On the β-invariant for graphs*

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Abstract

We discuss the relationship between Crapo's β-invariant and the expected number of connected components of a graph. Also, we list the 3-connected graphs G with $\beta(G) < 10$.

1 Introduction

In this paper we discuss the β-invariant, which was introduced for matroids by Crapo [3]. The relevant theory is a standard topic in matroid theory texts [9, 11], although most texts on graph theory (like [1]) do not mention it. If r is the rank function of a matroid M on a set E then

$$\beta(M) = (-1)^{r(M)} \sum_{S \subseteq E} (-1)^{|S|} r(S).$$

In particular, if G is a graph then the β-invariant of (the polygon matroid of) G is

$$\beta(G) = (-1)^{|V(G)| - \omega(G)} \sum_{S \subseteq E(G)} (-1)^{|S|} (|V(G)| - \omega(G:S)),$$

where $\omega(G)$ is the number of connected components in G and for $S \subseteq E(G)$, $G : S$ is the subgraph of G with $V(G:S) = V(G)$ and $E(G:S) = S$. Some interesting properties of the β-invariant are these: $\beta(M)$ is a non-negative integer; if e is not a loop or bond in M then $\beta(M) = \beta(M/e) + \beta(M-e)$; and if $\Pi_G(k)$ is the chromatic polynomial of a graph G then $\beta(G) = |\Pi_G'(1)|$.

There are other polynomial invariants of graphs and matroids that are related to the β-invariant, aside from the chromatic polynomial. To define one such invariant, we consider the edges of a graph G to be independently subject

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to failure, with every edge e having probability $p = p(e)$ of operating successfully. The expected number of connected components of G, after the failure of some of its edges, is then a polynomial in p; it is given by

$$E\omega(G) = \sum_{S \subseteq E(G)} p^{|S|} (1-p)^{|E(G)-S|} \omega(G:S).$$

It is apparent that if $E(G) \neq \emptyset$ then up to sign, $\beta(G)$ is equal to the coefficient of $p^{|E(G)|}$ in $E\omega(G)$. It may be less apparent that in fact, $E\omega(G)$ is almost completely determined by the β-invariants of G and its subgraphs; in Section 2 we prove that it is.

Theorem 1 *If $E(G) \neq \emptyset$ then*

$$E\omega(G) - |V(G)| = (-1)^{|V(G)|-1} \sum_{\emptyset \neq S \subseteq E(G)} (-1)^{|S|+\omega(G:S)} \beta(G:S) p^{|S|}.$$

Readers familiar with network reliability will observe that Theorem 1.1 resembles the result of Satyanarayana and Khalil [6] that gives an expansion of the reliability polynomial of a graph in terms of the reliability dominations of its subgraphs. The resemblance is made more striking by Huseby's work [4] relating the reliability domination of a graph to the β-invariant of a certain extension of the graph's polygon matroid. It turns out that this resemblance is part of a general relationship between reliability and expected value that we will not discuss in detail here; the interested reader is referred to [7].

Several authors have proven theorems characterizing the matroids with particular β-values. Crapo [3] proved that the only connected matroid with $\beta = 0$ consists solely of a single loop, and that all disconnected matroids have $\beta = 0$. Brylawski [2] proved that a matroid has $\beta = 1$ iff it is the polygon matroid of a series-parallel network, and noted that the β-invariant of a parallel connection of two matroids is the product of their β-invariants. Oxley [5] listed all the 3-connected matroids with $\beta \leq 4$, and observed that the number of nonisomorphic 3-connected matroids with any given value of β must be finite. (This follows inductively from Tutte's wheels and whirls theorem [8], because there are only two matroids M with a given value of β — the wheel and whirl of the appropriate size — that do not have at least one 3-connected minor $M - e$ or M/e with a smaller β-invariant.)

Our second principal theorem is the following partial extension of Oxley's list.

Theorem 2 *A simple, 3-connected graph G with $\beta(G) \leq 9$ must be isomorphic to one of the following: a wheel graph with no more than eleven vertices; $K_{3,3}$; K_5; or one of the other graphs appearing in Figures 1–6.*

The graphs pictured in Figure 1 are those with $\beta(G) \leq 4$. Those in Figures 2, 3, 4, 5 and 6 have $\beta(G) = 5$, 6, 7, 8 and 9, respectively.

2 The $E\omega$-polynomial

In this section we prove Theorem 1.1, and also discuss several other properties of $E\omega(G)$. All of these properties generalize, *mutatis mutandis*, to the polynomial that gives the expected rank of a randomly chosen subset of a matroid; we leave the formulation of these generalizations to the reader.

Proposition 1 *Let e be any edge of a graph G. Then $E\omega(G) = pE\omega(G/e) + (1-p)E\omega(G-e)$.*

Proof. This follows immediately from these two facts: if $e \notin S \subseteq E(G)$ then $\omega(G:S) = \omega((G-e):S)$, and if $e \in S \subseteq E(G)$ then $\omega(G:S) = \omega((G/e):(S-e))$. $\square$

We leave it to the reader to use Proposition 2.1 to verify two calculations: if G is a forest (i.e., all its connected components are trees) then $E\omega(G) = |V(G)| - p\,|E(G)|$, and if G is a circuit then $E\omega(G) = p^{|V(G)|} - p\,|V(G)| + |V(G)|$.

Proposition 2 *Let G be a graph with k isthmuses. Then the derivative $E\omega(G)'$ is $-k$ when $p = 1$.*

Proof. If $|E(G)| = 0$ then $E\omega(G) = |V(G)|$ is constant and $E\omega(G)'$ is identically zero.

Suppose $|E(G)| \geq 1$, and pick an edge $e \in E(G)$. Since $E\omega(G) = pE\omega(G/e) + (1-p)E\omega(G-e)$, $E\omega(G)' = pE\omega(G/e)' + E\omega(G/e) + (1-p)E\omega(G-e)' - E\omega(G-e)$. Evaluating at $p = 1$, we conclude that $E\omega(G)'(1) = E\omega(G/e)'(1) + E\omega(G/e)(1) - E\omega(G-e)(1)$. If e is not an isthmus, $E\omega(G/e)(1) - E\omega(G-e)(1) = \omega(G/e) - \omega(G-e) = 0$, and consequently $E\omega(G)'(1) = E\omega(G/e)'(1)$. If e is an isthmus, then $E\omega(G/e)(1) - E\omega(G-e)(1) = \omega(G/e) - \omega(G-e) = -1$, and consequently $E\omega(G)'(1) = E\omega(G/e)'(1) - 1$. The proposition follows, by induction. $\square$

In particular, if G has no isthmuses at all then $p = 1$ is a critical value for $E\omega(G)$; the polynomial can have either a local minimum (as does $E\omega(K_3)$) or an inflection point (as does $E\omega(K_4)$) at $p = 1$.

Note that the degree of $E\omega(G)$ can be no higher than $|E(G)|$; we let $b(G)$ denote the coefficient of $p^{|E(G)|}$ in $E\omega(G)$. Proposition 2.1 immediately implies that $b(G) = b(G/e) - b(G-e)$ for any edge e of G.

Proposition 3 *If $E(G) = \emptyset$ then $b(G) = |V(G)|$. If $E(G) \neq \emptyset$ then*

$$(-1)^{|E(G)|-|V(G)|+\omega(G)-1} \cdot b(G) = \beta(G).$$

Proof. The first assertion is obvious.

If G has a loop ℓ then $\beta(G) = 0$; also $E\omega(G) = E\omega(G-\ell)$ is of degree no higher than $|E(G)| - 1$, and consequently $b(G) = 0$.

If G has an isthmus i that isn't the only edge of G then $\beta(G) = 0$; also $E\omega(G) = E\omega(G-i) - p$ is of degree no higher than $|E(G)| - 1$, and consequently

$b(G) = 0$. If G has only one edge and it is an isthmus, then $E\omega(G) = |V(G)| - p$, so $b(G) = -1$; since $\beta(G) = 1$ and $|E(G)| - |V(G)| + \omega(G) - 1 = -1$, the second assertion is satisfied.

Suppose now that $|E(G)| \geq 2$ and G has no loops or isthmuses; choose any $e \in E(G)$. Since $b(G) = b(G/e) - b(G - e)$ and $\beta(G) = \beta(G/e) + \beta(G - e)$, the second assertion follows by induction. □

We turn now to the proof of Theorem 1.1 of the introduction, which by Proposition 2.3 is equivalent to the assertion that

$$E\omega(G) = \sum_{S \subseteq E(G)} b(G : S) p^{|S|}.$$

If $E(G) = \emptyset$ then $E\omega(G) = |V(G)|$ and Theorem 1.1 is satisfied.

If an edge ℓ of G is a loop then $E\omega(G) = E\omega(G - \ell)$ and $b(G : S) = 0$ for every $S \subseteq E(G)$ with $\ell \in S$, so both sides of the equality are the same for G as for $G - \ell$.

If $e \in E(G)$ is not a loop then observe that if $S \subseteq E(G)$ contains e, $b(G : S) = b((G : S)/e) - b((G : S) - e) = b((G/e) : (S - e)) - b((G - e) : (S - e))$. Presuming that Theorem 1.1 applies to G/e and $G - e$, we conclude that

$$\begin{aligned}
&\sum_{e \in S \subseteq E(G)} b(G : S) p^{|S|} \\
&= \sum_{e \in S \subseteq E(G)} \left(b((G/e) : (S - e)) - b((G - e) : (S - e)) \right) p^{|S|} \\
&= \sum_{e \in S \subseteq E(G)} b((G/e) : (S - e)) p^{|S - e| + 1} - \sum_{e \notin S \subseteq E(G)} b((G - e) : S) p^{|S| + 1} \\
&= pE\omega(G/e) - pE\omega(G - e).
\end{aligned}$$

Since $E\omega(G) = pE\omega(G/e) + (1 - p)E\omega(G - e)$, Theorem 1.1 follows by induction.

Corollary 1 *The constant term of $E\omega(G)$ is $|V(G)|$. The coefficient of p in $E\omega(G)$ is $-e(G)$, where $e(G)$ is the number of non-loop edges in G. The coefficient of p^2 in $E\omega(G)$ is the number of pairs of parallel non-loop edges in G.*

Proposition 4 *Let G and G^* be dual graphs, and let $E\omega(G; 1 - p)$ be the polynomial obtained from $E\omega(G)$ by replacing p by $1 - p$. Then*

$$E\omega(G^*) + p\,|E(G)| = E\omega(G; 1 - p) - |V(G)| + 1 + |E(G)|\,.$$

Consequently, $E\omega(G; 1 - p)$and $E\omega(G^*)$ are identical except for their constant and linear terms.

Proof. The rank functions r and r^* of the polygon matroids of G and G^* are related by $r^*(S) = |S| - r(E) + r(E - S)$. It follows from this and Euler's formula that for $S \subseteq E(G)$, $\omega(G^* : (E(G^*) - S)) = \omega(G : S) + |S| + 1 - |V(G)|$. Consequently

$$E\omega(G^*) - E\omega(G; 1-p)$$

$$= E\omega(G^*) - \sum_{S \subseteq E(G)} (1-p)^{|S|} p^{|E(G)-S|} \omega\,(G : S)$$

$$= E\omega(G^*) - \sum_{S \subseteq E(G)} (1-p)^{|S|} p^{|E(G)-S|} \left(\omega(G^* : (E(G^*) - S)) - |S| - 1 + |V(G)|\right)$$

$$= -\,|V(G)| + \sum_{S \subseteq E(G)} (1-p)^{|S|} p^{|E(G)-S|} (|S| + 1)$$

$$= -\,|V(G)| + \sum_{S \subseteq E(G)} p^{|S|} (1-p)^{|E(G)-S|} (|E(G) - S| + 1).$$

This last sum may be thought of as $E\omega(T)$, where T is a tree with $|E(G)|$ edges. It follows that

$$\begin{aligned} E\omega(G^*) - E\omega(G; 1-p) &= -\,|V(G)| + |V(T)| - p\,|E(T)| \\ &= -\,|V(G)| + 1 + |E(G)| - p\,|E(G)| \end{aligned}$$

as claimed. □

3 Low values of the β-invariant

As was mentioned in the introduction, Tutte's wheels and whirls theorem [8] implies that for any integer $\beta > 0$ there are only finitely many nonisomorphic 3-connected matroids with $\beta(M) = \beta$, namely the wheel and whirl of the appropriate size and some 3-connected matroids M with 3-connected minors $N = M - e$ or $N = M/e$ such that $0 < \beta(N) < \beta$. Given a list of the 3-connected matroids N with $0 < \beta(N) < \beta$, then, one could also list all the 3-connected matroids M with $\beta(M) = \beta$ by including every 3-connected M that results from inserting a single additional element into such an N so that N is a minor of M and $\beta(M) = \beta$, and then also including the appropriate wheel and whirl. Theorem 1.2 of the introduction can be proven in this way; we leave the details to the interested reader.

The proof of Theorem 1.2 developed during the research project which led to this paper actually did not involve Tutte's wheels and whirls theorem. Instead, *ad hoc* arguments — amounting, essentially, to proofs of special cases of Tutte's theorem — were devised to guarantee that the list given in Theorem 1.2 really does include all the graphs with $\beta < 10$. Most of these arguments are perhaps not of independent value, though Theorem 3.2 below, and its corollaries, may be of some interest.

It follows from Crapo's characterization of the matroids with $\beta = 0$ [3] that the β-invariant of a graph G is 0 iff at least one of the following is true of G: $E(G) = \emptyset$, some edge of G is a loop, or $|E(G)| \geq 2$ and $\kappa\,(G) = 1$, where $\kappa\,(G)$ denotes the vertex connectivity of G. Consequently, if $|E(G)| \geq 3$ then $\beta(G)$ is unchanged by either the deletion of a parallel edge or the contraction of an edge incident on a vertex of degree $d_G(v) = 2$. We refer to these operations as *simple series-parallel reductions.* We call a graph *reducible* if it admits a simple series-parallel reduction, and *reduced* if it does not. $\beta(G) = 1$ iff G is a series-parallel network, i.e., iff G is reducible to a single edge through some sequence of simple series-parallel reductions [2].

Suppose G_1 and G_2 are graphs with $V(G_1) \cap V(G_2) = \{u, v\}$. If the edge $e = \{u, v\}$ is present in both graphs then their union $G_1 \cup G_2$ is the *parallel connection* of G_1 and G_2 with respect to e, denoted $G_1 \parallel_e G_2$, or simply $G_1 \parallel G_2$ if no confusion can arise; we refer to e as the *edge of connection.* In this situation $\beta(G) = \beta(G_1)\beta(G_2)$, and if e is not the only edge of either G_1 or G_2 then $\beta(G - e) = \beta(G_1)\beta(G_2)$ also [2].

Lemma 1 *If G is a simple graph such that $\beta\,(G) = 1$ and $|V(G)| \geq 3$ then either $G \cong K_3$ or G has two non-adjacent vertices of degree* 2.

Proof. Since $\beta\,(G) = 1$, G is a simple series-parallel network. If $|V(G)| = 3$ then $G \cong K_3$.

Suppose $|V(G)| > 3$. Since G is a simple series-parallel network, G has at least one degree-2 vertex v; let e be an edge incident on v. In G/e there can be no more than one pair of parallel edges; for if there is one, it must arise from a 3-cycle of G containing e, and since $d_G(v) = 2$ there cannot be more than one such 3-cycle. Let H be the simple graph obtained from G/e by removing one of its two parallel edges, if indeed G/e has any parallel edges; observe that $d_H(w) = d_G(w)$ for every $w \in V(G)$ that is not adjacent to v. Since $|V(G)| > |V(H)| \geq 3$, it can be presumed that either $H \cong K_3$ or H has two non-adjacent degree-2 vertices. Either way, G must have a degree-2 vertex not adjacent to v. □

Suppose A is a simple graph. Then we will call A an *accordion graph* if there is a finite sequence $C_1, ..., C_n$ of 3- and 4-cycles in A such that $n \geq 2$, C_1 and C_n are 3-cycles, for $1 \leq i < n$ the intersection of C_i and C_{i+1} consists of one edge and its two vertices, the only vertices of degree 2 in $C_1 \cup ... \cup C_n$ are the vertices of C_1 and C_n which they do not share with C_2 and C_{n-1} (respectively), and A consists of $C_1 \cup ... \cup C_n$ together with a single additional edge, which connects these two vertices of C_1 and C_n. We call this additional edge a *strap.* Note that all wheel graphs are accordions, and that every edge on the "rim" of a wheel can be considered a strap; see Figures 1(e), 2(a), 3(d), 4(c), 5(i), 5(k), 6(e), 6(g) and 6(h) for other examples of accordions.

Theorem 3 *Suppose G is a simple graph, $\beta(G) > 1$, and either G has precisely one vertex of degree 2 or G has precisely two vertices of degree 2, which are adjacent. Then G is isomorphic to $G_r \parallel_e I$ or $(G_r \parallel_e I) - e$, where G_r is a reduced graph and there is an accordion graph A such that $I = C_2 \cup ... \cup C_{n-1}$ and e is in C_2.*

Proof. If G has precisely one degree-2 vertex v, let v be adjacent to u and w. Then the 3-cycle C with $V(C) = \{u, v, w\}$ has $G = G' \parallel_e C$ or $G = (G' \parallel_e C) - e$, where $e = \{u, w\}$ and $G' - e = G - v - e$. Similarly, if v and v' are the two adjacent degree-2 vertices in G, and they are also adjacent to u and w respectively, then the 4-cycle C with $V(C) = \{u, v, v', w\}$ has $G = G' \parallel_e C$ or $G = (G' \parallel_e C) - e$.

