G-intersecting Families

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Abstract

Let G be a graph on vertex set $[n]$, and for $X \subseteq [n]$ let $N(X)$ be the union of X and it's neighborhood in G. A family of sets $\mathcal{F} \subseteq 2^{[n]}$ is G-*intersecting* if $N(X) \cap Y \neq \emptyset$ for all $X, Y \in \mathcal{F}$. In this paper we study the cardinality and structure of the largest k-uniform G-intersecting families on a fixed graph G.

1 Introduction

An *intersecting family* is a collection of sets $\mathcal{F} \subseteq 2^{[n]}$ such that

$$X, Y \in \mathcal{F} \Rightarrow X \cap Y \neq \emptyset.$$

The classic theorem of Erdős-Ko-Rado [5] states that a k-uniform intersecting family on a ground set of size n contains at most $\binom{n-1}{k-1}$ sets. Furthermore, a family achieves this extremum if and only if the family consists of all k-sets containing some fixed element of $[n]$. For a general introduction to the theory of intersecting families see [3].

We consider a generalization of the notion of an intersecting family (for a survey of other generalizations of the Erdős-Ko-Rado theorem see [4]).

First consider the following: In an intersecting family $\mathcal{F}$ we require that $X, Y \in \mathcal{F}$ should intersect. Suppose that we relax this to simply require that $X, Y \in \mathcal{F}$ are "close" in some sense. Can something interesting be said? We believe that it can. For example suppose that we take "X, Y are close" to mean that there exist $x \in X, y \in Y$

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such that $\mathrm{dist}(x,y) = |x-y| \le 1$ (it will be convenient to have $\mathrm{dist}(n,1) = 1$ here). What can we say about the maximum size of a k-uniform family where every pair of distinct sets are close? A moments thought suggests that a candidate for a maximal $\mathcal{F}$ is the set of k-sets which contain 2 or 3. This is almost correct. Throwing in those sets which contain both 1 and 4 gives the correct answer, at least for n sufficiently large,

$$|\mathcal{F}| \le 2\binom{n-1}{k-1} - \binom{n-2}{k-2} + \binom{n-4}{k-2}$$

with equality only if $\mathcal{F}$ has the described structure, see Theorem 2.

Now for our full generalization. Let G be a graph on vertex set $[n]$. (The above example can be considered to be the case where G is a cycle.) For $X \subseteq [n]$ let

$$N(X) = X \cup \{y \in [n] : \exists x \in X \text{ such that } x \sim_G y\}.$$

We say that a family of sets, $\mathcal{F} \subseteq 2^{[n]}$, is *$G$-intersecting* if

$$X, Y \in \mathcal{F} \Rightarrow N(X) \cap Y \neq \emptyset.$$

In other words, $\mathcal{F}$ is G-intersecting if for all $X, Y \in \mathcal{F}$ there exist $x \in X$ and $y \in Y$ such that $x = y$ or $x \sim_G y$. For a fixed graph G how large can a k-uniform G-intersecting family be? For a graph G on vertex set $[n]$ and k a positive integer let

$$N(G,k) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq \binom{[n]}{k} \text{ and } \mathcal{F} \text{ is } G\text{-intersecting}\right\}.$$

This quantity can be easily computed in some cases. For example, for the complete graph we clearly have $N(K_n, k) = \binom{n}{k}$. Let $G = K_{a,n-a}$ be the complete bipartite graph having parts $A, B \subseteq [n]$. A G-intersecting family $\mathcal{F}$ can be partitioned into 3 families, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where $X \in \mathcal{F}_1$ implies X meets both A and B, $\mathcal{F}_2$ is an intersecting family of subsets of A, and $\mathcal{F}_3$ is an intersecting family of subset of B. It follows that

$$N(K_{a,n-a}, k) = \binom{n}{k} - \binom{a}{k} - \binom{n-a}{k} + \binom{a-1}{k-1} + \binom{n-a-1}{k-1}. \tag{1}$$

As another simple example consider the case of the union H_p of a clique on $\{1, 2, \dots, p\}$ and an independent set on $\{p+1, p+2, \dots, n\}$. Here we can show that

$$N(H_p, k) = \begin{cases} \binom{n}{k} - \binom{n-p}{k} : & k \le n/2 \\ \binom{n}{k} : & k > n/2 \end{cases}$$

The case $k > n/2$ is trivial and so assume that $k \le n/2$. We use induction on p. H_1 is an independent set and so $\mathcal{F}$ is H_1-intersecting iff it is intersecting. Thus $|\mathcal{F}| \le \binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}$. For $p \ge 2$ let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1 = \{X \in \mathcal{F} : p \notin X\}$. $\mathcal{F}_1$ is

H'_{p-1}-intersecting where $H'_{p-1} = H_p - \{p\}$ and so, by induction, $|\mathcal{F}_1| \leq \binom{n-1}{k} - \binom{n-p}{k}$. Clearly $|\mathcal{F}_2| \leq \binom{n-1}{k-1}$ and the result follows. Taking $\mathcal{F} = \{X : \ X \cap [p] \neq \emptyset\}$ shows the bound is tight.

The extremal G-intersecting families are analogous to the extremal intersecting families given by Erdős-Ko-Rado for sparse G and small k. To make this statement more precise, we first note that if $K \subseteq [n]$ is a clique in G then

$$\mathcal{F}(K) := \left\{ X \subseteq \binom{[n]}{k} : X \cap K \neq \emptyset \right\} \tag{2}$$

is a G-intersecting family. It follows that

$$N(G,k) \geq \binom{n}{k} - \binom{n-\omega}{k} \tag{3}$$

where $\omega = \omega(G)$ is the clique number of G. Note that if G is the empty graph then Erdős-Ko-Rado implies $N(G,k) = \binom{n-1}{k-1}$, $N(G,k)$ equals the bound given in (3) and any extremal family is of the form given in (2). However, for most graphs we cannot replace the inequality in (3) with equality. If K is a clique in G and $M_1, M_2, \ldots, M_r \subseteq [n] \backslash K$ satisfy

$$K \subseteq N(M_i) \text{ for } i = 1, \ldots, r \quad \text{and} \quad M_i \cap N(M_j) \neq \emptyset \text{ for } i \neq j \tag{4}$$

then the collection

$$\mathcal{F}(K; M_1, \ldots, M_r) = \left\{ X \subseteq \binom{[n]}{k} : X \cap K \neq \emptyset \text{ or } M_i \subseteq X \text{ for some } i \right\} \tag{5}$$

is G-intersecting. Thus, if there exists a maximum clique K and $M \subseteq [n] \backslash K$ such that $|M| \leq k$ and $K \subseteq N(M)$ then the bound given in (3) is not the truth. However, for some graphs we are able to show that if $\mathcal{F}$ is a large G-intersecting family then $\mathcal{F}$ is of the form given in (5).

