On the Irregularity Strength of Trees

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Abstract

For any graph G, let n_i be the number of vertices of degree i, and $\lambda(G) = \max_{i \leq j}\{\frac{n_i + \cdots + n_j + i - 1}{j}\}$. This is a general lower bound on the irregularity strength of graph G. All known facts suggest that for connected graphs, this is the actual irregularity strength up to an additive constant. In fact, this was conjectured to be the truth for regular graphs and for trees. Here we find an infinite sequence of trees with $\lambda(T) = n_1$ but strength converging to $\frac{11-\sqrt{5}}{8}n_1$.

1 Introduction

Let G be a graph with no components of size 1 or 2. Given a function $w : E(G) \to \{1, 2, \ldots, s\}$, the weight of a vertex v is $\omega(v) = \sum_{e \ni v} w(e)$. If the weight of each vertex is different we call this a function a *valid assignment.* The smallest possible s for which we can achieve a valid assignment is called the *irregularity strength* of graph G, which we denote $s(G)$. The notion if irregularity strength was first introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba in [2], where they noted that it constructs a multi-graph in which every edge is a copy of an edge from G and every vertex has a different degree.

For many graphs, the irregularity strength is known. In almost all of these cases, it is found to be within an additive constant of the following lower bound $\lambda(G)$, which was proven in [2]. Define

$$\lambda(G) \;=\; \max_{i \leq j} \frac{(n_i + \cdots + n_j) + i - 1}{j} \tag{1}$$

where n_i denotes the number of vertices of degree i. It was conjectured (see [11]) that $\lambda(G)$ is within an additive constant of the irregularity strength for connected graphs. This has been verified for particular families, among them are paths, complete graphs [2], cycles, most complete bipartite graphs, Turan graphs [6], wheels, k-cubes, and grids [4].

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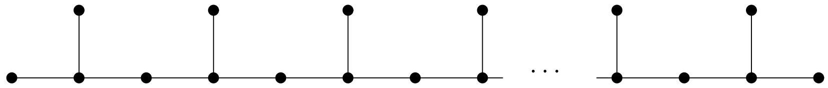
For r-regular graphs with n vertices, it is conjectured that $s(G) \leq \frac{n}{r} + c$ for some absolute constant c. This has been verified for $r \geq \frac{n}{2}$ [6] and for $r \geq \frac{n}{3}$ when the graph is 2-connected [7]. In [5], Frieze, Gould, Karoński, and Pfender show that $s(G) \leq c_1 \frac{n}{r}$ when $r \leq \sqrt{n}$, and $s(G) \leq c_2 \frac{n \log n}{r}$ when $r \geq \sqrt{n}$.

Until recently, the only case known with $s(G) - \lambda(G)$ larger than a constant was presented in [9]. There they proved that the strength of tP_3, the disconnected graph consisting of t copies of the path with 3 vertices has strength either $\frac{15}{14} n_1$ or $\frac{15}{14} n_1 + 1$.

It is easy to see that for a tree T with $n_1 \geq n_2$, we have $\lambda(T) = n_1$. The following conjecture appeared in [4]:

Conjecture 1 *The strength of any tree T is either $\lambda(T)$ or $\lambda(T) + 1$.*

This was verified for all full 2-ary and 3-ary trees [3], and trees with $n_2 = 0$ [1]. This conjecture was refuted in Kravitz's undergraduate thesis [10] where it is shown that for any constant μ, there exists trees T with $n_1 \geq n_2$ such that $s(T) > \lambda(T) + \mu$. Here we prove a stronger result for a very similar class of trees with $n_1 \geq n_2$.



Let us call this tree T_n, where n is the number of vertices. It has $n_1 = \frac{n+5}{3}$ and $n_1 \geq n_2$. Let V be the set of vertices, and for $i = 1, 2, 3$ we will use V_i to denote the set of vertices of degree i.

Here is the main result that we will prove in this paper:

Theorem 1 $\lim_{n \to \infty} \frac{s(T_n)}{n_1} = \frac{11 - \sqrt{5}}{8}$.

In Section 2, we prove the lower bound, and in Section 3 we prove the upper bound by constructing an assignment. Before proving this, we present a short proof that Conjecture 1 is false.

