Mathematical Sciences Department
Carnegie Mellon University

Differential Equations
21-260

Spring 2008 Exam 2

No calculator of any kind is permitted. Show all work and give clear explanations.

NAME: Solutions to Version 1 (10:30 lecture)

Question	Points	Score	Pres Pt
1	19+1		
2	19+1		
3	19+1		
4	19+1		
5	19+1		
Total	100		

1. (19+1 Points) Consider the differential equation

$$y'' - \frac{4y'}{t} + \frac{6y}{t^2} = 0, \ t > 0 \quad (*)$$

and the functions

$$u(t) = t^2, \ v(t) = t, \ w(t) = te^t, \ z(t) = t^3.$$

Find a fundamental set of solutions to $(*)$ from among these, prove that it is indeed a fundamental set, and give the general solution to $(*)$.

u is a solution: $u' = 2t$ and $u'' \equiv 2$, so

$$u'' - \frac{4u'}{t} + \frac{6u}{t^2} = 2 - \frac{8t}{t} + \frac{6t^2}{t^2} = 2 - 8 + 6 \equiv 0$$

z is also a solution, since $z' = 3t^2$ and $z'' = 6t$, so

$$z'' - \frac{4z'}{t} + \frac{6z}{t^2} = 6t - \frac{12t^2}{t} + \frac{6t^3}{t^2} = 6t - 12t + 6t \equiv 0.$$

Now we check the Wronskian of u & z: $uz' - u'z$

$= t^2(3t^2) - (2t)(t^3) = 3t^4 - 2t^4 = t^4$, which is not the zero function. So the pair u, z constitutes a fundamental set of solutions, and the gen. sol'n to $(*)$ is

$$At^2 + Bt^3$$

2. (19+1 Points) Solve the initial value problem using Laplace transform methods.

$$\begin{aligned} y'' + 9y &= 9t - 9 \\ y(0) &= -1 \\ y'(0) &= 7 \end{aligned}$$

$$\mathcal{L}\{y''\} + 9\,\mathcal{L}\{y\} = 9\,\mathcal{L}\{t\} - 9\,\mathcal{L}\{1\}$$

$$\Rightarrow \quad s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 9\,\mathcal{L}\{y\} = \frac{9}{s^2} - \frac{9}{s}$$

$$\Rightarrow \quad (s^2+9)\,\mathcal{L}\{y\} + s - 7 = \frac{9-9s}{s^2}$$

$$\Rightarrow \quad (s^2+9)\,\mathcal{L}\{y\} = \frac{9-9s}{s^2} - s + 7 = \frac{9 - 9s - s^3 + 7s^2}{s^2}$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{-s^3 + 7s^2 - 9s + 9}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow \quad -s^3 + 7s^2 - 9s + 9 = As(s^2+9) + B(s^2+9) + s^2(Cs+D)$$

$$= (A+C)s^3 + (B+D)s^2 + 9As + 9B$$

$$\Rightarrow \quad \left.\begin{aligned} A + C &= -1 \\ B + D &= 7 \\ 9A &= -9 \\ 9B &= 9 \end{aligned}\right] \Rightarrow \quad A = -1,\ B = 1,\ C = 0,\ D = 6$$

So $\mathcal{L}\{y\} = -\frac{1}{s} + \frac{1}{s^2} + \frac{6}{s^2+9} = \mathcal{L}\{-1 + t + 6\sin(3t)\}$

$$\Rightarrow \quad y(t) = -1 + t + 6\sin(3t)$$

3. (19+1 Points) Give the general solution to the differential equation

$$y'' - 2y' + 2y = 12\sin(3t) - 14\cos(3t)$$

① Solve $y'' - 2y' + 2y = 0$, with characteristic eq.