Theorem 1. *Let G be a graph on vertex set $[n]$ and k a positive integer such that*

$$n \geq \frac{2e(1+\Delta)\Delta'(k-1)^2 k(k+1)}{\omega}$$

where $\Delta = \Delta(G)$ is the maximum degree of G, $\omega = \omega(G)$ is the clique number of G, and $\Delta' = \Delta'(G)$ is the maximum size of a second neighborhood in G:

$$\Delta'(G) := \max_{v \in [n]} |\{i \in [n] : \mathrm{dist}_G(i,v) \leq 2\}| \, .$$

If $\mathcal{F}$ is a G-intersecting family containing more than $\binom{n}{k} - \binom{n-\omega}{k}$ sets, then there exists a clique K in G and $M_1, \ldots, M_r \subseteq [n] \backslash K$ such that (4) holds and $\mathcal{F} = \mathcal{F}(K; M_1, \ldots, M_r)$.

In some sense, Theorem 1 is analogous to the results of Hilton-Milner [6] and Hajnal-Rothschild [7] on intersecting systems which state that an intersecting family that is 'nearly' maximum is 'very similar' in structure to the maximum intersecting families.

Note that Theorem 1 contains information about neither the clique K (e.g. must it be a clique on $\omega(G)$ vertices?) nor the collection of sets $M_1, \ldots, M_r$. Therefore, when we apply this theorem we must optimize over all possible choices of these parameters. We do this for three classes of graphs. For an arbitrary graph G we define the p^{th} power of G, denoted G^p, to be the graph with vertex set $V(G)$ and an edge between vertices u and v if and only if there is a path of length at most p between u and v in G.

Corollary 2. *Let $G = C_n^p$ be the p^{th} power of a cycle on n vertices and k be a positive integer. If*

$$n \geq \frac{2e(2p+1)(4p+1)(k-1)^2 k(k+1)}{p+1} \tag{6}$$

then

$$N(G,k) = \binom{n}{k} - \binom{n-2p-1}{k} - p\binom{n-2p-2}{k-1}. \tag{7}$$

Furthermore, a G-intersecting family $\mathcal{F}$ has cardinality $N(G,k)$ if and only if $\mathcal{F} = \mathcal{F}\left(K, M_1, \ldots, M_{\binom{p+1}{2}}\right)$ where K is a clique on $p+1$ elements and

$$\left\{M_1, \ldots, M_{\binom{p+1}{2}}\right\} = \left\{\{i,j\} \in \binom{[n]\backslash K}{2} : i \not\sim_G j \text{ and } K \subseteq N(\{i,j\})\right\}.$$

Corollary 3. *Let G be a d-regular graph on vertex set $[n]$ having no cycle on 5 or fewer vertices. If*

$$\begin{aligned} k &< d, \\ n &\geq (k+1) + 2e(k-2)k(d+1)(d-1)^k \text{ and} \qquad (8) \\ n &\geq e(d+1)(d^2+d+1)(k-1)^2 k(k+1) \qquad (9) \end{aligned}$$

then

$$N(G,k) = \binom{n}{k} - \binom{n-2}{k} + \sum_{i=1}^{k-2}\left[\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{n-j(d-1)-i-2}{k-1}\right]. \tag{10}$$

Furthermore, a G-intersecting family has cardinality $N(G,k)$ if and only if it has the following form:

$$\mathcal{F} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{k-2}$$

where

$$\mathcal{G}_0 = \left\{X \in \binom{[n]}{k} : \ X \cap \{x,y\} \neq \emptyset\right\}$$

and for $1 \le i \le k-2$

$$\mathcal{G}_i = \left\{ X \in \binom{[n]}{k} : \begin{array}{l} X \cap \{x, y, v_1, v_2, \dots, v_{i-1}\} = \emptyset, \\ v_i \in X, \\ X \cap A_j \neq \emptyset,\ 1 \le j \le i-1, \\ X \cap B \neq \emptyset \end{array} \right\}.$$

Here

$$\begin{array}{l} \{x, y\} \textit{ is an edge of } G, \textit{ i.e., a maximum clique}, \\ N(\{x\}) \setminus \{x, y\} = \{v_1, v_2, \dots, v_{d-1}\}, \\ A_j = N(\{v_j\}) \setminus \{v_j, x\},\ 1 \le j \le d-1, \\ B = N(\{y\}) \setminus \{x, y\}. \end{array}$$

Remark 4. *Unfortunately, the* p*-cube* Q_p *on* $n = 2^p$ *vertices does not fit the framework of Corollary 3, it has cycles of length 4. It is possible to determine the maximal families for small values of* k*, but the complexity of their description soon gets out of hand. For example, if* $k \le 3$ *and*

$$n \ge e(p+1)\left(p + 1 + \binom{p+1}{2}\right)(k-1)^2 k(k+1)$$

then

$$N(Q_p, k) = \binom{n}{k} - \binom{n-2}{k} + \binom{n-3}{k-1} + \binom{n-4}{k-1} - 2\binom{n-p-2}{k-1}.$$

While we must rely on the structure of individual graphs to determine if families of the form given in (5) are extremal and to determine the value of $N(G, k)$ these families give, we would expect that for most graphs (not just sparse graphs) and small k the dominant term in $N(G, k)$ is $\binom{n}{k} - \binom{n-\omega}{k}$. In order to prove a version of this assertion, we appeal to the language of random graphs. Let $p, 0 < p < 1$, be a fixed real number. For the random graph $G_{n,p}$ we consider $N(G_{n,p}, k)$ where k is fixed and n goes to infinity. Since the clique number of $G_{n,p}$ is **whp**[1] $(2 + o(1)) \log_{1/p} n$ (see [2]) it follows from (3) that

$$N(G_{n,p}, k) \ge (2 + o(1)) \log_{1/p} n \cdot \binom{n - (2 + o(1)) \log_{1/p} n}{k-1}$$

whp. We show that a constant multiple of this lower bound is an upper bound on $N(G_{n,p}, k)$.

