On a List Coloring Conjecture of Reed

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Abstract

We construct graphs with lists of available colors for each vertex, such that the size of every list exceeds the maximum vertex-color degree, but there exists no proper coloring from the lists. This disproves a conjecture of Reed.

1 Introduction

Let $G = (V, E)$ be a graph, and let $\{L_v\}_{v \in V}$ be a family of sets. We think of L_v as the list of available colors at the vertex v. A *proper coloring of G from the lists* $\{L_v\}_{v \in V}$ is a function $f : V \to \bigcup_{v \in V} L_v$ such that $f(v) \in L_v$ for all $v \in V$, and $f(u) \neq f(v)$ whenever $\{u, v\} \in E$. This is an important variant of the standard graph coloring concept known as list coloring (standard graph coloring is equivalent to list coloring where lists are the same for every vertex).

Clearly, if the size of every L_v exceeds the maximum degree of G then we can greedily produce a proper coloring of G from the lists $\{L_v\}_{v \in V}$. This simple sufficient condition is not sensitive to the amount of overlap between lists of neighboring vertices. For example, if the lists of any two neighbors are disjoint, then a proper coloring exists whenever the lists are non-empty, regardless of the relation between their sizes and the maximum degree.

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In an attempt to establish a sufficient condition that takes into account the overlap of neighboring lists, Reed [4] considered the vertex-color degrees. These are the numbers $d_c(v)$, defined for $v \in V$ and $c \in L_v$ by

$$d_c(v) = |\{u \in V : \{u, v\} \in E \text{ and } c \in L_u\}|.$$

Reed made the following conjecture.

Conjecture (Reed). *Let $G = (V, E)$ be a graph, and let $\{L_v\}_{v \in V}$ be a family of sets. Suppose that for some non-negative integer d, the following two conditions hold:*

$$d_c(v) \leq d \text{ for every } v \in V \text{ and every } c \in L_v,$$
$$|L_v| \geq d + 1 \text{ for every } v \in V.$$

Then there exists a proper coloring of G from the lists $\{L_v\}_{v \in V}$.

In words, Reed conjectured that if the size of every L_v exceeds the maximum vertex-color degree of G then there exists a proper coloring of G from the lists $\{L_v\}_{v \in V}$.

In order to state some partial results concerning this conjecture, we denote by $g(d)$ the smallest integer for which the statement of the conjecture, with $d+1$ replaced by $g(d)$, holds true. It is easy to see that $g(d) \geq d+1$ (e.g., consider the complete graph). It follows from a straightforward application of the Lovász Local Lemma that $g(d) \leq \lceil 2ed \rceil$ for all positive integers d (see Alon [2], Reed [4]). Haxell [3] improved this to $g(d) \leq 2d$ for all positive integers d. For d tending to infinity, Reed and Sudakov [5] proved that $g(d) \leq (1 + o(1))d$.

In Section 2 we present a construction showing that $g(d) \geq d + 2$ for every integer $d \geq 2$. This disproves Reed's conjecture. We conclude this note in Section 3 with a few comments.

2 The construction

We will construct below, for every integer $d \geq 2$, a vertex-set V and a family of graphs $\{G_\alpha = (V_\alpha, E_\alpha)\}_{\alpha \in I}$, where I is a suitable index-set, such that $V_\alpha \subseteq V$ for all $\alpha \in I$, and the following properties are satisfied:

1. For every $\alpha \in I$, the maximum degree of G_α is d.

2. Every $v \in V$ belongs to $d + 1$ of the vertex-sets V_α.

3. For every $\alpha, \beta \in I$ and every $u, v \in V_\alpha \cap V_\beta$, we have $\{u, v\} \in E_\alpha$ if and only if $\{u, v\} \in E_\beta$.

4. There exists no choice of subsets $S_\alpha \subseteq V_\alpha$ for $\alpha \in I$, with S_α independent in G_α, so that $\bigcup_{\alpha \in I} S_\alpha = V$.

Assuming we have such a construction, let $G = (V, E)$ be the graph with edge-set $E = \bigcup_{\alpha \in I} E_\alpha$, and for every $v \in V$ let $L_v = \{\alpha \in I : v \in V_\alpha\}$ (i.e., each graph in the family corresponds to a color). It is straightforward to check that this will be a counterexample to the conjecture.

We proceed now to describe the construction. Let $d \geq 2$ be given. We take the vertex-set V to be

$$V = \bigcup_{i=1}^{2d+2} W_i,$$

where the W_i are pairwise disjoint sets, each of which is further decomposed as

$$W_i = X_i \cup Y_i,$$

with $X_i \cap Y_i = \emptyset$, and

$$|X_i| = d, \quad |Y_i| = d^2 - d + 1.$$

The index-set I is of size $(2d+2)d^2+d+3$, and the family $\{G_\alpha = (V_\alpha, E_\alpha)\}_{\alpha \in I}$ consists of three types of graphs:

- Type A: For every i, $1 \leq i \leq 2d+2$, we have d^2 graphs G_α whose vertex-sets are contained in W_i. Each of these graphs is a copy of the complete graph K_{d+1} having one vertex in X_i and d vertices in Y_i. The vertices of these d^2 graphs are chosen so that every vertex in X_i is covered d times, all vertices in Y_i but one are covered $d+1$ times, and the remaining vertex in Y_i, say y_i, is covered d times. This is possible because $d^3 = (d^2 - d)(d+1) + d$.

- Type B: We have two graphs G_α, each a copy of K_{d+1}, whose vertex-sets together cover the special vertices y_i, $1 \leq i \leq 2d+2$.

- Type C: For every j, $1 \leq j \leq d+1$, we have a graph G_α which is a copy of the complete bipartite graph $K_{d,d}$ with the sets X_{2j-1}, X_{2j} as the two parts of its vertex-set.

It is easy to check that properties 1–3 above are satisfied. To verify property 4, suppose that the subsets $S_\alpha \subseteq V_\alpha$, for $\alpha \in I$, are independent in the respective graphs G_α. The union of those S_α corresponding to G_α of type A leaves at least one vertex in each W_i, $1 \leq i \leq 2d+2$, uncovered, because for each i we have d^2 complete graphs and $|W_i| = d^2 + 1$. Taking into account also the two S_α corresponding to G_α of type B, there remain uncovered vertices in at least $2d$ of the sets W_i. Each of the $d+1$ sets S_α corresponding to G_α of type C is contained in one of the sets W_i, and therefore the union of all S_α still leaves uncovered vertices in at least $d-1$ of the sets W_i. As $d \geq 2$, this means that $\bigcup_{\alpha \in I} S_\alpha \neq V$. Thus our construction has all the required properties.

3 Comments

1. Reed [4] also proposed a weaker version of his conjecture, in which he considered the quantities $d_c(v)$ also for $c \notin L_v$, requiring that $d_c(v) \leq d+1$ for every $v \in V$ and every $c \notin L_v$ (in addition to the other two conditions). It may be checked that our construction disproves this weaker version as well.

2. It remains an interesting problem to evaluate or better estimate the function $g(d)$, now known to satisfy $d+2 \leq g(d) \leq 2d$ for $d \geq 2$. The first open instance of this problem asks whether there must exist a proper list coloring when $d = 3$ (that is, the vertex-color degrees are at most 3) and the lists have size 5. For d tending to infinity, it would be interesting to know whether $g(d) - d$, now known to be $o(d)$, actually tends to infinity.

3. Another direction for further research is to look for special classes of graphs for which Reed's conjecture does hold true. This is known to be the case for chordal graphs (see Aharoni et al. [1]).

References

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