$r^2 - 2r + 2 = 0 \Rightarrow$ roots $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm \frac{1}{2}\sqrt{-4}$

$= 1 \pm i$

Gen. sol'n is $e^t(A\cos t + B\sin t)$

② Look for a particular sol'n to (*) of the form

$y_p = a\cos(3t) + b\sin(3t) \Rightarrow y_p' = -3a\sin(3t) + 3b\cos(3t)$

$\Rightarrow y_p'' = -9a\cos(3t) - 9b\sin(3t)$

Then $y_p'' - 2y_p' + 2y_p = (-9a - 6b + 2a)\cos(3t) + (-9b + 6a + 2b)\sin(3t)$

$= (-7a - 6b)\cos(3t) + (6a - 7b)\sin(3t)$. To match (*) we need.

$$\left.\begin{aligned} -7a - 6b &= -14 \\ 6a - 7b &= 12 \end{aligned}\right] \Rightarrow a = 2,\ b = 0$$

So $y_p = 2\cos(3t)$

③ The gen. sol'n to (*) is

$$y(t) = 2\cos(3t) + e^t(A\cos t + B\sin t)$$

4. (19+1 Points) The displacement of a load on a spring is modeled by the equation

$$2u'' + \gamma u' + 3y = 0.$$

Suppose the load is pulled down 4 cm from its equilibrium position and then set in motion with an upward strike imparting an initial speed of 6 cm per second.

(a) Choose a particular numerical value for γ which will guarantee that after the load is set in motion, it will pass through the equilibrium position no more than once. Justify and explain your choice.

(b) Keep your choice of γ from part (a). Is it possible that by replacing the load with one having a different mass, the same initial conditions as above result in the load passing through the equilibrium position at least five times before its motion is no longer detectable? Explain.

(a) "Critical damping" occurs when the roots of the characteristic eq. are the same, i.e., when $\gamma^2 - 4mk = 0$, or $\gamma^2 - 4(2)(3) = 0$ $\Rightarrow \gamma^2 = 24 \Rightarrow \gamma = \sqrt{24}$. If γ is any smaller, then oscillation occurs, but if $\gamma \geq \sqrt{24}$, there is too much fluid resistance to allow for oscillation. So $\gamma = 5$ will work. Then the displacement function will be of the form $\underbrace{u(t) = Ae^{r_1 t} + Be^{r_2 t}}$, with r_1, r_2 both negative, and such a function cannot satisfy $u(t) = 0$ for more than one $t \geq 0$.

(b) Yes. The key would be to change m so that the displacement function ends up having the form $e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$ instead of belonging to this family. u will belong to this family instead if the characteristic eq. has complex roots, which occurs when $\gamma^2 - 4mk < 0$. So with $\gamma = 5$, I need $25 - 4(m)(3) < 0 \Rightarrow m > \frac{25}{12}$. Then any initial conditions except $u(0) = u'(0) = 0$ will produce the scenario described.

5. (19+1 Points) Give the general solution to the differential equation

$$y'' + y' - 6y = e^{2t} \qquad (*)$$

① Solve $y'' + y' - 6y = 0$, with characteristic equation $r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \Rightarrow$ roots $-3, 2$

So the gen. sol'n to the homogeneous eq. is $Ae^{-3t} + Be^{2t}$

② Try to find a particular solution y_p to the given equation.

Our first thought is to try y_p of the form ae^{2t}. But this won't work, as every function of that form is a sol'n to the homog. eq., so it can't be a sol'n to $(*)$. So instead try

$$y_p = ate^{2t} \Rightarrow y_p' = a\left[2te^{2t} + e^{2t}\right] = ae^{2t}(2t+1).$$

Then $y_p'' = a\left[2e^{2t}(2t+1) + 2e^{2t}\right] = ae^{2t}(4t+4)$

So $y_p'' + y_p' - 6y_p = ae^{2t}(4t+4+2t+1-6t) = 5ae^{2t}$,

and this should yield e^{2t}, so we need $5a = 1 \Rightarrow a = \frac{1}{5}$,

and $y_p = \frac{t}{5}e^{2t}$

③ Combine the results above and find the gen. sol'n is

$$y(t) = \frac{t}{5}e^{2t} + Ae^{-3t} + Be^{2t}$$