Theorem 5. *Let* $p, 0 < p < 1$, *be a fixed real number and* $k \ge 2$ *be a positive integer. Then*

$$\Pr\left(N(G_{n,p}, k) \ge c_{k,p} \binom{n}{k-1} \lambda_n\right) = o(1)$$

[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability,* **whp**, if $\Pr(\mathcal{E}_n) = 1 - o(1)$.

where $\lambda_n = \max\{\ln n, \log_{1/p} n\}$ *and*

$$c_{k,p} = \frac{8}{(1-p)^{k-1}} \left[\frac{4k(k-1)}{(1-p)^{(k-1)^2}} + 1 \right].$$

Note that if $k = 1$ then $N(G_{n,p}, 1)$ is given by the clique number of $G_{n,p}$ for any $0 < p < 1$.

For many graphs $N(G,k)$ is nearly equal to $\binom{n}{k}$ when k is large.

Theorem 6. *Let* $\delta = \delta(G)$ *be the minimum degree of the graph* G*. Then*

$$k(\delta+1) > cn \ln n \quad \Rightarrow \quad N(G,k) \geq \left(1 - n^{1-c}\right) \binom{n}{k}$$

Theorem 7. *Let* δ *and* c *be constants satisfying*

$$r := c - (1-c)^{\delta+1} > 0.$$

If G *is a graph on vertex set* $[n]$ *having minimum degree at least* δ *and having maximum degree* $\Delta(G) = \Delta < \frac{r}{s}\sqrt{\frac{n}{\log n}}$ *and* $k \geq cn$ *then*

$$N(G,k) \geq \left(1 - n^{-s^2/2}\right) \binom{n}{k}.$$

Now, it follows from (1) that

$$N(K_{\delta,n-\delta}, \lfloor cn \rfloor) \sim \left[1 - (1-c)^{\delta}\right] \binom{n}{k}$$

for δ, c fixed constants and $n \to \infty$. Hence, it is *not* the case that $N(G,k)$ is nearly $\binom{n}{k}$ whenever k is on the order of n.

We have seen that for most graphs $N(G,k)$ is determined by extremal families that 'cluster' around cliques when k is small and is nearly $\binom{n}{k}$ when k is large. As there is a considerable gap between the values of k for which these two types of behavior have been observed, this leaves the obvious question: what happens for k in between? Is there a sharp phase transition between these two types of extrema? Do other types of extrema exist? As it seems likely that the answers to these questions will depend on the graph in question, we hazard a conjecture for only one graph, the cycle.

Conjecture 8. *There exists a constant* c *such that for any fixed* $\epsilon > 0$

$$k \leq (c-\epsilon)n \quad \Rightarrow \quad N(C_n, k) = 2\binom{n-1}{k-1} - \binom{n-2}{k-2} + \binom{n-4}{k-2}.$$

$$k \geq (c+\epsilon)n \quad \Rightarrow \quad N(C_n, k) = (1 - o(1))\binom{n}{k}$$

Note that it follows from Theorem 7 that $N(C_n, k)$ is nearly $\binom{n}{k}$ for $k > .32n$.

The remainder of this paper is organized as follows. Theorems 6 and 7 are proved in the next section. In Section 3 we first prove Theorem 1 and then prove Corollaries 2 and 3. Finally, in Section 4, we prove Theorem 5.

2 Large k

Both proofs in this section are probabilistic, and for both proofs we restrict our attention to the probability space $\binom{[n]}{k}$ with the uniform measure. Let $X_{n,k}$ denote a random element of this probability space.

Proof of Theorem 6. Let

$$\mathcal{F} = \left\{ X \in \binom{[n]}{k} : N(X) = [n] \right\}.$$

Clearly, $\mathcal{F}$ is a G-intersecting family. For a fixed vertex $v \in [n]$ we have

$$\begin{aligned} \mathrm{P}(v \notin N(X_{n,k})) &= \prod_{i=0}^{deg_G(v)} \left(1 - \frac{k}{n-i}\right) \\ &\leq \left(1 - \frac{k}{n}\right)^{\delta+1} \\ &< e^{-k(\delta+1)/n} . \end{aligned}$$

Therefore, by the first moment method,

$$\mathrm{P}(X_{n,k} \notin \mathcal{F}) < ne^{-k(\delta+1)/n}.$$

Hence, $|\mathcal{F}|$ is at least

$$\binom{n}{k}\left(1 - ne^{-k(\delta+1)/n}\right).$$

□

Proof of Theorem 7. Let

$$\mathcal{F} = \left\{ X \in \binom{[n]}{k} : |N(X)| > n - k \right\}.$$

Clearly, $\mathcal{F}$ is a G-intersecting family. For $1 \leq i \leq n$ let Y_i be the random variable

$$Y_i(X) = \mathbf{E}\left[|\overline{N(X_{n,k})}| : X_{n,k} \cap [i] = X \cap [i] \right]$$

(here $\overline{N(X_{n,k})} = [n] \setminus N(X_{n,k})$), and let Y_0 be the (constant) random variable $Y_0 = \mathbf{E}[|\overline{N(X_{n,k})}|]$. $Y_0, \ldots, Y_n$ is a martingale. Since

$$\mathbf{E}\left[|\overline{N(X_{n,k})}| \right] \leq \sum_{i=1}^{n} \left(1 - \frac{k}{n}\right)^{deg_G(i)+1} \leq (1-c)^{\delta+1} n$$

and

$$|Y_i - Y_{i-1}| \leq \Delta,$$

it follows from Azuma's inequality (see [1, page 85]) that for $\lambda > 0$

$$\mathrm{P}\left(Y_n > (1-c)^{\delta+1} n + \lambda\Delta\sqrt{n}\right) < e^{-\lambda^2/2}.$$

Since $\mathrm{P}(X_{n,k} \notin \mathcal{F}) \leq \mathrm{P}(Y_n \geq cn)$, the result follows. □

3 Sparse graphs, small k

In this section we prove Theorem 1 and Corollaries 2 and 3. We first prove two general lemmas from which Theorem 1 follows directly, and then go on to prove the corollaries. Throughout this section we assume G is a graph on vertex set $[n]$ and k is a positive integer satisfying

$$n \geq \frac{2e(1+\Delta)\Delta'(k-1)^2 k(k+1)}{\omega} \tag{11}$$

where $\Delta = \Delta(G)$ is the maximum degree of G, $\omega = \omega(G)$ is the clique number of G, and $\Delta' = \Delta'(G)$ is the maximum size of a second neighborhood in G:

$$\Delta'(G) := \max_{v \in [n]} \left|\{i \in [n] : \mathrm{dist}_G(i,v) \leq 2\}\right| .$$

