Mathematical Sciences Department
CARNEGIE MELLON UNIVERSITY

Differential Equations
21-260

Fall 2007 Exam 3

No calculator of any kind is permitted. Show all work and give clear explanations.

NAME: Solutions

Question	Points	Score	Pres Pt
1	25+1		
2	21+1		
3	25+1		
4	25+1		
Total	100		

1. [redacted] Let f be the function which is identically equal to 1 over the interval $[0, 3]$. Find a Fourier sine series representation for f, good at least over $(0, 3)$.

To get a F.S.S., we should extend f as an <u>odd</u> function to $[-3,3]$.

Let's call the extension $\bar{F}$. Then $\bar{F}(x) = \begin{cases} -1 & \text{for } -3 \le x < 0 \\ 0 & \text{at } x = 0 \\ 1 & \text{for } 0 < x \le 3 \end{cases}$

Now the Fourier coefficients are $b_n = \frac{1}{3}\int_{-3}^{3} \bar{F}(x) \sin\left(\frac{n\pi x}{3}\right) dx$

$$= \frac{1}{3}\left[\int_{-3}^{0} (-1)\sin\left(\frac{n\pi x}{3}\right)dx + \int_{0}^{3} \sin\left(\frac{n\pi x}{3}\right) dx\right]$$

$$= \frac{1}{3}\left(\left[\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_{-3}^{0} + \left[-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_{0}^{3}\right)$$

$$= \frac{1}{3}\left(\frac{3}{n\pi} - \frac{3}{n\pi}\underbrace{\cos(-n\pi)}_{\text{same as } \cos(n\pi)} - \frac{3}{n\pi}\cos(n\pi) + \frac{3}{n\pi}\right)$$

$$= \frac{1}{3}\left(\frac{6}{n\pi} - \frac{6}{n\pi}\cos(n\pi)\right) = \frac{2}{n\pi} - \frac{2}{n\pi}\cos(n\pi) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

So $f(x) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{3}\right)$ for $0 < x \le 3$.

$$\left(\text{or, } f(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{3}\right)\right)$$

2. ([redacted]) Suppose we seek solutions $u = u(x,t)$ to the equation $u_x - t^2 u_{tt} = 0$.

(a) Use the separation of variables technique to reduce the PDE to a pair of ODEs.

(b) Show that if the boundary condition

$$u(0,t) = 0 \text{ for all } t \geq 0$$

is imposed, then there are no nontrivial solutions.

(a) Suppose $u(x,t) = X(x)T(t)$. Then $u_x = X'T$ and

$u_{tt} = XT''$, so $u_x - t^2 u_{tt} = X'T - t^2 X T'' = 0$

$$\Longrightarrow \frac{X'}{X} = \frac{t^2 T''}{T} \text{ for all } (x,t).$$

Hence these are each constant functions. Call the constant $-\sigma$, and

we have $\dfrac{X'}{X} \equiv -\sigma$ and $\dfrac{t^2 T''}{T} \equiv -\sigma$

$$\Longrightarrow X' + \sigma X = 0 \quad (*)$$

$$t^2 T'' + \sigma T = 0$$

(b) $u(0,t) = X(0)T(t)$, and if this equals 0 for all $t \geq 0$,

then either the number $X(0) = 0$, or the function $T(t)$ is identically 0.

the latter would mean $u \equiv 0$. So to get $u \not\equiv 0$, we'd need $X(0) = 0$

But the general solution to $(*)$ is $X(x) = Ae^{\sigma x}$, and $X(0) = 0$

would yield $A = 0 \Rightarrow X \equiv 0 \Rightarrow u \equiv 0$.

So there are no nontrivial solutions u.

3. [redacted] Solve the initial value problem

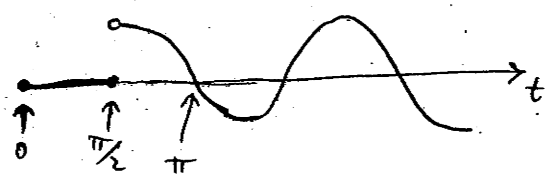
$$\begin{aligned} y'' + y' &= f(t) \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

where

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t < \pi/2 \\ \sin t & \text{for } t \ge \pi/2 \end{cases}$$

Hint: Recall that the graphs of sine and cosine are horizontal translates of one another.

The graph of f looks like this →



Notice that that ~~is~~ what you'd get by shifting the cosine function (for $t \ge 0$) to the right $\pi/2$ units. So $f(t) = u_{\pi/2}(t)\cos\left(t - \frac{\pi}{2}\right)$.

$$s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0) + s\,\mathcal{L}\{y\} - y(0) = \mathcal{L}\{f\} = e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\cos t\}$$

$$\Rightarrow (s^2+s)\,\mathcal{L}\{y\} - s - 1 = e^{-\frac{\pi}{2}} \cdot \frac{s}{s^2+1}$$

$$\Rightarrow s(s+1)\,\mathcal{L}\{y\} = s + 1 + e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1}$$

Divide ~~multiply~~ through by $s(s+1)$:

$$\mathcal{L}\{y\} = \frac{1}{s} + e^{-\frac{\pi}{2}s} \cdot \frac{1}{(s+1)(s^2+1)}$$

Now $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1} \rightarrow 1 = A(s^2+1) + (Bs+C)(s+1)$

$$\Rightarrow 1 = As^2 + Bs^2 + (B+C)s + (A+C)$$

So $A + B = 0$, $A + C = 0$, and $B + C = 1$, whence we get $B = C = \frac{1}{2}$, $A = -\frac{1}{2}$.