Claim 1 *For all n, we have $s(T_n) \geq \frac{13}{12} n_1 - 3$.*

Proof: Suppose we have a valid assignment w of strength $(1 + c)n_1$. For $i = 1, 2, 3$, define S_i as follows:

$$S_i \ := \sum_{v_i \in V_i} \omega(v_i).$$

Since there are $3n_1 - 5$ vertices in the graph, we have

$$S_1 + S_2 + S_3 \ = \ \sum_{v \in V} \omega(v) \ \geq \ \sum_{i=1}^{3n_1 - 5} i \ > \ \frac{9}{2} n_1^2 - 14 n_1$$

Next, note that because of the structure of the tree, $S_1 + S_2 = S_3$, because each of these sums includes the weight of each edge exactly once. Thus,

$$S_1 + S_2 \ > \ \frac{9}{4} n_1^2 - 7 n_1$$

Now, the following are true because the weight of a vertex of degree 1 is at most $(1+c)n_1$, degree 2 totals are similarly at most $2(1+c)n_1$, and there are at most n_1 of either, and they are all different.

$$S_1 \leq \sum_{i=cn_1+1}^{(1+c)n_1} i \quad \text{and} \quad S_2 \leq \sum_{i=(1+2c)n_1+1}^{(2+2c)n_1} i \quad \Rightarrow \quad S_1 + S_2 \leq [(2+3c)n_1 + 1]n_1$$

Putting these together, we get

$$\tfrac{9}{4}n_1^2 - 7n_1 \ < \ S_1 + S_2 \ \leq \ (2+3c)n_1^2 + n_1 \quad \Rightarrow \quad (1+c)n_1 > \tfrac{13}{12}n_1 - \tfrac{8}{3}$$

□

2 Lower Bound

Fix some $c < \frac{1}{8}(3-\sqrt{5})$, and suppose for the sake of contradiction that we have a valid assignment of strength $(1+c)n_1$. Note that all degree one vertices have weights at most $(1+c)n_1$, degree two vertices at most $(2+2c)n_1$, and degree three vertices at most $(3+3c)n_1$. The following four sets will partition $\{1, 2, \ldots, (3+3c)n_1\}$:

$\mathbf{A} = \{1, 2, \ldots, (1+c)n_1\}$.

$\mathbf{B} = \{(1+c)n_1 + 1, (1+c)n_1 + 2, \ldots, (2+2c)n_1\}$.

$\mathbf{C} = \{(2+2c)n_1 + 1, \ldots, 3n_1 - 5\}$.

$\mathbf{D} = \{3n_1 - 4, \ldots, (3+3c)n_1\}$.

Partition the weights of vertices of V_2 into $V_{2A} \cup V_{2B}$ and V_3 into $V_{3A} \cup V_{3B} \cup V_{3C} \cup V_{3D}$ depending on where their weights meet these four sets. So, we let $V_{2A} = \{w(V) : v \in V_2 \text{ and } w(v) \in \mathbf{A}\}$, $V_{2B} = \{w(v) : v \in V_2 \text{ and } w(v) \in \mathbf{B}\}$, and similarly define $V_{3A}, V_{3B}, V_{3C}, V_{3D}$. We will also think of V_1 as a set of weights. There are $3n_1 - 5$ vertices, so we can draw a 1-1 correspondence between weights in V_{3D} and numbers between 1 and $3n_1 - 5$ with no vertex having that weight. Define $\mathbf{A}_\emptyset$ as the values in $\mathbf{A}$ with no weights equal to that value, i.e. $\mathbf{A}_\emptyset = \mathbf{A} \setminus (V_1 \cup V_{2A} \cup V_{3A})$. Similarly define $\mathbf{B}_\emptyset, \mathbf{C}_\emptyset$. Let V_{2B*} be the smallest $|\mathbf{A}_\emptyset|$ elements in V_{2B}.

We are now going to take these sets and define new sets of weights V_1', V_2', V_3'. We are not defining a new weight function, we are only moving vertex weights between the sets. These new sets will fill all values in $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and have no values in $\mathbf{D}$. Basically, we are going to fill all holes in $\mathbf{A}$ by moving the smallest possible degree 2 totals from $\mathbf{B}$ to fill them, then move weights from V_{3D} to fill those vacancies, then any other vacancies in $\mathbf{B}, \mathbf{C}$ will also be filled by the remaining weights in V_{3D}.

$V_1' = V_1$, $V_{2A}' = V_{2A} \cup A_\emptyset$, and $V_{3A}' = V_{3A}$.

$V_{2B}' = V_{2B} \setminus V_{2B*}$ and $V_{3B}' = V_{3B} \cup V_{2B*} \cup \mathbf{B}_\emptyset$.

$V'_{3C} = V_{3C} \cup \mathbf{C}_\emptyset = \mathbf{C}$.

$V'_{3D} = \emptyset$.