Note that the assumptions of Corollaries 2 and 3 imply (11). We further assume $\mathcal{F}$ is a large G-intersecting family; in particular, we assume $\mathcal{F}$ is a G-intersecting family satisfying

$$|\mathcal{F}| > \binom{n}{k} - \binom{n-\omega}{k} . \tag{12}$$

At this point, we must introduce some notation and make some preliminary observations. For $i \in [n]$ let

$$\mathcal{F}_i = \{X \in \mathcal{F} : i \in X\} \text{ and } \nu_i = |\mathcal{F}_i| .$$

We set $\nu_{\max}$ to be the largest of the ν_i's. Since, for $X \in \mathcal{F}$ fixed, every set in $\mathcal{F}$ intersects $N(X)$ we have

$$|\mathcal{F}| \leq |N(X)|\nu_{\max} \leq (\Delta+1)k\nu_{\max},$$

and so

$$\nu_{\max} > \frac{1}{k(\Delta+1)}\left[\binom{n}{k} - \binom{n-\omega}{k}\right] . \tag{13}$$

Finally, we say a vertex $i \in [n]$ is *saturated* if

$$\nu_i > \frac{1}{k(\Delta+1)}\left[\binom{n}{k} - \binom{n-\omega}{k}\right] .$$

Note that it follows from (13) that there exists at least one saturated vertex. We will make use of the following inequality:

$$a, b \geq 2, m > ab \quad \Rightarrow \quad \binom{m-2}{b-2} < \frac{2e(b-1)}{m}\binom{m-a}{b-1} . \tag{14}$$

Proof of (14). Let a, b, m be integers such that $a, b \geq 2$ and $ab < m$.

$$\begin{aligned}
\frac{\binom{m-2}{b-2}}{\binom{m-a}{b-1}} &< \frac{b-1}{m-a}\left(\frac{m-b+1}{m-a-b+2}\right)^{b-2} \\
&< \frac{2(b-1)}{m}\left(1+\frac{a-1}{m-a-b+2}\right)^{b-2} \\
&< \frac{2(b-1)}{m} e^{\frac{(a-1)(b-2)}{m-a-b+2}} \\
&< \frac{2e(b-1)}{m} .
\end{aligned}$$

$\square$

Lemma 9. *If v is a saturated vertex then*

$$v \in N(X) \quad \forall X \in \mathcal{F}.$$

Proof. Let $\mathcal{F}'_v = \{X \backslash \{v\} : X \in \mathcal{F}_v\}$. We will call a collection $\mathcal{H} \subseteq \mathcal{F}'_v$ *separated* if

$$\forall X, Y \in \mathcal{H} \quad N(X) \cap N(Y) = \emptyset .$$

Let $\mathcal{H}$ be a maximum separated collection in $\mathcal{F}'_v$. The number of vertices at a distance less than or equal to 2 from an element of $\mathcal{H}$ is at most

$$M = |\mathcal{H}|(k-1)\Delta'.$$

We have

$$\begin{aligned}
|\mathcal{F}_v| &\leq \binom{n-1}{k-1} - \binom{n-M-1}{k-1} && \text{(by the maximality of } \mathcal{H}) \\
&\leq M\binom{n-2}{k-2} \\
&\leq \frac{M2e(k-1)}{n}\binom{n-\omega}{k-1} && \text{(by (11) and (14))} \\
&< \frac{M2e(k-1)}{n\omega}\left[\binom{n}{k} - \binom{n-\omega}{k}\right] \\
&< \frac{|\mathcal{H}|}{k(k+1)(\Delta+1)}\left[\binom{n}{k} - \binom{n-\omega}{k}\right] && \text{(by (11))} .
\end{aligned}$$

Since v is a saturated vertex we have

$$|\mathcal{H}| \geq k+1.$$

Now consider an arbitrary $X \in \mathcal{F}$. Since $\mathcal{F}$ is G-intersecting, X intersects $H \cup \{v\}$ for all $H \in \mathcal{H}$. Thus, by the pigeonhole principle, some vertex of X must be adjacent to at least two of the sets of the form $H \cup \{v\}$. Since $\mathcal{H}$ is separated this vertex is a neighbor of v. $\square$

Lemma 10. *If u and v are saturated vertices then $u \sim_G v$.*

Proof. Let u and v be saturated vertices and assume for the sake of contradiction that $u \not\sim_G v$. It follows from Lemma 9 that every element of $\mathcal{F}_u$ intersects $N(\{v\})$. Therefore,

$$\begin{aligned}
|\mathcal{F}_u| &\leq (\Delta+1)\binom{n-2}{k-2} && \text{(by Lemma 9)} \\
&\leq \frac{(\Delta+1)2e(k-1)}{n}\binom{n-\omega}{k-1} && \text{(by (11) and (14))} \\
&< \frac{(\Delta+1)2e(k-1)}{n\omega}\left[\binom{n}{k}-\binom{n-\omega}{k}\right] \\
&< \frac{1}{(1+\Delta+\Delta^2)(k-1)k(k+1)}\left[\binom{n}{k}-\binom{n-\omega}{k}\right] && \text{(by (11))} \\
&< \frac{1}{k(\Delta+1)}\left[\binom{n}{k}-\binom{n-\omega}{k}\right].
\end{aligned}$$

$\square$

It follows from Lemma 10 that the saturated vertices form a clique K in G, and it follows from Lemma 9 that any set $X \in \mathcal{F}$ satisfies $K \subseteq N(X)$. Thus, we have proved Theorem 1 (note that we can simply take the collection $\{M_1, \ldots, M_r\}$ to be those sets in $\mathcal{F}$ that do not intersect K).

Proof of Corollary 2. For notational convenience we identify the vertex set $[n]$ of $G = C_n^p$ with the cyclic group $\mathbf{Z}_n$ in the natural way; to be precise,

$$\alpha \sim_G \beta \Leftrightarrow \alpha - \beta \in \{1, \ldots, p, n-p, \ldots, n-1\}$$

where the difference is taken in $\mathbf{Z}_n$. Let $\mathcal{F}$ be a maximum G-intersecting family. We may assume that $\mathcal{F}$ is of the form given in (5).

If $|K| = p+1$ then $K = \{\alpha, \alpha+1, \ldots, \alpha+p\}$ for some $\alpha \in \mathbf{Z}_n$. Furthermore, for each set $M \in \{M_1, \ldots, M_r\}$

$$\exists \alpha+p+x, \alpha-y \in M \text{ such that } x, y \in \{1, \ldots, p\} \text{ and } x+y \leq p+1. \tag{15}$$

However, if M_i and M_j satisfy (15) then $M_i \cap N(M_j) \neq \emptyset$. So, we can take the family $\{M_1, \ldots, M_r\}$ to be the collection of all 2 element sets that satisfy (15), and it suffices to show $|K| = p+1$.

Assume for the sake of contradiction that there exists $\alpha \in K$ such that $\alpha+1 \notin K$ and $K \cap \{\alpha+2, \ldots, \alpha+p\} \neq \emptyset$. Since $\mathcal{F}$ is maximum and $\alpha+1$ is not saturated, the set $\mathcal{F}' := \{X \in \mathcal{F} : \alpha+1 \notin N(X)\}$ is nonempty. Therefore,

$$|\mathcal{F}_{\alpha+1}| \leq k(2p+1)\binom{n-2}{k-2}.$$

Since an arbitrary element of $\mathcal{F}'$ must contain $\alpha - p$ and some element of $\{\alpha+p+2,\ldots,\alpha+2p\}$ we have

$$|\mathcal{F}'| \le (p-1)\binom{n-2}{k-2}.$$

It follows from (14) and (6) that

$$|\mathcal{F}_{\alpha+1}| + |\mathcal{F}'| \le [k(2p+1)+p-1]\binom{n-2}{k-2} < \binom{n-1}{k-1}.$$

Since the family

$$(\mathcal{F}\backslash\mathcal{F}_{\alpha+1}\backslash\mathcal{F}') \cup \left\{X \in \binom{[n]}{k-1} : \alpha+1 \in X\right\}$$

is G-intersecting, this is a contradiction.

Thus, we may assume $K = \{\alpha, \alpha+1, \ldots, \alpha+l\}$ for some $\alpha \in \mathbf{Z}_n$ and $l \le p$. Assume for the sake of contradiction that $l < p$. Let $\mathcal{F}'' = \{X \in \mathcal{F} : \alpha - 1 \notin N(X)\}$. Since an arbitrary element of $\mathcal{F}''$ covers α without covering $\alpha-1$, we have $\mathcal{F}'' \subseteq \mathcal{F}_{\alpha+p}$. Since neither $\alpha-1$ nor $\alpha+p$ is saturated we have

$$|\mathcal{F}_{\alpha-1}| + |\mathcal{F}_{\alpha+p}| \le \frac{2}{k(2p+1)}\left[\binom{n}{k} - \binom{n-p-1}{k}\right] < \frac{2}{k}\binom{n-1}{k-1}.$$

This yields a contradiction because the family

$$(\mathcal{F}\backslash\mathcal{F}_{\alpha-1}\backslash\mathcal{F}_{\alpha+p}) \cup \left\{X \in \binom{[n]}{k} : \alpha-1 \in X\right\}$$

is G-intersecting and is larger than $\mathcal{F}$.

To establish (7) we assume that $K = \{1, 2, \ldots, p+1\}$. Then we can take

$$\mathcal{F} = \binom{[n]}{k} \setminus \left(\bigcup_{i=0}^{p} \mathcal{G}_i\right)$$

where

$$\mathcal{G}_0 = \left\{X \in \binom{[n]}{k} : X \cap (K \cup \{n-p+1, \ldots, n\}) = \emptyset\right\}$$

and

$$\mathcal{G}_i = \left\{X \in \binom{[n]}{k} : n-i+1 \in X, X \cap \{n-i+2, \ldots, 2p+3-i\} = \emptyset\right\}.$$

Equation (7) follows immediately. □

Proof of Corollary 3. Assume $\mathcal{F}$ is a maximum G-intersecting family. We may assume $\mathcal{F}$ is of the form given in (5). Let $\mathcal{M}$ be the collection $\{M_1, \ldots, M_r\}$ defined in (5).

We first note that $|K| = 2$: If x is the unique saturated vertex then

$$\begin{aligned}
|\mathcal{F}| &\leq |\mathcal{F}_x| + \sum_{v:v\sim x} |\mathcal{F}_v| \\
&< \binom{n-1}{k-1} + \frac{d}{k(d+1)}\left[\binom{n}{k} - \binom{n-2}{k}\right] \\
&= \left(\frac{n-1}{2n-k-1} + \frac{d}{k(d+1)}\right)\left[\binom{n}{k} - \binom{n-2}{k}\right] \\
&< \binom{n}{k} - \binom{n-2}{k},
\end{aligned}$$

which is a contradiction.

Now, let $K = \{x, y\}$,

$$A = N(\{x\})\backslash\{x, y\} \text{ and } B = N(\{y\})\backslash\{x, y\}.$$

For $v \in A \cup B$ let

$$A_v = N(\{v\})\backslash\{x, y, v\}.$$

Note that since G contains no small cycle, these sets are pairwise disjoint. Furthermore, the only edges between these sets are those connecting $v \in A \cup B$ with A_v and possibly edges connecting A_v and A_u where $u \in A$ and $v \in B$.

It remains to determine the size and structure of

$$\mathcal{F}_0 := \{X \in \mathcal{F} : X \cap K = \emptyset\}.$$

It follows from Theorem 1 that all elements of $\mathcal{F}_0$ intersect both A and B (it follows immediately that $\mathcal{F}_0$ is empty for $k = 1$; we henceforth assume $k \geq 2$). We proceed in stages. In the first stage we show that there exists a vertex $v_1 \in A \cup B$ (we assume without loss of generality that $v_1 \in A$) such that

$$\mathcal{M}_1 := \{\{v_1, b\} : b \in B\}$$

is a subset of $\mathcal{M}$ *and* that all sets in

$$\mathcal{F}_1 := \{X \in \mathcal{F}_0 : \nexists Y \in \mathcal{M}_1 \text{ such that } Y \subseteq X\}$$

must intersect $A_1 := A_{v_1}$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_0$ all sets in $\mathcal{F}_1$ must intersect A_1, $A\backslash\{v_1\}$ and B (note that any set in $\mathcal{F}_0$ that contains v_1 must contain some set in $\mathcal{M}_1$). In stage $2 \leq i \leq k-1$ we show that there exists a vertex $v_i \in A\backslash\{v_1, \ldots, v_{i-1}\}$ such that

$$\mathcal{M}_i := \left\{Y \in \binom{[n]}{i+1} : v_i \in Y \text{ and } Y \text{ intersects } B, A_1, \ldots, A_{i-1}\right\}$$

is a subset of $\mathcal{M}$ *and* that all sets in

$$\mathcal{F}_i := \{X \in \mathcal{F}_{i-1} : \nexists Y \in \mathcal{M}_i \text{ such that } Y \subseteq X\}$$

must intersect A_i. Thus, all sets in $\mathcal{F}_i$ must intersect $A\backslash\{v_1,\dots,v_i\}$, B, A_1, … , A_i. We have a complete characterization of $\mathcal{F}$ after the $(k-1)$st stage because $\mathcal{F}_{k-1}$ is empty: an element of $\mathcal{F}_{k-1}$ has k elements and intersects $A\backslash\{v_1,\dots,v_{k-1}\}$, B, A_1, … , A_{k-1}.