So $\mathcal{L}\{y\} = \dfrac{1}{s} + e^{-\frac{\pi s}{2}}\left(\dfrac{-1/2}{s-(-1)} + \dfrac{1}{2}\cdot\dfrac{s}{s^2+1} + \dfrac{1}{2}\cdot\dfrac{1}{s^2+1} \right)$

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So the solution is

$$y(t) = 1 + u_{\frac{\pi}{2}}(t)\left[-\frac{1}{2}e^{-(t-\frac{\pi}{2})} + \frac{1}{2}\cos\left(t-\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(t-\frac{\pi}{2}\right)\right]$$

That form is good enough for an answer, but notice this is $\sin t$,

and $\sin(t-\frac{\pi}{2}) = -\cos t$, so we could write

$$y(t) = 1 + \frac{1}{2}u_{\frac{\pi}{2}}(t)\left[e^{-(t-\frac{\pi}{2})} + \sin t - \cos t\right]$$

[redacted] Consider the system

$$\begin{aligned} x_1' &= -5x_1 + \alpha x_2 \\ x_2' &= -3x_1 + x_2 \end{aligned}$$

Where α is some positive parameter.

(a) Would you characterize this as a cooperative system, a parasitic system, or a competitive system? Clearly explain.

(b) Argue that if $\alpha > 3$, then solution trajectories spiral inward toward the origin.

(c) Let $\alpha = 10/3$. Give the general solution to the system. Sketch the trajectory which begins at the point $(1, 3)$. (An extra page is provided if needed.)

(a) This is parasitic, because the survival of x_1 depends on x_2, since without x_2 present, we just have $x_1' = -5x_1$, and x_1 would die out. But the presence of x_1 has a detrimental impact on the growth of x_2. without x_1 present, we'd have $x_2' = x_2$, and x_2 would grow without bound.

(b) With $A = \begin{bmatrix} -5 & \alpha \\ -3 & 1 \end{bmatrix}$, $\det(A - \lambda I) = (-5-\lambda)(1-\lambda) + 3\alpha$

$= \lambda^2 + 4\lambda - 5 + 3\alpha = 0$ when $\lambda = \dfrac{-4 \pm \sqrt{16 - 4(3\alpha - 5)}}{2}$.

Whether these are real or complex, the real part is -2, which is negative. To get spiral trajectories, we need complex eigenvalues, so we need

$16 - 4(3\alpha - 5) = 16 - 12\alpha + 20 = 36 - 12\alpha < 0 \iff \alpha > 3.$

Trajectories spiral <u>inward</u>, not outward because the real part is negative.

(c) well, $\frac{10}{3} > 3$, so we know we get inward-spiralling trajectories, and we can actually sketch the one starting at $(1,3)$ now!

Consider $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 & \frac{10}{3} \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix}$

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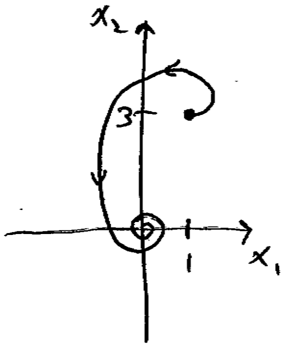
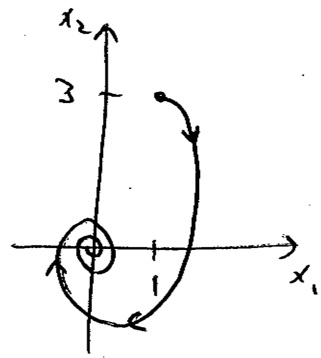
So $x_1'(0) > 0$ and $x_2'(0) = 0$. So there's a horizontal tangent line at $(1,3)$ and the trajectory initially moves to the right. Does it go like this?



No, because then the backward trajectory, for $t < 0$, would have to cross the forward trajectory. So it goes like this:



Now then ... to get the general solution, we find the eigenvalues:

$$\frac{-4 \pm \sqrt{36 - 12\left(\frac{10}{3}\right)}}{2} = \frac{-4 \pm \sqrt{36-40}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

Let's find an eigenvector $\xi = u + iv$ for $-2+i$:

$$A - (-2+i)I = A + (2-i)I = \begin{bmatrix} -3-i & \frac{10}{3} \\ -3 & 3-i \end{bmatrix} \sim \begin{bmatrix} -3(3+i) & 10 \\ -3(3+i) & 10 \end{bmatrix}$$

$$\Rightarrow \quad -3(3+i)\xi_1 + 10\xi_2 = 0 \quad \Rightarrow \quad \xi = \begin{bmatrix} 10 \\ 3(3+i) \end{bmatrix} \text{ works.}$$

Then $\xi = \begin{bmatrix} 10 \\ 9 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, so $u = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, and

$$x(t) = e^{-2t}\left(c_1\left(\begin{bmatrix} 10 \\ 9 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\cos t + \begin{bmatrix} 10 \\ 9 \end{bmatrix}\sin t\right)\right)$$