trajectory moves up and to the right initially. This means the orientation is counterclockwise:

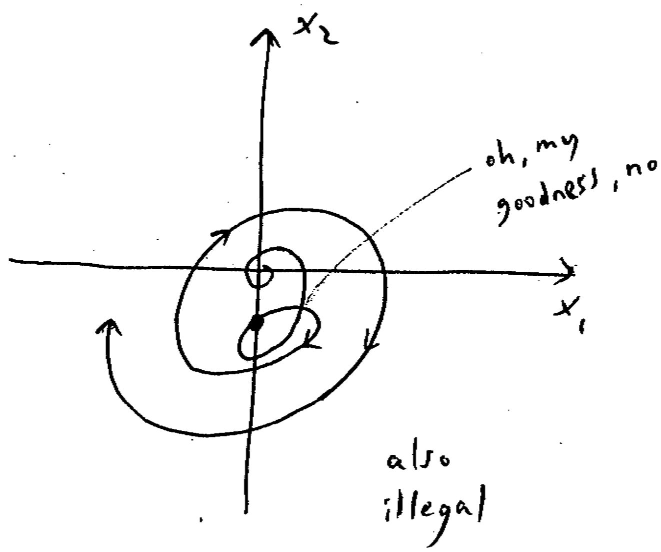


Now you might wonder why we couldn't have a clockwise trajectory like this:



This curve still moves up and to the right initially. But this couldn't be the trajectory for t starting at 0 and then increasing, and the reason is that if you consider $x(t)$ for $t \to -\infty$, you see that $x(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. And if we consider t starting at 0 and then decreasing, thereby generating a "backward trajectory", we see that the curve must move down and to the left initially, from the point $(0, -1)$. There is no way to do this and then spiral toward the origin without crossing the "forward trajectory." See what I mean, Verne? Lookee:





(10) $\det(A - rI) = \det\begin{bmatrix} -3-r & 2 \\ -1 & -1-r \end{bmatrix} = (-3-r)(-1-r) + 2$

$= (r+3)(r+1) + 2 = r^2 + 4r + 5 = 0 \Rightarrow r = \dfrac{-4 \pm \sqrt{16-20}}{2}$

$= -2 \pm i$. Now $A - (-2+i)I = \begin{bmatrix} -3+2-i & 2 \\ -1 & -1+2-i \end{bmatrix}$

$= \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \sim \begin{bmatrix} -(1+i) & 2 \\ -(1+i) & (1-i)(1+i) \end{bmatrix} = \begin{bmatrix} -(1+i) & 2 \\ -(1+i) & 1-i^2 \end{bmatrix}$

$= \begin{bmatrix} -(1+i) & 2 \\ -(1+i) & 2 \end{bmatrix} \Rightarrow -(1+i)\xi_1 + 2\xi_2 = 0$, so $\xi = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$ works.

Writing $\xi = u + iv = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we arrive at the general solution

$$x(t) = e^{-2t} \left[c_1 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin t \right) \right],$$

or

$$x(t) = e^{-2t} \left(c_1 \begin{bmatrix} 2\cos t \\ \cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 2 \sin t \\ \cos t + \sin t \end{bmatrix} \right).$$

Now with $x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, we have

$$x(0) = e^0 \left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2c_1 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Longrightarrow c_1 = \tfrac{1}{2}, \quad c_2 = -\tfrac{5}{2}$$

So the solution is given by

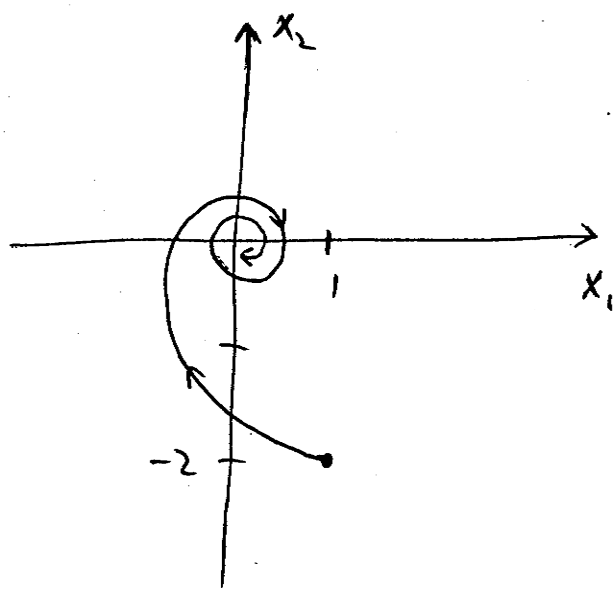
$$x(t) = e^{-2t} \left(\frac{1}{2} \begin{bmatrix} 2\cos t \\ \cos t - \sin t \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 2 \sin t \\ \cos t + \sin t \end{bmatrix} \right)$$

$$= e^{-2t} \begin{bmatrix} \cos t - 5 \sin t \\ -2 \cos t - 3 \sin t \end{bmatrix}$$

The trajectory starts at $(1, -2)$ and spirals toward the origin as $t \to \infty$. Does it spiral clockwise or counterclockwise? Well, let's check $x'(0) = A\,x(0)$

$$= \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}.$$ So $x_1'(0) < 0$ and $x_2'(0) > 0$, and note also that $|x_1'|$ is quite a bit larger than $|x_2'|$. So we initially move up and to the left; the tangent line has slope $-\frac{1}{7}$. I claim that the trajectory is clockwise. (Again, by looking at the "backward trajectory" as well, you can tell that the opposite orientation is impossible.)

