Mathematical Sciences Department
Carnegie Mellon University

Differential Equations
21-260

Fall 2007 Final Exam

No calculator of any kind is permitted. Show all work and give clear explanations.

NAME:

Question	Points	Score	Pres Pt
1	18+1		
2	18+1		
3	19+1		
4	21+1		
5	17+1		
6	22+1		
7	18+1		
8	19+1		
9	19+1		
10	19+1		
Total	200		

1. (18+1 Points) Solve the initial value problem

$$\begin{aligned} y'' - 3y' - 4y &= 0 \\ y(0) &= 3 \\ y'(0) &= 17 \end{aligned}$$

2. (18+1 Points) Find the general solution to each differential equation.

   (a) $y' + ty + \dfrac{y}{t} - 1 = 0$

   (b) $y' - ty^2 - y^2 - t - 1 = 0$

3. (19+1 Points) Provide a clear solution portrait for the differential equation

$$y' = (y^3 + 2y)(y^2 - 10y + 25)$$

Identify any and all equilibrium solutions and classify each as stable, unstable, or semistable.

4. (21+1 Points) Find the general solution to the differential equation

$$y'' + 4y' + 3y = 15e^{2t} + 6\sin t$$

5. (17+1 Points) The motion of a load on a spring is governed by the equation

$$2u'' + \gamma u' + 9u = 0,$$

where $u = u(t)$ is the displacement of the load at time t, measured in seconds. A number of tests are performed in which the load is displaced from its equilibrium position and and released. In each experiment, the load is observed to pass through its equilibrium position every $\pi/2$ seconds. Is it possible to determine the exact value of γ from this information? If so, find it. If not, what *can* you say about γ?

6. (22+1 Points) Solve the initial value problem

$$\begin{aligned} y'' + y' &= f(t) \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

where

$$f(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < \pi/2 \\ -\cos t & \text{for} \quad t \ge \pi/2 \end{cases}$$

7. (18+1 Points) Find any and all solutions to the two point boundary value problem

$$\begin{aligned} y'' - 3y' + 2y &= 0 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

8. (19+1 Points) Consider the system

$$\begin{aligned} x_1{}' &= -4x_1 + 4x_2 \\ x_2{}' &= 3x_1 \end{aligned}$$

Give the general solution to the system. Provide a solution portrait. Then, in a separate picture, sketch the trajectory which begins at the point $(6, -1)$.

9. (19+1 Points) Let f be given by

$$f(t) = \begin{cases} 1 & \text{for} \quad -2 \leq t < 1 \\ 0 & \text{for} \quad 1 \leq t \leq 2 \end{cases}$$

Find a Fourier series representation for f.

10. (19+1 Points) Find the solution to the following heat conduction problem.

$$\begin{array}{rcll} u_{xx} & = & u_t & \text{for } 0 < x < 3, t > 0 \\ u(0,t) & = & 1 & \text{for all } t > 0 \\ u(3,t) & = & 4 & \text{for all } t > 0 \\ u(x,0) & = & x & \end{array}$$