Mathematical Sciences Department
Carnegie Mellon University

Differential Equations
21-260

Fall 2007 Exam 2

No calculator of any kind is permitted. Show all work and give clear explanations.

NAME: Solutions

Question	Points	Score	Pres Pt
1	21+1		
2	21+1		
3	21+1		
4	16+1		
5	16+1		
Total	100		

1. (21+1 Points) Solve the initial value problem using Laplace transform methods.

$$y'' + 3y' + 2y = 4, \; y(0) = 0, \; y'(0) = 1$$

Apply $\mathcal{L}$ to both sides of this $\uparrow$: $\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{4\}$

$$\Rightarrow \quad s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} = \mathcal{L}\{4\}$$

$$\Rightarrow \quad (s^2+3s+2)\mathcal{L}\{y\} - s\cdot 0 - 1 - 3\cdot 0 = \frac{4}{s}$$

$$\Rightarrow \quad (s^2+3s+2)\mathcal{L}\{y\} = \frac{4}{s} + 1 = \frac{4+s}{s}$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{s+4}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

So $\quad s + 4 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

$$= (A+B+C)s^2 + (3A+2B+C)s + 2A$$

$$\Rightarrow \text{ need } \left.\begin{aligned} A+B+C &= 0 \\ 3A+2B+C &= 1 \end{aligned}\right]$$

$$2A = 4 \Rightarrow \underline{A = 2}, \text{ so these become}$$

$$\left[\begin{aligned} B+C &= -2 \\ 2B+C &= -5 \end{aligned}\right] \Rightarrow C = -2-B \Rightarrow 2B - 2 - B = -5 \Rightarrow \underline{B = -3} \Rightarrow \underline{C = 1}$$

So $\mathcal{L}\{y\} = \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+2}$, or $\frac{2}{s} - \frac{3}{s-(-1)} + \frac{1}{s-(-2)}$

So $y(t) = 2 - 3e^{-t} + e^{-2t}$

2. (21+1 Points) Find the general solution to the differential equation.

$$y'' - 2y' + y = 3\sin 2t - 7\cos 2t$$

Solve $\underline{y'' - 2y' + y = 0}$: roots of $r^2 - 2r + 1 = (r-1)^2 = 0$

$\Rightarrow$ only $r = 1$

So the general solution to this is $e^t(At + B)$.

Next we seek a particular solution y_p to the given equation, of the form $y_p(t) = a\cos 2t + b\sin 2t$

$y_p' = -2a\sin 2t + 2b\cos 2t$

$y_p'' = -4a\cos 2t - 4b\sin 2t$

So $y_p'' - 2y_p' + y_p = -4a\cos 2t - 4b\sin 2t + 4a\sin 2t - 4b\cos 2t$

$+ a\cos 2t + b\sin 2t = (-4b + 4a + b)\sin 2t + (-4a - 4b + a)\cos 2t$

$= (4a - 3b)\sin 2t + (-3a - 4b)\cos 2t = 3\sin 2t - 7\cos 2t$

$\Rightarrow$ $4a - 3b = 3 \longrightarrow a = \frac{3+3b}{4}$, so $-3\left(\frac{3+3b}{4}\right) - 4b = -7$

$-3a - 4b = -7$

$\Rightarrow -\frac{9}{4} - \frac{9}{4}b - 4b = -7$

$\Rightarrow -9 - 9b - 16b = -28$

$\Rightarrow -25b = -19$

$\Rightarrow b = \frac{19}{25}$

So $a = \dfrac{3 + \frac{57}{25}}{4} = \dfrac{75 + 57}{100} = 13.2$

$\Rightarrow y(t) = 13.2\cos 2t + \frac{19}{25}\sin 2t + e^t(At + B)$

is the general solution

3. (21+1 Points) Solve the initial value problem.

$$4y'' + 4y' + 5y = 0, \ y(0) = -2, \ y'(0) = 0$$

The characteristic equation is $4r^2 + 4r + 5 = 0$. The roots are $\frac{-4 \pm \sqrt{16-80}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = -\frac{1}{2} \pm i$

So the general solution is $y(t) = e^{-\frac{t}{2}}(A \cos t + B \sin t)$

Now $y(0) = e^0(A \cdot 1 + B \cdot 0) = \underline{A = -2}$

Then $y(t) = e^{-\frac{t}{2}}(-2\cos t + B \sin t)$, so

$$y' = e^{-\frac{t}{2}}(2\sin t + B\cos t) - \frac{1}{2}e^{-\frac{t}{2}}(-2\cos t + B\sin t)$$

So $y'(0) = e^0(B) - \frac{1}{2}e^0(-2) = B + 1 = 0 \Rightarrow \underline{B = -1}$

So $y(t) = e^{-\frac{t}{2}}(-2\cos t - \sin t)$

4. (16+1 Points) Exactly one of the two families below is the general solution to $4t^2y'' + 8ty' + y = 0$. Which is it? Justify.

(i) $\frac{1}{\sqrt{t}}(A + B\ln t)$

(ii) $At^2 + \frac{B}{\sqrt{t}}$

(ii) can't be right because if $A=1$ and $B=0$, we get $y(t) = t^2$, and this is not even a solution to the diff. eq: Note $y' = 2t$ and $y'' = 2$, so $4t^2y'' + 8ty' + y = 8t^2 + 16t^2 + t^2 = 25t^2$, but we are supposed to get the zero function. So (ii) isn't right, and we could actually stop here and conclude that (i) is the general solution, because the problem statement assures us that exactly one of the choices is right. (This argument indirectly establishes that (i) is correct.)

To establish directly that (i) is correct, let $y_1 = \frac{1}{\sqrt{t}}$ and $y_2 = \frac{\ln t}{\sqrt{t}}$. Then $y_1' = -\frac{1}{2}t^{-3/2}$ and $y_1'' = \frac{3}{4}t^{-5/2}$, so

$4t^2y_1'' + 8ty_1' + y_1 = 3t^{-1/2} - 4t^{-1/2} + t^{-1/2} \equiv 0 \Rightarrow y_1$ is a solution.

Then writing $y_2 = t^{-1/2}\ln t$, $y_2' = t^{-1/2}\cdot\frac{1}{t} - \frac{1}{2}t^{-3/2}\ln t$

$\Rightarrow y_2' = t^{-3/2}(1 - \frac{1}{2}\ln t) \Rightarrow y_2'' = t^{-3/2}(-\frac{1}{2t}) - \frac{3}{2}t^{-5/2}(1 - \frac{1}{2}\ln t)$

$\Rightarrow y_2'' = -\frac{1}{2}t^{-5/2}[1 + 3(1 - \frac{1}{2}\ln t)] = -\frac{1}{2}t^{-5/2}[4 - \frac{3}{2}\ln t]$

So $4t^2y_2'' + 8ty_2' + y_2 = -2t^{-1/2}(4 - \frac{3}{2}\ln t) + 8t^{-1/2}(1 - \frac{1}{2}\ln t) + t^{-1/2}\ln t$

$= t^{-1/2}(-8 + 3\ln t + 8 - 4\ln t + \ln t) \equiv 0$. So y_2 is also a solution.

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Finally, we need y_1 & y_2 to be a fundamental pair of solutions, so we check the Wronskian $y_1 y_2' - y_1' y_2$

$$= t^{-1/2} \cdot t^{-3/2}\left(1 - \tfrac{1}{2}\ln t\right) + \tfrac{1}{2} t^{-3/2} \cdot t^{-1/2} \ln t$$

$$= t^{-2}\left(1 - \tfrac{1}{2}\ln t + \tfrac{1}{2}\ln t\right) = \frac{1}{t^2},$$ which is not the zero function. So we do have a fundamental pair, and (i) is $Ay_1 + By_2$, so (i) is the general solution.

5. (16+1 Points) A load with mass of 4 units is attached to a spring with spring constant 3. The load is pulled down one inch from its equilibrium position and then is set in motion with an upward strike that imparts an initial speed of one inch per second. [Over the next ten seconds, the load oscillates, passing through its equilibrium position several times.] Write down an initial value problem consistent with this information. Justify.

$$4u'' + \gamma u' + 3u = 0$$
$$u(0) = 1$$
$$u'(0) = -1$$

Letting $u = u(t)$ = displacement at time t; with $u > 0$ denoting positions below equilibrium, the initial displacement should be 1. Then since $u' < 0$ denotes upward motion, we need $u'(0) = -1$. The damping constant γ should be small enough to be consistent with this fact.

If γ is so large that oscillation is not possible, then the load would pass through the equilibrium position at most one time. So we need the characteristic equation $4r^2 + \gamma r + 3 = 0$ to have complex roots, which occurs when $\gamma^2 - 4(4)(3)$ is negative. So we need $\gamma^2 < 48$. $\gamma = 6$, say, would work, and the IVP

$$4u'' + 6u' + 3u = 0$$
$$u(0) = 1$$
$$u'(0) = -1$$

would be consistent with the given information.