We begin with the first stage. As noted above, every element of $\mathcal{F}_0$ must intersect both A and B. For $(a,b)\in A\times B$ let

$$\mathcal{F}_{a,b}=\{X\in\mathcal{F}_0:\{a,b\}\subseteq X\}.$$

Let $\mathcal{P}$ be the set of pairs (a,b) for which $\mathcal{F}_{a,b}$ is nonempty. We say that the pair $(a,b)\in\mathcal{P}$ *covers* $(a',b')\in\mathcal{P}$ if

$$X\in\mathcal{F}_{a,b}\Rightarrow N(X)\cap\{a',b'\}\neq\emptyset.$$

Now, if $(a,b)\in\mathcal{P}$ does not cover (a',b') then there exists $X\in\mathcal{F}_{a,b}$ such that $N(X)\cap\{a',b'\}=\emptyset$, and

$$|\mathcal{F}_{a',b'}|\leq|N(X)|\binom{n-5}{k-3}\leq(d+1)k\binom{n-5}{k-3}. \tag{16}$$

On the other hand, if (a,b) covers (a',b') and $\{a,b\}\cap\{a',b'\}=\emptyset$ then every set in $\mathcal{F}_{a,b}$ contains some element of $A_{a'}\cup A_{b'}$, and

$$|\mathcal{F}_{a,b}|\leq 2(d-1)\binom{n-5}{k-3}. \tag{17}$$

We call a pair $(a,b)\in\mathcal{P}$ *good* if every pair in $\mathcal{P}$ covers (a,b) and

$$(a,b)\text{ covers }(a',b')\Rightarrow\{a,b\}\cap\{a',b'\}\neq\emptyset.$$

Note that if (a,b) is good then we may assume that $\mathcal{F}$ contains all sets that contain $\{a,b\}$. A pair $(a,b)\in\mathcal{P}$ that is not good will be called *bad*. It follows from (16) and (17) that

$$|\mathcal{F}_{a,b}|\leq(d+1)k\binom{n-5}{k-3}$$

for any bad pair (a,b). Thus, the number of sets in $\mathcal{F}$ containing at least one bad pair is at most

$$d^2(d+1)k\binom{n-5}{k-3}.$$

By (8) and (14) this quantity is less than $\binom{n-2d}{k-2}$ which is a lower bound on the number of sets containing some fixed good pair (and no other good pair). Thus, $\mathcal{F}$ has a maximal number of good pairs.

How many good pairs could $\mathcal{F}$ have? Note that if (a, b) and (a', b') are good pairs then $\{a, b\} \cap \{a', b'\} \neq \emptyset$. Clearly, we optimize the number of good pairs by taking all pairs that contain some fixed vertex. Therefore, there exists a vertex v_1 such that the collection

$$\mathcal{M}_1 = \{\{v_1, b\} : b \in B\}$$

is a subset of $\mathcal{M}$. Set $\mathcal{F}_1 = \{X \in \mathcal{F}_0 : v_1 \notin X\}$. By the definition of a good pair and the condition $k < d$, any set in $\mathcal{F}_1$ must intersect A_1. We have completed the first stage.

We now proceed by induction. Suppose stage l for $1 \leq l \leq k-2$ has been completed as described above. Each set in $\mathcal{F}_l$, the collection of sets in $\mathcal{F}$ that have not yet been characterized (i.e. sets in $\mathcal{F}$ that neither intersect K nor contain a set in $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l$), intersects $A \backslash \{v_1, \ldots, v_l\}$, B, and each of $A_1, \ldots, A_l$. We proceed as above: For

$$\mu = (a, b, x_1, \ldots, x_l) \in (A \backslash \{v_1, \ldots, v_l\}) \times B \times A_1 \times \cdots \times A_l$$

we define

$$\mathcal{F}_\mu = \{X \in \mathcal{F}_l : \{a, b, x_1, \ldots, x_l\} \subseteq X\} .$$

and set

$$\mathcal{P}_l = \{\mu \in A \times B \times A_1 \times \cdots \times A_l : \mathcal{F}_\mu \neq \emptyset\} .$$

We say that $\mu \in \mathcal{P}_l$ *covers* $\mu' = (a', b', x'_1, \ldots, x'_l) \in \mathcal{P}_l$ if

$$X \in \mathcal{F}_\mu \Rightarrow N(X) \cap \{a', b', x'_1, \ldots, x'_l\} \neq \emptyset$$

and that $\mu = (a, b, x_1, \ldots, x_l) \in \mathcal{P}_l$ is *good* if every $\mu' \in \mathcal{P}_l$ covers μ and

$$\mu \text{ covers } (a', b', x'_1, \ldots x'_l) \Rightarrow \{a, b, x_1, \ldots, x_l\} \cap \{a', b', x'_1, \ldots x'_l\} \neq \emptyset .$$

Following the argument above, we conclude that $\mathcal{F}_l$ must have the maximum number of good vectors. This maximum is achieved by taking the set of good vectors to be all vectors that contain some fixed vertex. As the number of good vectors is then the product of the cardinalities of the sets that do not contain this fixed vertex, we must take the fixed vertex to be in the smallest of the sets: $A \backslash \{v_1, \ldots, v_l\}$.

Now put $\mathcal{G}_0 = \mathcal{F} \setminus \mathcal{F}_0$ and $\mathcal{G}_i = \mathcal{F}_{i-1} \setminus \mathcal{F}_i$ for $1 \leq i \leq k-2$ to obtain the collection described in the statement of the theorem.

Then to obtain (10) we see that

$$|\mathcal{G}_0| = \binom{n}{k} - \binom{n-2}{k}$$

and

$$|\mathcal{G}_i| = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{n - j(d-1) - i - 2}{k-1}$$

for $1 \le i \le k-2$. $\square$

4 Random graphs

In this section we prove Theorem 5. We begin with definitions and notation. For finite sets A, B, not necessarily disjoint, let $K(A, B)$ be the graph on vertex set $A \cup B$ having edge set

$$\{\{a, b\} : a \in A, b \in B\}.$$

If $A, B \subseteq V(G)$ are disjoint vertex sets in a graph G then we set

$$e_G(A, B) = e(A, B) = |\{\{x, y\} \in E(G) : x \in A, y \in B\}|.$$

For an arbitrary hypergraph $\mathcal{H}$ we set $\nu(\mathcal{H})$ equal to the size of the largest matching in $\mathcal{H}$ (i.e. collection of pairwise disjoint edges) and $\tau(\mathcal{H})$ equal to the size of the smallest vertex cover in $\mathcal{H}$ (i.e. collection of vertices that meets all edges). Clearly,

$$\mathcal{H} \quad k\text{-uniform} \Rightarrow \tau(\mathcal{H}) \le k\nu(\mathcal{H}) . \tag{18}$$

Let $0 < p < 1$ and a positive integer $k \ge 2$ be fixed. Let $G = G_{n,p}$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a G-intersecting family. For $X \subseteq [n]$ let

$$\mathcal{F}_X = \{Y \subseteq [n] \setminus X : Y \cup X \in \mathcal{F}\} \qquad \text{and} \qquad \nu_X = |\mathcal{F}_X| .$$

In other words, ν_X is the number of sets in $\mathcal{F}$ that contain X. If $|X| = k-1$ we will think of $\mathcal{F}_X$ as a set (rather than a collection of one element sets). We say that $X \in \binom{[n]}{k-1}$ is *saturated* if

$$\nu_X \ge 8k(1-p)^{-k+1}\lambda_n$$

(note that this definition of saturated differs from that of Section 3). We form the $(k-1)$ -uniform hypergraph

$$\mathcal{H} = \mathcal{H}_{\mathcal{F}} = \left\{ X \in \binom{[n]}{k-1} : X \text{ is saturated} \right\} .$$

We prove Theorem 5 by showing that for an arbitrary G-intersecting family $\mathcal{F}$, the hypergraph $\mathcal{H}_{\mathcal{F}}$ is not too 'spread-out.' One measure of the degree to which a hypergraph is 'spread-out' is the vertex cover number. We first observe that if $\tau(\mathcal{H})$

is small then we have the desired bound on $\mathcal{F}$: If there exists a vertex cover A of $\mathcal{H}$ such that

$$|A| < 32k(k-1)(1-p)^{k-k^2}\lambda_n$$

then we have

$$\begin{aligned} |\mathcal{F}| &= \frac{1}{k}\sum_{X:|X|=k-1} \nu_X \\ &= \frac{1}{k}\left[\sum_{X \text{ saturated}} \nu_X + \sum_{X \text{ not saturated}} \nu_X\right] \\ &\le \frac{1}{k}\left[k\sum_{v\in A} \nu_{\{v\}} + \sum_{X \text{ not saturated}} \nu_X\right] \\ &\le |A|\binom{n}{k-1} + \frac{1}{k}\binom{n}{k-1}\cdot 8k(1-p)^{-k+1}\lambda_n \\ &< \left[32k(k-1)(1-p)^{k-k^2} + 8(1-p)^{-k+1}\right]\binom{n}{k-1}\lambda_n. \end{aligned} \tag{19}$$

Now, let $\mathcal{F}$ to be a G-intersecting family such that

$$|\mathcal{F}| \ge \left[32k(k-1)(1-p)^{k-k^2} + 8(1-p)^{-k+1}\right]\binom{n}{k-1}\lambda_n .$$

We show that such a family $\mathcal{F}$ exists only with probability $o(1)$. It follows from (19) that $\tau(\mathcal{H}) \ge 32k(k-1)(1-p)^{k-k^2}\lambda_n$, and by (18) there exists a matching $X_1,\dots,X_r$ in $\mathcal{H} \subseteq \binom{[n]}{k-1}$ where

$$r = \lceil 32k(1-p)^{k-k^2}\cdot\lambda_n\rceil .$$

We show that such a matching exists only with probability $o(1)$.

Why can there not exist a large collection of disjoint saturated sets with high probability? We begin with an observation about pairs of sets in the matching. Consider disjoint $X_i \ne X_j$ for which $e(X_i,X_j)=0$. For

$$x \in \mathcal{F}_{X_i} \quad \text{and} \quad y \in \mathcal{F}_{X_j}$$

it follows from the G-intersecting property that one of the following holds:

$$x \in N(X_j),\quad y \in N(X_i) \quad \text{or} \quad x \sim_G y .$$

In more global terms, if $A = \mathcal{F}_{X_i}\backslash N(X_j)$ and $B = \mathcal{F}_{X_j}\backslash N(X_i)$ then

$$K(A,B) \subseteq G . \tag{20}$$

However, if A and B are large then **whp** such subgraphs do not appear in the random graph $G_{n,p}$.

Lemma 11. *If $\mathcal{Q}$ is the event that there exists $A, B \subseteq [n]$ such that*

$$|A|, |B| > 2\log_{1/p} n \text{ and } K(A,B) \subseteq E(G),$$

then $\Pr(\mathcal{Q}) = o(1)$.

The proof of Lemma 11 is deferred to the end of this section. It follows from Lemma 11 and (20) that we can assume that for $i \neq j$

$$e(X_i, X_j) = 0 \Rightarrow \min\{|\mathcal{F}_{X_i} \backslash N(X_j)|, |\mathcal{F}_{X_j} \backslash N(X_i)|\} \leq 2\log_{1/p} n. \tag{21}$$

We shall see, using (21), that there exists a set X_a in the matching for which many of the sets $N(X_j)$ have a large intersection with $\mathcal{F}_{X_a}$. We will then finish the proof by showing that **whp** neighborhood intersections of this kind do not occur in the random graph $G_{n,p}$.

In order to establish the existence of the set X_a mentioned above, we form a digraph D on vertex set $[r]$. We connect i to j with an arc if

$$e(X_i, X_j) = 0 \quad \text{and} \quad |\mathcal{F}_{X_i} \backslash N(X_j)| < |\mathcal{F}_{X_j} \backslash N(X_i)|$$

It follows from (21) that

$$(i,j) \in D \Rightarrow |\mathcal{F}_{X_i} \backslash N(X_j)| \leq 2\log_{1/p} n. \tag{22}$$

Since G is random, the following lemma shows that D has **whp** many arcs.

Lemma 12. *If $\mathcal{R}$ is the event that there exist disjoint sets $Y_1, \ldots, Y_r \in \binom{[n]}{k-1}$ such that*

$$\left|\left\{\{i,j\} \in \binom{[r]}{2} : e(Y_i, Y_j) = 0\right\}\right| \leq \frac{1}{2}\binom{r}{2}(1-p)^{(k-1)^2}.$$

then $\Pr(\mathcal{R}) = o(1)$.

The proof of Lemma 12 is deferred to the end of this section. It follows from Lemma 12 that there exists $a \in [r]$ having out degree in D at least

$$s := \left\lceil \frac{1}{r}\left(\frac{1}{2}\binom{r}{2}(1-p)^{(k-1)^2}\right)\right\rceil \geq 8k(1-p)^{-k+1}\lambda_n - \frac{1}{4}(1-p)^{(k-1)^2}.$$

It follows from (22) and the fact that X_a is saturated that the set $S := \mathcal{F}_{X_a}$ and the collection of disjoint $(k-1)$-sets

$$\{Y_1, \ldots, Y_s\} := \{X_i : (a,i) \in D\}$$

satisfy

(1) $|S| \geq s$, and

(2) $|N(Y_i) \cap S| \geq s - 2\log_{1/p} n \qquad \text{for } i = 1, \ldots, s.$

We now complete the proof by showing that **whp** such a collection of sets does not appear in G.

Lemma 13. *If $\mathcal{S}$ is the event that there exist $S \subseteq [n]$ and a collection of disjoint sets $Y_1, Y_2, \ldots, Y_s \in \binom{[n]}{k-1}$ satisfying* (1) *and* (2) *then* $\Pr(\mathcal{S}) = o(1)$.

The proof of Lemma 13 is deferred to the end of this section.

Proof of Lemma 11. Let $q = 2\log_{1/p} n$, and for $j = 1, \ldots, q$ let $\mathcal{Q}_j$ be the event that there exists $A, B \subseteq [n]$ such that

$$|A|, |B| = q, |A \cap B| = j \text{ and } K(A, B) \subseteq E(G).$$

Consider a fixed j and set $l = q - j$. An application of the first moment method yields

$$\begin{aligned}
\Pr(\mathcal{Q}_j) &\leq \binom{n}{2l+j}\binom{2l+j}{j} \cdot p^{l^2+2lj+\binom{j}{2}} \\
&\leq \frac{n^{2l+j}}{(2l+j)!}\frac{(2l+j)!}{(2l)!j!} \cdot p^{q^2-j^2/2-j/2} \\
&= \left(n^q p^{q^2}\right) \cdot n^l p^{-j^2/2} \cdot p^{-j/2} \cdot \frac{1}{(2l)!j!} \\
&\leq p^{-j} \cdot \frac{1}{(2l)!j!} \\
&= o\left(\frac{1}{\log n}\right) .
\end{aligned}$$

□

Proof of Lemma 12. For disjoint sets $Y_1, \ldots, Y_r$ the random variable

$$E := \left|\left\{\{i, j\} \in \binom{[r]}{2} : e(Y_i, Y_j) = 0\right\}\right|$$

is distributed as $\mathrm{B}\left(\binom{r}{2}, (1-p)^{(k-1)^2}\right)$. The Chernoff bound then implies that

$$\Pr\left[E \leq \frac{1}{2}\binom{r}{2} \cdot (1-p)^{(k-1)^2}\right] < \exp\left\{-\frac{r(r-1)}{16} \cdot (1-p)^{(k-1)^2}\right\} .$$

Therefore,

$$\begin{aligned}
\Pr(\mathcal{R}) &\leq \binom{n}{k-1}^r \exp\left\{-\frac{r(r-1)}{16} \cdot (1-p)^{(k-1)^2}\right\} \\
&\leq n^{r(k-1)} \exp\left\{-2rk(1-p)^{-k+1}\lambda_n + \frac{r}{16}(1-p)^{(k-1)^2}\right\} \\
&\leq n^{r(k-1)} \exp\left\{-2rk(1-p)^{-k+1}\lambda_n + 2k(1-p)^{-k+1}\lambda_n + 1\right\} \\
&\leq O(n^{r(k-1)-(2rk-2k)(1-p)^{-k+1}}) \\
&= o(1) .
\end{aligned}$$

$\square$

Proof of Lemma 13. Let $\hat{s} = s - k + 1$. Consider $A, B \subseteq [n]$ such that $|A| = s$ and $|B| = k - 1$. The random variable

$$M_{A,B} = |\{x \in A \backslash B : e(\{x\}, B) = 0\}|$$

is stochastically dominated by $\mathrm{B}(\hat{s}, (1-p)^{k-1})$. It follows from the Chernoff bound that

$$\begin{aligned} \Pr(M_{A,B} \leq 2 \log_{1/p} n) &\leq \Pr\left[\mathrm{B}(\hat{s}, (1-p)^{k-1}) \leq \frac{1}{2} \cdot \hat{s}(1-p)^{k-1}\right] \\ &\leq \exp\left\{-\frac{1}{8} \cdot \hat{s}(1-p)^{k-1}\right\} \end{aligned}$$

Therefore, the probability that fixed $S, Y_1, \ldots, Y_s$ satisfy (1) and (2) is at most

$$\exp\left\{-\frac{1}{8} \cdot \hat{s}s(1-p)^{k-1}\right\}.$$

Thus,

$$\begin{aligned} \Pr(\mathcal{S}) &\leq \binom{n}{s}\binom{n}{k-1}^{s} \exp\left\{-\frac{1}{8} \cdot \hat{s}s(1-p)^{k-1}\right\} \\ &< n^{s+(k-1)s} \exp\left\{-\frac{1}{8} \cdot \hat{s}s(1-p)^{k-1}\right\}\left(\frac{1}{s!}\right) \\ &\leq \exp\left\{sk\lambda_n - \frac{1}{8} \cdot \hat{s}s(1-p)^{k-1}\right\}\left(\frac{1}{s!}\right) \\ &\leq \exp\left\{\frac{s}{8}\left(\frac{(1-p)^{k^2-k}}{4} + (k-1)(1-p)^{k-1}\right)\right\}\left(\frac{1}{s!}\right) \\ &= o(1) . \end{aligned}$$

$\square$

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