$V'_2 = V'_{2A} \cup V'_{2B}$ and $V'_3 = V'_{3A} \cup V'_{3B} \cup V'_{3C}$.

$V' = V'_1 \cup V'_2 \cup V'_3$, so the notation is consistent with these new sets.

As in the proof of Claim 1 in Section 1, we will define sums S', so for example:

$$S'_{3C} \quad := \quad \sum_{i \in V'_{3C}} i$$

First note that $V' = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$. Therefore

$$S'_1 + S'_2 + S'_3 = \sum_{i=1}^{3n_1-5} i = \frac{9}{2} n_1^2 + O(n_1) \tag{2}$$

Claim 2 *We have*

$$S'_1 + S'_2 - S'_3 \geq 0 \tag{3}$$

Proof: First note that $S_1 + S_2 - S_3 = 0$ because of the structure of the tree. Clearly $S_1 = S'_1$ because $V_1 = V'_1$. So, it suffices to show $(S_3 - S'_3) - (S_2 - S'_2) \geq 0$.

Let V_{3D*} be the smallest $|\mathbf{A}_\emptyset|$ elements in V_{3D}. First, we have

$$S_{3D*} + \sum_{i \in A_\emptyset} i \quad \geq \quad \sum_{j=3n_1-4}^{3n_1-5+|\mathbf{A}_\emptyset|} j \; + \; \sum_{i=1}^{|\mathbf{A}_\emptyset|} i \quad = \quad (3n_1 - 4 + |\mathbf{A}_\emptyset|)|\mathbf{A}_\emptyset|$$

Now, since $|V_{2A}| + |V_{2B}| = n_1 - 3$, and $|V_{2A}| \leq |\mathbf{A}| - |V_{1A}| \leq cn_1$, we have that $|V_{2B}| \geq (1-c)n_1 - 3$. Since $V_{2B} \subseteq \mathbf{B}$, this tells us that the smallest element of V_{2B} is no larger than $n_1 + 3cn_1 + 4$. So,

$$S_{2B*} \;\leq\; \sum_{i=n_1+3cn_1+4}^{n_1+3cn_1+4+|\mathbf{A}_\emptyset|} i \;\leq\; [n_1 + 3cn_1 + |\mathbf{A}_\emptyset| + 4]|\mathbf{A}_\emptyset|$$

Now, putting these together and using $|\mathbf{A}_\emptyset| \leq cn_1$,

$$S_{3D*} + \sum_{i \in A_\emptyset} i \; - \; 2\, S_{2B*} \;\geq\; (n_1 - 7cn_1 - 12)|\mathbf{A}_\emptyset| \;>\; 0 \tag{4}$$

¿From the definition of the sets, we have

$$(S_3 - S'_3) - (S_2 - S'_2) \;=\; \left(S_{3D} - S_{2B*} - \sum_{i \in B_\emptyset \cup C_\emptyset} i \right) - \left(S_{2B*} - \sum_{i \in A_\emptyset} i \right),$$

and using (4) this becomes

$$(S_3 - S'_3) - (S_2 - S'_2) \;\geq\; [S_{3D} - S_{3D*}] - \sum_{i \in B_\emptyset \cup C_\emptyset} i \;\geq\; 0.$$

The last inequality is true because both sides count the same number of elements, but elements in V_{3D} are bigger than elements in $\mathbf{B} \cup \mathbf{C}$. $\square$

It follows from (2) and (3) that for n_1 sufficiently large we have

$$S'_3 \quad \leq \quad \tfrac{9}{4} n_1^2$$

We also have

$$S'_{3C} \quad = \quad \sum_{i=(2+2c)n_1+1}^{3n_1-5} i \quad = \quad (\tfrac{5}{2} - 4c - 2c^2) n_1^2 + O(n_1)$$

thus

$$S'_{3A} + S'_{3B} \quad \leq \quad (2c^2 + 4c - \tfrac{1}{4}) n_1^2 + O(n_1) \quad < \quad (6c^2 + c) n_1^2 \tag{5}$$

for n_1 sufficiently large. The last inequality in (5) holds because $c < \frac{1}{8}(3 - \sqrt{5})$ implies $0 < \frac{1}{4} + 4c^2 - 3c$.

Claim 3 *We have $|V'_{2A}| < |V'_{3A}|$.*

Proof: The following is a trivial bound.

$$S'_{3A} + S'_{3B} \geq \sum_{i=1}^{|V'_{3A}|} i \; + \sum_{i=(1+c)n_1}^{(1+c)n_1 + |V'_{3B}|} i.$$

If, by contradiction, $|V'_{2A}| \geq |V'_{3A}|$, then since $|V'_{2A}| + |V'_{3A}| = cn_1$, we have $|V'_{3A}| \leq \frac{1}{2} cn_1$, and since $|V'_{3A}| + |V'_{3B}| = 2cn_1 + 3$, we have $|V'_{3B}| \geq \frac{3}{2} cn_1 + 3$. However, this leads to

$$S'_{3A} + S'_{3B} \quad \geq \quad \sum_{i=1}^{\frac{1}{2} cn_1} i + \sum_{i=(1+c)n_1}^{(1+\frac{5}{2}c)n_1+3} i = (\tfrac{3}{2} c + \tfrac{11}{4} c^2) n_1^2 + O(n_1)$$

This violates condition (5) for any $c < \frac{2}{13}$. $\square$

Let $\mathbf{S}$ be the set of $2|V_{3A}|$ non-pendant edges with smallest weights.

Claim 4 *There exists partitions of V_{2A}, V_{2B}, into $V_{2A_0} \cup V_{2A_1} \cup V_{2A_2}$ and $V_{2B_0} \cup V_{2B_1}$, respectively, such that*

$$|\mathbf{S}| \quad = \quad |V_{2A_1}| + 2|V_{2A_2}| + |V_{2B_1}| \tag{6}$$

and

$$\sum_{e \in \mathbf{S}} w(e) \quad \geq \quad S_{2A_2} + S_{2B_1} - (1+c) n_1 |V_{2B_1}|. \tag{7}$$

Proof: For $i = 0, 1, 2$ and any $v \in V_{2A}$, we put $w(v)$ into set V_{2A_i} if exactly i edges incident with v are in $\mathbf{S}$. For $j = 0, 1, 2$ and any $v \in V_{2B}$, put $w(v)$ into set W_{2B_j} if exactly j of its edges are in $\mathbf{S}$. From counting edges, we see that

$$|\mathbf{S}| \quad = \quad |V_{2A_1}| + 2|V_{2A_2}| + |W_{2B_1}| + 2|W_{2B_2}|.$$

We have $|W_{2B_0}| > |V_{2B}| - 2|\mathbf{S}| > (1-c)n_1 - 4cn_1 > 4cn_1 > |W_{2B_2}|$. Let W_0 be a set of $|W_{2B_2}|$ weights from W_{2B_0}. Define the following new partition of V_{2B}.

$$V_{2B_0} := W_{2B_0} \setminus W_0$$
$$V_{2B_1} := W_{2B_1} \cup W_0 \cup W_{2B_2}$$

Note that (6) holds.

For an arbitrary symbol $*$, define

$$S_* = \sum_{v \in V_*} w(v) \quad \text{and} \quad T_* = \sum_{v \in W_*} w(v)$$

We will continue to represent the sum of all vertex totals in V_* with S_* for all sets, and now we will use W_* to represent the sum of vertex totals in T_*. From our partition, we have

$$\sum_{e \in \mathbf{S}} w(e) \quad = \quad S_{2A_2} + T_{2B_2} + \overline{S}_{2A_1} + \overline{T}_{2B_1}, \tag{8}$$

where the bars in the last two sums indicate the sum of the smaller edges only from those vertices. Note that

$$\overline{S}_{2A_1} \geq 0 \quad \text{and} \quad \overline{T}_{2B_1} \geq T_{2B_1} - (1+c)n_1|W_{2B_1}| \tag{9}$$

Combining (8) and (9), then substituting $T_{2B_2} + T_{2B_1} = S_{2B_1} - T_0$,

$$\sum_{e \in \mathbf{S}} w(e) \quad \geq \quad S_{2A_2} + S_{2B_1} - T_0 - (1+c)n_1|W_{2B_1}| \tag{10}$$

Certainly $T_0 \leq (2+2c)n_1|W_0|$ since it is a sum of $|W_0|$ elements less than $(2+2c)n_1$. So,

$$T_0 \leq 2|W_0|(1+c)n_1 = (1+c)n_1(|W_0| + |W_{2B_2}|) = (1+c)n_1(|V_{2B_1}| - |W_{2B_1}|)$$

This with (10) gives (7). $\square$

Corollary 1 *There exists partitions of V'_{2A}, V'_{2B} into $V'_{2A_0} \cup V'_{2A_1} \cup V'_{2A_2}$ and $V'_{2B_0} \cup V'_{2B_1}$, respectively, such that*

$$|\mathbf{S}| \;\leq\; |V'_{2A_1}| + 2|V'_{2A_2}| + |V'_{2B_1}| \tag{11}$$

$$\sum_{e \in \mathbf{S}} \omega(e) \;\geq\; S'_{2A_2} + S'_{2B_1} - (1+c)n_1|V'_{2B_1}| \tag{12}$$

Proof: Take the partitions from Claim 4 and define the following:

$$V'_{2B_1} = V_{2B_1} \setminus V_{2B*} \qquad V'_{2A_1} = V_{2A_1} \cup \mathbf{A}_\emptyset \qquad V'_{2A_2} = V_{2A_2}$$

$$V'_{2A_0} = V_{2A_0} \qquad V'_{2B_0} = V_{2B_0} \setminus V_{2B*}$$

Equation (6) and the following relations give (11):

$$|V'_{2A_1}| = |V_{2A_1}| + |\mathbf{A}_\emptyset| \qquad |V'_{2B_1}| \geq |V_{2B_1}| - |V_{2B*}| \qquad |V_{2B*}| = |\mathbf{A}_\emptyset|.$$

Note that the following is true because $V_{2B*} \cap V_{2B_1} \subseteq \mathbf{B}$:

$$\sum_{i \in V_{2B*} \cap V_{2B_1}} i - (1+c)n_1 |V_{2B*} \cap V_{2B_1}| \quad > \quad 0$$

Combining this with equation (7) gives (12). □

Let V'_{1*} be the set of the $|V'_{3A}|$ degree 1 vertices with the smallest weights.

Corollary 2 *The following hold for the partitions from Corollary 1:*

$$2|V'_{3A}| \quad \leq \quad |V'_{2A}| + |V'_{2A_2}| + |V'_{2B_1}| \tag{13}$$

and

$$S'_{3A} \quad \geq \quad S'_{1*} + S'_{2A_2} + S'_{2B_1} - (1+c)n_1 |V'_{2B_1}| \tag{14}$$

Proof: Equation (13) is just (11) with substitutions $|\mathbf{S}| = 2|V'_{3A}|$ and $|V'_{2A}| \geq |V'_{2A_1}| + |V'_{2A_2}|$. The sum $S'_{3A} = S_{3A}$ is the sum of $|V_{3A}|$ pendant edges and $2|V_{3A}|$ non-pendant edges, which have sums at least S_{1*} and $\sum_{e \in \mathbf{S}} w(e)$, respectively. □

Just for easier reading, we will define λ so that $|V'_{3A}| = \lambda c n_1$. This gives $|V'_{2A}| = (1-\lambda)cn_1$ and $|V'_{3B}| = (2-\lambda)cn_1$. Similarly, define α so $|V'_{2A_2}| = \alpha c n_1$, and of course $\alpha \leq 1 - \lambda$ because $|V'_{2A_2}| \leq |V'_{2A}| = (1-\lambda)cn_1$. We can rewrite (13) as

$$|V'_{2B_1}| \geq 2|V'_{3A}| - |V'_{2A}| - |V'_{2A_2}| = (3\lambda - 1 - \alpha)cn_1.$$

Define $x \geq 3\lambda - 1 - \alpha$ so that $|V'_{2B_1}| = cxn_1$.

Now, we have the following trivial bounds:

$$S'_{3B} + S'_{2B_1} \geq \sum_{i=(1+c)n_1+1}^{(1+c)n_1 + |V'_{3B}| + |V'_{2B_1}|} i \;=\; \sum_{i=(1+c)n_1+1}^{(1+3c-\lambda c+xc)n_1} i$$

and

$$S'_{1*} + S'_{2A_2} \geq \sum_{i=1}^{|V'_{1*}|+|V'_{2A_2}|} i \;=\; \sum_{i=1}^{(\lambda+\alpha)cn_1} i$$

Using these with (14) gives

$$S'_{3A} + S'_{3B} \quad \geq \quad \sum_{i=1}^{(\lambda+\alpha)cn_1} i + \sum_{i=(1+c)n_1+1}^{(1+3c-\lambda c+xc)n_1} i - (1+c)cxn_1^2$$

The right side evaluates to

$$\left[\tfrac{c^2}{2}x^2 + c^2(2-\lambda)x + \tfrac{c}{2}\left(4 - 2\lambda - 6\lambda c + c\alpha^2 + 2\lambda^2 c + 2c\lambda\alpha + 8c\right)\right] n_1^2 + O(n_1)$$

The coefficient of n_1^2 is increasing over all x in our range, so it is minimized at $x = 3\lambda - 1 - \alpha$. After substituting this in, we see that the coefficient of n_1^2 is minimized with respect to α when $\alpha = 1 - \lambda$, its largest possible value. So, we can now say

$$S'_{3A} + S'_{3B} \quad \geq \quad \left[\tfrac{9}{2}c^2\lambda^2 - (c^2+c)\lambda + \tfrac{5}{2}c^2 + 2c\right] n_1^2 + O(n_1) \tag{15}$$

The coefficient of n_1^2 is minimized with respect to λ at $\lambda = 1$, which leads to

$$S'_{3A} + S'_{3B} \quad \geq \quad [6c^2 + c]n_1^2 + O(n_1) \tag{16}$$

This combined with (5) gives our contradiction for n_1 sufficiently large. □

Note: If we take $\alpha = 0$, $\lambda = 1$, and $c = \frac{3-\sqrt{5}}{8}$, then this bound would be tight. This is reflected in the construction we give in the next section.

3 Upper Bound

Suppose we are given sufficiently large T_n. Let $m = n_1 + 313$. Define c so that $(1+c)m = \lfloor \frac{11-\sqrt{5}}{8} m \rfloor$. We can then say $c = \frac{3-\sqrt{5}}{8} - \frac{\alpha}{m}$ with $0 < \alpha < 1$. We are going to prove that

$$s(T_n) \quad \leq \quad (1+c)m \quad \leq \quad \tfrac{11-\sqrt{5}}{8}\, n_1 \; + \; 313(\tfrac{11-\sqrt{5}}{8}).$$

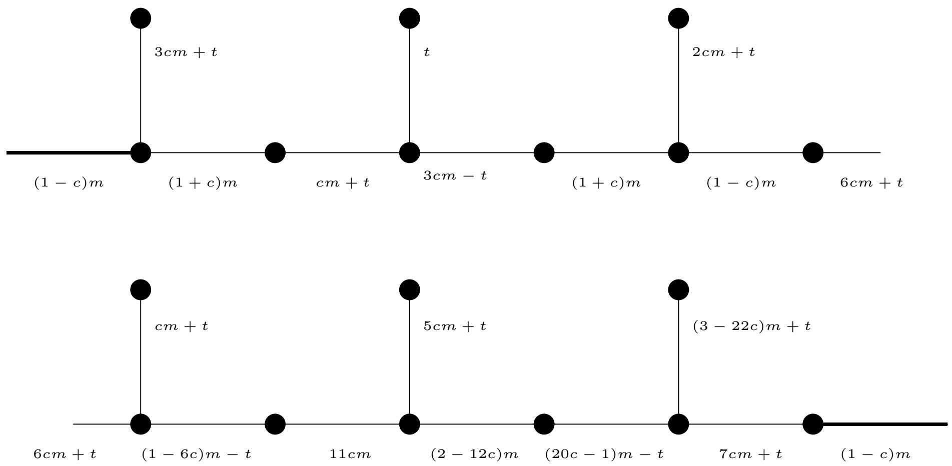
Note that the only time we will use the fact n_1 is sufficiently large is in the proof of Lemma 1.

We make an assignment that respects the following rules:

1. $d(v) = 1 \Rightarrow \omega(v) \in (0, 4cm) \cup (5cm, (1+c)m)$.
2. $d(v) = 2 \Rightarrow \omega(v) \in ((1+2c)m, (2+2c)m)$.
3. $d(v) = 3 \Rightarrow \omega(v) \in (4cm, 5cm) \cup ((1+c)m, (1+2c)m) \cup ((2+2c)m, 3m)$.

In words, we assign each degree a collection of intervals so that every vertex of that degree is in one of the assigned intervals.

We will assign in 4 phases by dividing T_n into a number of small pieces. The pieces all fit together because the edges between pieces all get weight $(1-c)m$.
Phase 1: We have one piece of the following type for each value of t between 1 and $cm - 1$. The labels on the edges indicate the weights.



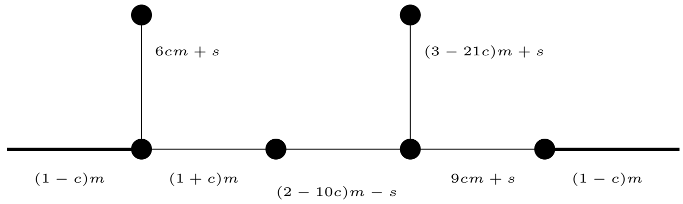
The first edge will be the first pendant edge on the tree. Here are the weights obtained by vertices of degree 1,2,3, respectively as t goes from 1 to $cm-1$. All are listed in the order they occur above, from left to right with the top row before the bottom, except the last entry for degree 1 vertices is from the first pendant vertex.

Degree 1
$3cm+1$ to $4cm-1$
1 to $cm-1$
$2cm+1$ to $3cm-1$
$cm+1$ to $2cm-1$
$5cm+1$ to $6cm-1$
$(3-22c)m+1$ to $(3-21c)m-1$
$(1-c)m$

Degree 2
$(1+2c)m+1$ to $(1+3c)m-1$
$(1+3c)m+1$ to $(1+4c)m-1$
$(1+5c)m+1$ to $(1+6c)m-1$
$(1+4c)m+1$ to $(1+5c)m-1$
$(1+7c)m+1$ to $(1+8c)m-1$
$(1+6c)m+1$ to $(1+7c)m-1$

Degree 3
$(2+3c)m+1$ to $(2+4c)m-1$
$4cm+1$ to $5cm-1$
$(2+2c)m+1$ to $(2+3c)m-1$
$(1+c)m+1$ to $(1+2c)m-1$
$(2+4c)m+1$ to $(2+5c)m-1$
$(2+5c)m+1$ to $(2+6c)m-1$

Phase 2: The next set of pieces will be constructed as follows, where s will take all but one of the values from 1 to $(22c-2)m-1$. The value skipped will be determined later; it will be skipped to avoid a situation in which one of the degree one vertices assigned here has the same weight as the last vertex.

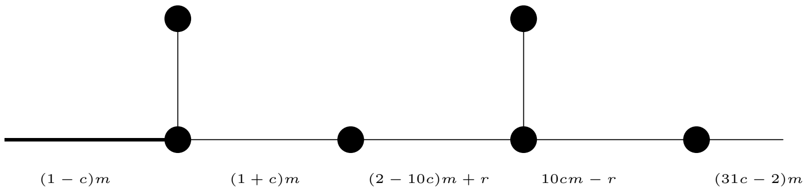


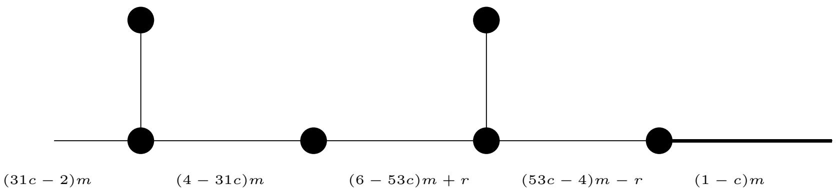
All weights obtained from these pieces as s runs from 1 to $(22c - 2)m - 1$ are in the following corresponding intervals:

Degree 1	**Degree 2**
$(3 - 21c)m + 1$ to $(1 + c)m - 1$	$(1 + 8c)m + 1$ to $(30c - 1)m - 1$
$6cm + 1$ to $(28c - 2)m - 1$	$(5 - 31c)m + 1$ to $(3 - 9c)m - 1$

Degree 3
$(5 - 22c)m + 1$ to $3m - 1$
$(2 + 6c)m + 1$ to $28cm - 1$

Phase 3: For the next pieces, r will run from 1 to $(11c - 1)m$.





The $4(11c-1)m$ vertices with degree 1 will take all weights from $(28c-2)m+1$ to $(72c-6)m$. It does not matter which goes where because all degree 3 vertices have their two non-pendant edges with a sum of $2m$. We get the following totals as r moves from 1 to $(11c - 1)m$.

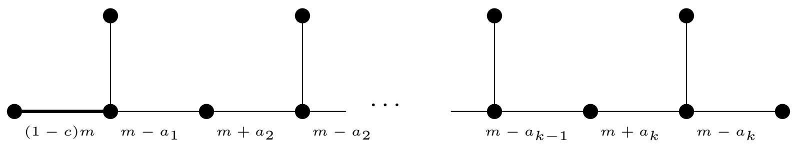
Degree 1	**Degree 2**
$(28c - 2)m + 1$ to $(72c - 6)m$	$(3 - 9c)m + 1$ to $(2 + 2c)m$
	$(30c - 1)m$ to $(41c - 2)m - 1$
Degree 3	$(10 - 84c)m + 1$ to $(9 - 73c)m$
$(28c)m + 1$ to $(72c - 4)m$	$(41c - 2)m$ to $(52c - 3)m - 1$

