Sam – Linear stochastic homogenization – Part 2

- Let $A^\varepsilon(x,\omega) = a(T_{\frac{x}{\varepsilon}}\omega)$, with $a : \Omega \to \mathbb{R}^{d\times d}_{sym}$
  be s.t. $(*)$ $\nu_1 I \le a \le \nu_2 I$ a.s.

- $\overline{f}(x,\omega) = f(T_x\omega)$ $\rightarrow$ stationary extension : assume $(f(x_1+h), \dots, f(x_n+h))$
  has the same law on $\mathbb{R}^n$, $\forall h \in \mathbb{R}^d$.

- consider the pb:

$$-\,\mathrm{div}\big(A^\varepsilon(x,\omega)\nabla u^\varepsilon(x,\omega)\big) = f(x)$$

By $(*)$ the above eq. has a solution a.s..

- Recall that: T_x is ergodic w.r.t. μ if
  $\forall x$ $f(T_x\omega) = f(\omega)$ a.s. $\iff$ f constant a.s.

- (Deterministic) mean value: $M(f)$ if $\forall K \subseteq \mathbb{R}^d$ bdd, meas.

$$M(f) = \lim_{t\to\infty} \frac{1}{t^d|K|}\int_{tK} f(x)\,dx = \lim_{\varepsilon\to 0} \frac{1}{|K|}\int_K f\left(\frac{x}{\varepsilon}\right)dx.$$

- Ptwise ergodicity thm:
  Let $f \in L^2(\Omega)$, If T_x is ergodic, then $\overline{f}$ has a mean value a.s., which is invariant under T_x.

- Weyl's decomposition:

- Def: • $L^2_{pot}(\Omega) := \{ f \in L^2(\Omega)^d ;\ x \mapsto f(T_x\omega)$ is an L^2-gradient a.s.$\}$

  • $L^2_{sol}(\Omega) := \{ f \in L^2(\Omega)^d ;\ x \mapsto f(T_x\omega)$ is an L^2-gradient & divergence free a.s.$\}$

  • $V^2_{pot}(\Omega) := L^2_{pot}(\Omega) \cap \{ f : \mathbb{E}[f] = 0 \}$

  • $V^2_{sol}(\Omega) := L^2_{sol}(\Omega) \cap \{ f : \mathbb{E}[f] = 0 \}$

- **Thm:**

$$L^2(\Omega)^d = V^2_{pot}(\Omega) \oplus V^2_{sol}(\Omega) \oplus \mathbb{R}^d.$$

- **Thm:** It holds $A^\varepsilon(\cdot, \omega)$ a.s. H-converges to a deterministic constant matrix A°. Furthermore, A° is characterized by

$$A^\circ \xi = \mathbb{E}[a(\omega)(\xi + V_\xi(\omega))] \quad \forall \xi \in \mathbb{R}^d,$$

where V_ξ solves

$$\begin{cases} \mathbb{E}[a(\omega)(\xi + V_\xi(\omega)) \cdot \varphi(\omega)] = 0 & \forall \varphi \in V^2_{pot}(\Omega), \\ V_\xi \in V^2_{pot}(\Omega). \end{cases}$$

Proof:

- $u^\varepsilon(\cdot,\omega) \longrightarrow u^0(\cdot,\omega)$ in $H_0^1(U)$
- let $\sigma^\varepsilon(\cdot,\omega) := A^\varepsilon(\cdot,\omega)\nabla u^\varepsilon(\cdot,\omega) \xrightarrow{w} \sigma^0$ in $L^2(\Omega)$.
- we want to show that $\sigma^0 = A^0(\cdot,\omega)\nabla u^0(\cdot,\omega)$.
- let: $\cdot\, p = \xi + v_\xi$, $\mathbb{E}[p] = \xi$,
  - $q(\omega) = a(\cdot,\omega)p(\omega)$, $\mathbb{E}[q] = A^0\xi$

- $\begin{cases} p^\varepsilon(\cdot,\omega) = p(\tau_{\frac{x}{\varepsilon}}\omega) \longrightarrow \xi & w\text{-}L^2(U) \\ q^\varepsilon(\cdot,\omega) \longrightarrow A^0\xi & w\text{-}L^2(U) \end{cases}$ a.s.

- σ^ε have a fixed divergence, while p^ε have gradients converging weakly in L^2. ↳ and converge w^* σ^0

Then

$$\sigma^\varepsilon(x,\omega)\, p^\varepsilon(x,\omega) \longrightarrow \sigma_0 \xi$$

$$\shortparallel$$

$$A^\varepsilon \nabla u^\varepsilon p^\varepsilon \longrightarrow \nabla u^0 A^0 \xi \qquad \Rightarrow \sigma^0 = A^0 \nabla u^0$$

$\square$

- Examples :

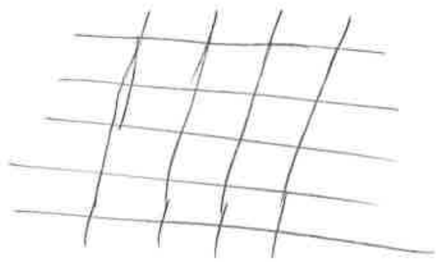
1). Periodic case : $\Omega = \mathbb{T}^d$, Lebesgue measure, and

$$\underbrace{T_x}\ \omega = \omega + x \pmod 1$$

$\hookrightarrow$ measure preserving, ergodic

We recover the usual cell problem

2) Consider a partition of $\mathbb{R}^2$ into unit cells



in each block, the coeff. of the matrix are

$A_1 \rightsquigarrow$ prob. p

$A_2 \rightsquigarrow$ prob $1-p$