So we get:



(11) (a) $\det(A - rI) = \begin{bmatrix} \frac{3}{4} - r & -2 \\ 1 & -\frac{5}{4} - r \end{bmatrix} = (\frac{3}{4} - r)(-\frac{5}{4} - r) + 2$

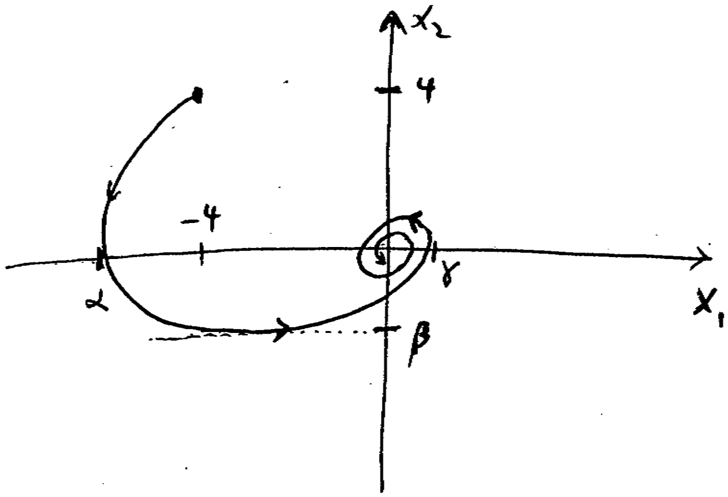
$= (r - \frac{3}{4})(r + \frac{5}{4}) + 2 = r^2 + \frac{1}{2}r - \frac{15}{16} + 2 = r^2 + \frac{1}{2}r + \frac{17}{16}$

$= 0 \Rightarrow r = \dfrac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{17}{4}}}{2} = \dfrac{-\frac{1}{2} \pm \sqrt{-4}}{2} = -\frac{1}{4} \pm i.$

Since the real part (of the eigenvalues) is negative, trajectories spiral inward.

(b) Let's suppose $x(0) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$, and let's consider the trajectory of the corresponding solution. We have $x'(0) = Ax(0)$

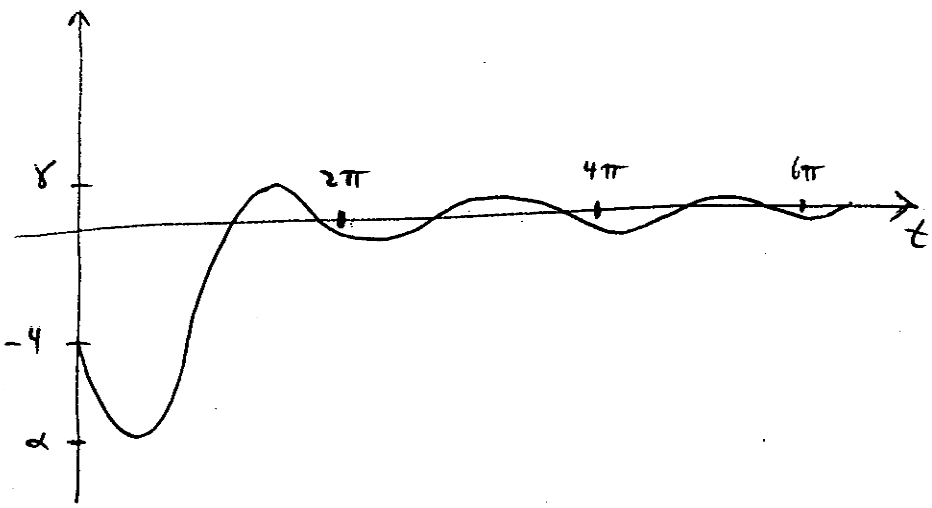
$$= \begin{bmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ -9 \end{bmatrix}.$$ So x_1 and x_2 are both decreasing initially, which means that the trajectory must be oriented counterclockwise and look roughly like so:



(c) So we can see that x_2 has an absolute max. value of 4 and an abs. min. value of ... something, I don't know ... some negative number; call it β. And x_1 has abs. min. value $\alpha < 0$ and abs. max. value $\gamma > 0$.

Now, if it weren't for the factor of $e^{-\frac{1}{4}t}$ in the solution, the solution would be periodic with period 2π (because $\beta = 1$ here, so the solution involves terms with $\cos t$ and $\sin t$). So if the exponential factor weren't there, the trajectory would come around to the point $(-4,4)$ again when $t = 2\pi$ (and 4π, 6π, etc.) So the trajectory hits the line segment between $(0,0)$ and $(-4,4)$ whenever t is an integer multiple of 2π.

So here is a rough sketch of $x_1 = x_1(t)$:



And $x_2 = x_2(t)$ looks roughly like this: