Midterm Exam
Math 371 Fall 2010

Instructions: This is an untimed take home exam. You may consult the class textbook (Naive Lie Theory by John Stillwell) and your notes. You may not consult any other books, or sources of any kind. You may not discuss the problems on the midterm with any person. Show all of your work on each problem.

Name:

Honor Pledge: On my honor, I have neither given nor received any unauthorized aid on this exam. I have followed the instructions outlined above.

Signature:

Problem	Score
1	
2	
3	
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Total	

1. (10pts) Let H be a normal subgroup of a Lie group G. Recall, that G is called path-connected if $\forall g, h \in G$ there exists a continuous map $f : [0,1] \to G$ such that $f(0) = g$ and $f(1) = h$.

   (a) Prove that if G is path connected, then so is G/H.

   (b) Prove that if G/H and H are path-connected then so is G.

   (c) Give an example, where G and G/H are path-connected, but H is not. (Hint: we have seen such an example in class.)

   You can assume that the projection $G \to G/H$ is a continuous map.

2. (10pts) Recall, that we proved that $O(2n+1)/\{\pm I\}$ is isomorphic to $SO(2n+1)$. Give a geometric argument why $O(2n)/\{\pm I\}$ is not isomorphic to $SO(2n+1)$. Try to be as rigorous as you can.

3. (10pts) Let $q(t)$ be a path in one of the matrix Lie groups $SO(n), SU(n)$ or $SP(n)$, such that $q(0) = I$. Find $(q(t)^{-1})'|_{t=0}$ in terms of $q'(0)$.

4. (10pts) Explain why a maximal torus of $O(n)$ is also a maximal torus in $SO(n)$.

5. (20pts) The Lie group $SU(n)$ consists of certain $n \times n$ matrices with complex entries. Such matrices can be identified with points in $\mathbb{C}^{n^2}$ and thus with points in $\mathbb{R}^{2n^2}$. Using this identification $SU(n)$ becomes a subset of $\mathbb{R}^{2n^2}$. Prove that this subset is closed and bounded. I.e. prove the following two statements.

   (a) If a sequence A_n is in $SU(n)$ and the limit $A = \lim_{n\to\infty} A_n$ exists in $\mathbb{R}^{2n^2}$ then the limit A is in $SU(n)$.

   (b) There exists a positive number r, such that under the identification above, $SU(n)$ is a subset of the ball of radius r around the origin in $\mathbb{R}^{2n^2}$.

   This proves that the Lie group $SU(n)$ is compact. The same is true about $U(n)$, $O(n)$, $SO(n)$ and $Sp(n)$ but not about $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $GL(n,\mathbb{H})$, $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$.

6. (20pts) Prove that the quotient group $O(2)/\{\pm I\}$ is isomorphic to $O(2)$ by considering the map $\phi : O(2) \to O(2)$ given by

$$\phi(A) = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\frac{1-det(A)}{2}} A.$$

7. (20pts) Which of the following groups are path-connected?

   (a) $SL(n,\mathbb{R})$ (b) $SL(n,\mathbb{C})$ (c) $GL(n,\mathbb{R})$ (d) $GL(n,\mathbb{C})$

   (e) $GL(n,\mathbb{H})$ (f) $O(n)$ (g) $U(n)$

   Prove that your answer is correct for $SL(n,\mathbb{R})$, $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. You do not need to justify your answers for the other groups. In your justifications you can use your answers to the other parts of the problem, even the ones which you didn't have to justify.