On Partitions of Discrete Boxes

Noga Alon* Tom Bohman† Ron Holzman‡

Daniel J. Kleitman§

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Abstract

We prove that any partition of an n-dimensional discrete box into nontrivial sub-boxes must consist of at least 2^n sub-boxes, and consider some extensions of this theorem.

1 The theorem

A set of the form

$$A = A_1 \times A_2 \times \cdots \times A_n,$$

where $A_1, A_2, \ldots, A_n$ are finite sets with $|A_i| \geq 2$, will be called here an *n-dimensional discrete box.* A set of the form $B = B_1 \times B_2 \times \cdots \times B_n$, where $B_i \subseteq A_i$, $i = 1, \ldots, n$, is a *sub-box* of A. Such a set B is said to be *nontrivial* if $\emptyset \neq B_i \neq A_i$ for every i.

The following theorem answers a question posed by Kearnes and Kiss [1, Problem 5.5].

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Research supported in part by a USA-Israeli BSF grant, by the Israel Science Foundation and by the Herman Minkowski Minerva Center for Geometry at Tel Aviv University. Email: noga@math.tau.ac.il

†Department of Mathematics, Carnegie Mellon University, Pittsburgh, USA, and Department of Mathematics, Massachusetts Institute of Technology, Cambridge, USA. Email: tbohman@moser.math.cmu.edu

‡Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel. Work done while this author was visiting the Department of Mathematics, Massachusetts Institute of Technology. Research supported by the M. and M. L. Bank Mathematics Research Fund and by the Fund for the Promotion of Research at the Technion. Email: holzman@tx.technion.ac.il

§Department of Mathematics, Massachusetts Institute of Technology, Cambridge, USA. Email: djk@math.mit.edu

Theorem 1 *Let A be an n-dimensional discrete box, and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of A into nontrivial sub-boxes. Then $m \geq 2^n$.*

Proof. Let

$$B^j = B_1^j \times B_2^j \times \cdots \times B_n^j, \ j = 1, \ldots, m.$$

Let us call a sub-box C of A odd if its cardinality is odd. Let $\mathcal{O}(A)$ denote the collection of all odd sub-boxes of A. For $j = 1, \ldots, m$, define:

$$\mathcal{O}_j(A) = \{C \in \mathcal{O}(A) \mid C \cap B^j \text{ is odd}\}.$$

A sub-box is odd if and only if each of its n factors has odd cardinality, and the nontriviality of the B^j implies that half of the odd cardinality subsets of A_i intersect B_i^j in an odd number of elements. This implies

$$\frac{|\mathcal{O}_j(A)|}{|\mathcal{O}(A)|} = \frac{1}{2^n}, \ j = 1, \ldots, m. \tag{1}$$

For each $C \in \mathcal{O}(A)$ the partition $\{B^1, B^2, \ldots, B^m\}$ induces a partition of C in which at least one of the parts must have odd cardinality, which implies

$$\bigcup_{j=1}^{m} \mathcal{O}_j(A) = \mathcal{O}(A). \tag{2}$$

It follows from (1) and (2) that $m \geq 2^n$.

2 Extensions and non-extensions

2.1 Infinite boxes

The theorem remains true if in the definition of an n-dimensional discrete box we allow the sets $A_1, A_2, \ldots, A_n$ to be infinite. This follows by considering the finitely many atoms induced by the partition at hand.

2.2 Partitions mod 2

The theorem remains true, with the same proof, if $\{B^1, B^2, \ldots, B^m\}$ is only assumed to be a partition mod 2, that is, $\{B^1, B^2, \ldots, B^m\}$ is a multi-family of nontrivial sub-boxes of A such that every point of A is covered an odd number of times.