If G' has no degree-2 vertex then it is reduced and the theorem is satisfied.

The only possible degree-2 vertices of G' are the vertices of e, and $\beta(G') = \beta(G) > 1$, so if G' is not reduced then the inductive hypothesis implies that the theorem is true for G'. Since C is a 3- or 4-cycle, this immediately implies that the theorem is true for G. □

Corollary 2 *Suppose $\kappa(G) = 2$ and G is reduced. Then there are reduced graphs G_1 and G_2 and an I as specified in the previous theorem such that G is isomorphic to $G_1 \parallel I \parallel G_2$, possibly with one or both edges of connection removed.*

Proof. Let G be isomorphic to $H_1 \parallel_e H_2$ or $(H_1 \parallel_e H_2) - e$. H_1 and H_2 must meet the criteria of the previous theorem if they are not reduced, for the only possible degree-2 vertices in either are the vertices of e, and if either has $\beta = 1$ then it must have two non-adjacent degree-2 vertices, one of which would be of degree 2 in G, an impossibility. It follows that $H_1 = G_1 \parallel I_1$ and $H_2 = G_2 \parallel I_2$ (possibly with one or both edges of connection removed), where G_1 and G_2 are reduced and I_1 and I_2 are as described in the previous theorem. Consequently $G \cong H_1 \parallel I_1 \parallel I_2 \parallel H_2 \cong G_1 \parallel I \parallel G_2$, possibly with one or both edges of connection removed. □

Corollary 3 *If $\beta(G) > 1$, G is reduced and there is an $e \in E(G)$ such that $\beta(G - e) = 1$, then G is an accordion with e as a strap.*

Proof. Let $e = \{u, v\}$ and let $w \neq u$ be a vertex adjacent to v. Let $\tilde{G} = K_4 \parallel_{\{v,w\}} (G - e)$. If $\tilde{G}$ is reduced then $G - e$ cannot have two nonadjacent degree-2 vertices; Lemma 4.1 then implies that $|V(G)| = |V(G - e)| \leq 3$. This is impossible, though, since $\beta(G) > 1$.

Hence $\tilde{G}$ is not reduced; clearly its only vertex of degree 2 is u. By Theorem 3.2, $\tilde{G} = \tilde{G}_r \parallel I$, possibly with the edge of connection removed; since $\beta(\tilde{G}_r) = 2$, Oxley's theorem implies that $\tilde{G}_r \cong K_4$. Indeed, $\tilde{G}_r$ must be the same copy of K_4 that appears in the original description $\tilde{G} = K_4 \parallel_{\{v,w\}} (G - e)$, for if not then it's a subgraph of $G - e$, an impossibility since $\beta(G - e) = 1$. Hence $G - e = I$. The "end cycles" of I must be 3-cycles because G is reduced. □

4 Characterizing graphs using the β-invariant

Certain collections of information about a graph and the β-invariant characterize the graph up to isomorphism. For instance, if G is 3-connected and one is given all the subsets $S \subseteq E(G)$ with $\beta(G : S) = 1$ then one can find the circuits of G, and by Whitney's theorem [12] this determines G up to isomorphism. It is natural to wonder what other such combinations of information serve to characterize a graph.

For instance, is it possible to characterize a 3-connected graph up to isomorphism by prescribing the sets $b_n(G) = \{b \mid \exists S \subseteq E(G) \text{ with } |S| = n \text{ and } b = \beta(G : S)\}$, $0 \leq n \leq |E(G)|$? Inspecting the graphs of Figures 1-6, we see that when $\beta(G) < 10$, $b_{|E(G)|-1}(G)$ alone is enough to characterize a graph up to isomorphism. This is not true in general, though. For instance, the graphs pictured in Figure 7 have the same $b_{|E(G)|-1}(G)$; indeed, for every $b \in b_{|E(G)|-1}(G)$ they have the same number of subsets $S \subseteq E(G)$ with $|S| = |E(G)| - 1$ and $b = \beta(G : S)$.

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