Phase 4: Putting all pieces so far end-to-end, we get a large structure with both ends having weight $(1-c)m$. In Phase 4, we are going to assign weights to the remaining edges, and all weights of vertices will be in in the following intervals:

Degree 1	**Degree 2**
$(72c-6)m+1$ to $(1-c)m-1$	$(52c-3)m$ to $(10-84c)m$
$(1-c)m+1$ to $(3-22c)m-305$	$(9-73c)m$ to $(5-31c)m$

Degree 3
$(72c-4)m+1$ to $(3-c)m-1$
$(3-c)m+1$ to $(5-22c)m-305$

Let $k=(9-94c)m-305$. We are going to find $a_1,\ldots,a_k$ such that $-cm \leq a_j \leq cm$ for all $j=1,2,\ldots,k$, and assign like this:



This piece will then be attached to our big piece. There are k remaining degree 1 vertices other than the one with weight $m-a_k$. These k vertices will get all weights from $(72c-6)m+1$ to $(3-22c)m-305$ except for $(1-c)m$. Since every degree 3 vertex above has its non-pendant edges adding up to $2m$, we know that the degree 3 vertices have weights from $(72c-4)m+1$ to $(5-22c)m-305$, all unused so far.

Claim 5 *There exist integers $a_1,\ldots,a_k$ such that*

1. *$|a_i| \leq cm$ for all $i=1,2,\ldots,k$,*
2. *When $a_1,\ldots,a_k$ are assigned to edges of the final piece, all degree 2 vertices have weights which are integers in the following set:*

$$[(52c-3)m,(10-84c)m] \cup [(9-73c)m,(5-31c)m]$$

These intervals are approximately $(1.9656m, 1.979m)$ *and* $(2.029m, 2.040m)$.

3. *$m \leq m-a_k < (1+c)m$.*

¿From this, we see that no edge has weight more than $(1+c)m$ because weights are $m+a_i$ for $i=1,2,\ldots,k$. Checking totals from all 4 phases verifies that every vertex has different weight. Also, $(3-21c)m \leq m-a_k < (1+c)m$, so we may skip some value of s and ensure that no other vertex has weight $m-a_k$.

The graph we then create has this many vertices of degree 1:

$$1+6(cm-1)+2([22c-2]m-2)+4(11c-1)m+k+1 \;=\; m-313 \;=\; n_1$$

So, we are done once we prove this claim. The following Lemma will do most of this work.

Lemma 1 *There exist k distinct elements $y_1, y_2, \ldots, y_k$ of the set*

$$S \;:=\; \{(52c-5)m, \ldots, (8-84c)m\} \;\cup\; \{(7-73c)m, \ldots, (3-31c)m\}$$

such that

$$0 \le y_1 + y_2 + \cdots + y_j \le 2cm \quad \text{for } j = 1, 2, \ldots, k, \tag{17}$$

and

$$0 \le y_1 + y_2 + \cdots + y_k \le cm$$

Proof: Since $-c < 52c - 5 < 8 - 84c < 0 < 7 - 73c < 3 - 31c < c$, we have $|s| \le cm$ for all $s \in S$. Provided we can find k integers from S whose sum is between $-cm$ and cm, it is easy to find a greedy algorithm to order them as $y_1, y_2, \ldots, y_k$ and still satisfy (17).

Set S has $k+303$ elements. Suppose we take k elements from S by taking the y_i's to be all elements from S except for the smallest x and the largest $303 - x$. Then,

$$y_1 + \cdots + y_k \;=\; \sum_{i=(52c-5)m+x}^{(8-84c)m} i + \sum_{i=(7-73c)m}^{(3-31c)m-302+x} i$$

Recall how $c = \frac{3-\sqrt{5}}{8} - \frac{\alpha}{m}$ with $0 < \alpha < 1$. Let x be the integer between $162 + 60.2\alpha$ and $163 + 60.2\alpha$. Then, we get $y_1 + \cdots + y_k = km + f(\alpha, x)$, where k is a constant such that $|k| < c$. □

To prove Claim 5, first let $a_1 = -cm$. This way $m + a_1 = (1-c)m$, making the first degree 3 vertex consistent with the rest of the piece. Now, note that because the degree 2 sums are $2m - a_j + a_{j+1}$ for all $j = 1, 2, \ldots, k$, we can say $y_j = -a_j + a_{j+1}$. Solving this for the a_i's yields

$$a_j = y_1 + y_2 + \cdots + y_{j-1} - cm \text{ for } j = 2, 3, \ldots, k.$$

So, Lemma 1 ensures *1,3* from Claim 5, and *2* comes directly from our choice of S. □

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