Old exam problems and practice problems

1. Do problems from section 8.1, exercises 1-20.

2. Determine whether the sequence $\{(n^3+n)e^{-n^2}\}_{n=1}^{\infty}$ converges. If so, find the limit.

3. Consider the sequence

$$\left\{\frac{2}{5}, -\frac{4}{10}, \frac{6}{17}, -\frac{8}{26}, \frac{10}{37}, -\frac{12}{50}, \dots\right\}.$$

   (a) Give a formula for the general term a_n of the sequence. (Hint: The first few perfect squares are 1, 4, 9, 16, 25, 36.)

   (b) Use the squeeze theorem to show that $\lim_{n\to\infty} a_n = 0$.

4. Determine whether the sequence $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$ converges. If so, find the value of the limit.

5. Determine whether the sequence $\{ne^{-n^2}\}_{n=1}^{\infty}$ converges. If so, find the value of the limit.

6. Determine whether the sequence $\left\{\frac{e^n}{n!}\right\}_{n=1}^{\infty}$ converges. If so, find the value of the limit.

7. Suppose you are a prisoner confined to your cell 23 hours per day, and your only luxury is an unlimited supply of pencils and paper. (No computer or calculator allowed.) So to pass the time, you decide to create a table of the fourth roots of all integers from $k = 1$ to $k = 100$, accurate to ten decimal places.

   (a) Explain how you can use Newton's Method to obtain the iterative formula

$$x_{n+1} = \frac{1}{4}\left(3x_n + \frac{k}{x_n^3}\right),$$

   involving operations you can do by hand, enabling you to find $\sqrt[4]{k}$.

   (b) Suppose you keep track of how long you've been imprisoned as follows: Each time you begin to compute $\sqrt[4]{k}$ for a new value of k, you choose your initial guess x_1 equal to the number of days you've been locked up. Is it possible that this practice will lead to any erroneous results in your table of fourth roots? Explain.

8. Solve the following differential equations. *Give the general solution explicitly whenever possible*; when this is not possible, indicate that you have found the implicit general solution.

(a) $y' = \dfrac{1}{y - \sin y}$

(b) $y' - 2xy - x = 0$

(c) $y'' + 2y' + y = 0$

(d) $xy' + xy + y - e^{-x} = 0, \quad x > 0$

(e) $y' = \dfrac{xy + y}{\ln y}$

(f) $y' = \dfrac{x \sec y}{\sqrt{x^2 + 1}}$

(g) $y' = \dfrac{x - e^x}{y + e^y}$

(h) $y'' + 2y' + 26y = 0$

(i) $y' - y^2 + 2y = 0$

(j) $y' - 2y\left(1 - \dfrac{1}{x}\right) = \dfrac{1}{x}$

(k) $y' - y^2\left(1 - \dfrac{1}{x}\right) = 0$

(l) $y'' - 6y' + 5y = 0$

(m) $y' = \dfrac{x \ \csc y}{(x^2 + 1)^2}$

(n) $y' = \dfrac{1}{3y^7 - \sin y}$

(o) $y' = \dfrac{x^2}{1 - y^2}$

(p) $y'' = 4y$

(q) $y' = \dfrac{y \cos x}{1 + 2y^2}$

9. (a) Verify that $y = -\sqrt{x^2 - 4}$ is a solution to the differential equation $yy' - x = 0$ on the interval $(2, \infty)$.

   (b) Determine the general solution to the equation $yy' - x = 0$.

   (c) How should you choose the constant(s) in your general solution in part (b) to obtain the particular solution $y = -\sqrt{x^2 - 4}, \quad x > 2$?

10. Recall that the simplest population growth model assumes only that the population of a species grows at a rate proportional to its current size. This assumption leads to the differential equation

$$\frac{dP}{dt} = kP,$$

where $P = P(t)$ is the population at time t. We saw in class that the solutions to this equation are exponential functions.

Now suppose that the Earth is facing the imminent invasion of alien beings from the planet Gonnagitcha. The Gonnagitchens have conducted clandestine reconnaisance missions and found that they very much enjoy our vegetation and Charleston Chew bars. Their own scientists have determined that because of our superior nutritional offerings, they will thrive quite well here, with a growth rate proportional to the *square* of the current size of the alien population at any given time. They plan to arrive with 100 colonists.

It is your job to appeal to the Gonnagitchens and convince them not to carry out their plan. The argument that they are hideous with regard to both appearance and personal hygiene probably will not work; rather, you should convince them that colonizing Earth would be as bad for them as it would be for us. *Hint:* Choose an exceedingly conservative proportionality constant, such as 0.001, and consider $\lim_{t \to 10^-} P(t)$. Sketch your solution for $t \geq 0$.

11. Solve each initial value problem.

    (a) $y' - \dfrac{\sin x}{\sin y} = 0, \;\; y(\pi) = 1$

    (b) $y'' + 25y = 0, \;\; y(\pi) = 1, \;\; y'(\pi) = 10$

    (c) $y' = \dfrac{3x^2 + 4x + 2}{2(y-1)}, \;\; y(0) = -1$

12. The differential equation

$$y' = \frac{y - 4x}{x - y}$$

is not separable. However, by defining a new variable $u = y/x$, one can replace the equation above with a differential equation involving u and x. The new equation *is* separable. Use this method to solve the equation above.

13. (a) Evaluate $\displaystyle\int_0^1 \sqrt{1 + 4x^2}\; dx$.

    (b) Identify the integral in part (a) as the length of a curve.

    (c) Argue, using your results in parts (a) and (b), that the following inequality is true:

$$\sqrt{2} < \frac{\sqrt{5}}{2} + \frac{\ln(2 + \sqrt{5})}{4}$$

14. Find the value(s) of c for which the function $f(x) = xe^{cx}$ is a solution to the differential equation

$$f''(x) - 8f'(x) + 16f(x) = 0$$

15. Find the value(s) of c for which the function $f(x) = e^{cx}$ is a solution to the differential equation

$$f''(x) + 5f'(x) + 6f(x) = 0$$

16. Three students are given the formula for a function $f(x)$ but are not given graphical information. The students are told that at least one root of f is guaranteed to lie in the interval $[-2, 4]$. Each student applies Newton's Method. Gertrude chooses initial approximation $x_1 = 0$; Ethyl chooses $x_1 = 2$; and Harlan chooses $x_1 = 4$. Sketch a graph, which will play the role of f here, and would lead to Gertrude and Harlan finding *different* roots of f, and would cause Newton's Method to fail for Ethyl. (You don't need a *formula* for f; you just need a picture.) Explain why your sketch "works".

17. Determine whether each sequence converges. In case of convergence, find the limit.

    (a) $\left\{\sqrt[n]{n}\right\}_{n=1}^{\infty}$

    (b) $\left\{\dfrac{\cos(n^3 - n + 2n - 11)}{n^2}\right\}_{n=1}^{\infty}$

    (c) $\left\{\dfrac{e^n}{n!}\right\}_{n=1}^{\infty}$

    (d) $\left\{\left(\sin\left[\dfrac{(4n+1)\pi}{2}\right]\right)^n\right\}_{n=0}^{\infty}$

18. (a) Show that $\left\{\dfrac{2^n}{(2n)!}\right\}_{n=1}^{\infty}$ is a monotonic sequence.

    (b) Does $\left\{\dfrac{2^n}{(2n)!}\right\}_{n=1}^{\infty}$ converge? Explain.