2.3 Conditions for equality

An obvious example of equality in the theorem is obtained by splitting each A_i into two nonempty parts, and taking $B^1, B^2, \ldots, B^{2^n}$ to be the corresponding cells. One can derive from the above proof some conditions that any example of equality must satisfy, and one might hope that these will lead to a characterization of all such examples. In particular, one might naively conjecture that every n-dimensional example of equality may be obtained by splitting one factor into two parts, and further partitioning each of the two resulting boxes according to some $(n-1)$-dimensional examples of equality. However, the following partition of a $3 \times 3 \times 3$ box into 8 nontrivial sub-boxes, in which none of the factors is split into just two parts, seems to indicate that examples of equality do not obey a simple construction rule:

$$\begin{array}{ccccccc}
A & = & \{1,2,3\} & \times & \{a,b,c\} & \times & \{\alpha,\beta,\gamma\} \\
B^1 & = & \{1\} & \times & \{a\} & \times & \{\alpha\} \\
B^2 & = & \{1\} & \times & \{a\} & \times & \{\beta,\gamma\} \\
B^3 & = & \{1\} & \times & \{b,c\} & \times & \{\alpha,\beta\} \\
B^4 & = & \{1,2\} & \times & \{b,c\} & \times & \{\gamma\} \\
B^5 & = & \{2,3\} & \times & \{a,b\} & \times & \{\alpha,\beta\} \\
B^6 & = & \{2,3\} & \times & \{a\} & \times & \{\gamma\} \\
B^7 & = & \{2,3\} & \times & \{c\} & \times & \{\alpha,\beta\} \\
B^8 & = & \{3\} & \times & \{b,c\} & \times & \{\gamma\}
\end{array}$$

2.4 Partition numbers of hypergraphs

If $\mathcal{H} = (V, E)$ is a hypergraph (i.e., E is a family of subsets of V), let us define the *partition number* $\pi(\mathcal{H})$ as the least p such that E contains a partition $\{B^1, B^2, \ldots, B^p\}$ of V (letting $\pi(\mathcal{H}) = \infty$ if there is no such p). If $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ are two hypergraphs, let us define their *product* $\mathcal{H}_1 \times \mathcal{H}_2$ to be the hypergraph with vertex-set $V_1 \times V_2$ and edge-set consisting of all sets of the form $B_1 \times B_2$, $B_1 \in E_1$, $B_2 \in E_2$.

Clearly, if E consists of all the proper subsets of V and $|V| \geq 2$, then the partition number of $\mathcal{H} = (V, E)$ is 2. Our theorem asserts that the product of n such hypergraphs has partition number 2^n. This raises the question whether the partition number is multiplicative with respect to hypergraph product. It is easy to see that $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq \pi(\mathcal{H}_1) \cdot \pi(\mathcal{H}_2)$, but the following example shows that in general equality need not hold.

Let $k > 4$ be an integer, and let V_1 and V_2 be two sets of cardinality $3k$. Let E_1 consist of all subsets of V_1 of cardinality 1 or $k+1$, and let E_2 consist of all subsets of V_2 of cardinality 1 or $2k-1$. Then $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 =$

(V_2, E_2) satisfy $\pi(\mathcal{H}_1) = k$ and $\pi(\mathcal{H}_2) = k + 2$. However, $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq 6k$. In order to see this, identify the vertex-set of $\mathcal{H}_1 \times \mathcal{H}_2$ with the edge-set $E(K_{3k,3k})$ of a complete bipartite graph with $3k$ vertices on each side. Find a $(k+1)$-regular subgraph G of $K_{3k,3k}$, and partition the edge-sets of G and its bipartite complement into $3k$ stars each, centered on opposite sides. As $6k < k(k+2)$ for $k > 4$, this is a counterexample to the multiplicativity of the partition number with respect to hypergraph product.

One may define the *mod 2 partition number* $\overline{\pi}(\mathcal{H})$ in a similar way, by considering partitions mod 2 (as in subsection 2.2) instead of partitions. Here, too, multiplicativity fails in general. Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two copies of a Fano plane (vertices are points, edges are lines). Then $\overline{\pi}(\mathcal{H}_i) = 3$ for $i = 1, 2$, but $\overline{\pi}(\mathcal{H}_1 \times \mathcal{H}_2) \leq 7$, as shown by the mod 2 partition of $V_1 \times V_2$ formed by taking the product of each line with itself.

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References

[1] K. A. Kearnes and E. W. Kiss, Finite algebras of finite complexity, *Discrete Math.* **207** (1999), 89-135.