The use of variational techniques in the study of sequences of elliptic operators

Working group
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- **A survey of Γ-convergence:**

1) Motivation:

given a functional F on a set X, we would like to study what happen to the minimum value

$$\inf_X F =: m_F$$

and the minimum pts

$$\{x \in X : F(x) = \inf_X F\} =: M_F$$

when we ~~move~~ perturb F.

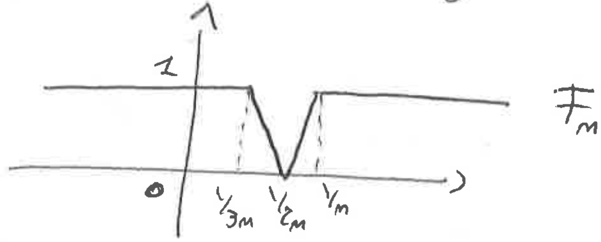
In particular we would like a notion of "perturbation", i.e., convergence, such that:

i) it is not that much restrictive, so it can be used in the applications

ii) is such that: $F_n \to F \Rightarrow \begin{cases} m_{F_n} \to m_F \\ M_{F_n} \to M_F \end{cases}$

2) Definition:

A good example that helps to understand the definition of Γ-convergence is the following:



We have:

- $\min F_m = 0$
- $M_{F_m} = \{ 1/2m \}$

Because of requirement ii), we would like our limiting function F to be:

$$F(x) := \begin{cases} 0 & x = 0, \\ 1 & x \neq 0. \end{cases}$$

This is because the sequence of minimizers of F_m is approaching $x=0$. How to capture this behaviour?
The idea is the following:

i) we fix a pt x and one ngbh $U \in \mathcal{N}(x)$

ii) we look to the behaviour of $\inf_U F_m$

iii) we localize "at x"

That is:

$$\sup_{U\in\mathcal{N}(x)} \left(\lim_m{}^{\inf}_{\sup}\right) \inf_{y\in U} F_m(y)$$

pb: they are <u>not</u> always the same!

$\Downarrow$

- <u>Def</u>: given a <u>topological</u> space (X,τ) and $F_m : X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ we define:

  - $\Gamma\text{-}\liminf_m F_m(x) := \sup_{U\in\mathcal{N}(x)} \liminf_m \inf_{y\in U} F_m(y)$,

  - $\Gamma\text{-}\limsup_m F_m(x) := \sup_{U\in\mathcal{N}(x)} \limsup_m \inf_{y\in U} F_m(y)$.

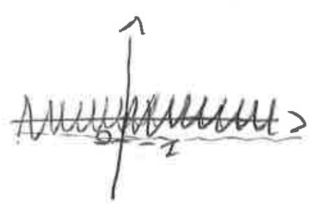
  If the above are equal, we define the common value as $\Gamma\text{-}\lim_m F_m(x) =: F(x)$.

  If this happens $\forall x\in X$, we say that $F_m \xrightarrow{\Gamma\tau} F$.

- <u>Remark</u>: the <u>topology</u> we are working with is fundamental!

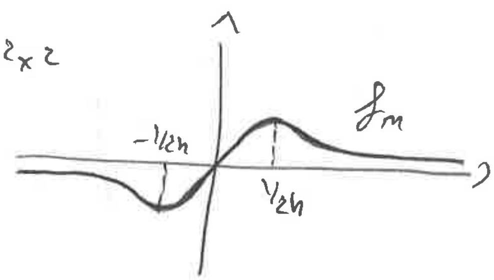
- <u>Examples</u>:

  i) $f_m(x) := \sin(mx)$



$f_m \xrightarrow{\Gamma} -1$

  ii) $f_m(x) := n x e^{-2n^2x^2}$



$f_m \xrightarrow{\Gamma} \begin{cases} -\frac{1}{2}e^{-1/2} & x=0 \\ 0 & x\neq 0.\end{cases}$

iii) assume $F_n \equiv F : X \to \overline{\mathbb{R}}$; the we have:

$$\exists\ \Gamma\text{-}\lim_m F(x) = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y) = \underbrace{\overline{F}(x)}_{\text{the relaxed functional}}$$

$$\overline{F}(x) := \sup_{G \in \mathcal{G}(F)} G(x),$$

where $\mathcal{G}(F)$ is the set of all lower-semi-continuous functions G s.t. $G \leq F$.

· Def: we say that $F : X \to \overline{\mathbb{R}}$ is l.s.c. if $\forall x \in X$ $\forall t \in \mathbb{R}$ s.t. $t < F(x)$, $\exists\, U \in \mathcal{U}(x)$ s.t. $F(y) > t$ $\forall y \in U$.

→ it is possible to see that:

$$F \text{ l.s.c. at } x \in X \iff F(x) = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y)$$

[it always holds that: $F(x) \geq \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y)$]

→ So a constant sequence can Γ-converge to something else ⇒ | Γ-convergence is **not** metrizable!

3) Sequential characterization:

The topological definition is not a very good one to work with. By assuming something more on the space X, it is possible to have an equivalent one.

- **Thm**: (strong top. - first axiom)

  Assume X satisfies the first axiom of countability [i.e. every pt has a countable base]

  Then:

$$F_n \xrightarrow{\Gamma} F \iff \begin{array}{l} i) \ \forall x, \forall x_n \to x: F(x) \leq \liminf_n F_n(x_n) \\ ii) \ \forall x \ \underbrace{\exists x_n \to x}_{\text{the recovery sequence}} \text{ s.t. } F(x) = \lim_n F_n(x_n) \end{array}$$

- **Thm**: (weak top. - separable dual)

  Assume X is a Banach space with a separable dual. Moreover, assume $\exists\, \Psi: X \to \overline{\mathbb{R}}$ with $\lim_{\|x\| \to +\infty} \Psi(x) = +\infty$ s.t. $F_n \geq \Psi$.

  Then:

$$F_n \xrightarrow{\Gamma_w} F \iff \begin{array}{l} i) \ \forall x, \forall x_n \to x: F(x) \leq \liminf_n F_n(x_n) \\ ii) \ \forall x \ \exists x_n \to x \text{ s.t. } F(x) = \lim_n F_n(x_n) \end{array}$$

- **Def:** a set $K \subset X$ is said <u>countably compact</u> if every sequence in K has a cluster pt in K

- **Def:** a function $F : X \to \overline{\mathbb{R}}$ is said <u>coercive</u> if $\overline{\{F \le t\}}$ is countably compact, $\forall t \in \mathbb{R}$.

- **Def:** $(F_n)_n$ is said to be <u>equi-coercive</u> if $\forall t \in \mathbb{R}$ $\exists K_t$ closed & countably compact s.t. $\overline{\{F_n \le t\}} \subseteq K_t$ $\forall n$.

- <u>Thm</u>: (<u>weak top. - reflexive</u>)

Let X be a reflexive Banach space.
Assume $(F_n)_n$ is equi-coercive in the weak topology.

Then:

- i) $\forall x$ $\forall x_n \rightharpoonup x$: $F(x) \le \liminf_n F_n(x_n)$
  ii) $\forall x$ $\exists x_n \rightharpoonup x$ s.t. $F(x) = \lim_n F_n(x_n)$ $\Rightarrow F_n \xrightarrow{\Gamma\text{-}w} F$

- $F_n \xrightarrow{\Gamma\text{-}w} F \Rightarrow$ i) $\forall x$ $\forall x_n \rightharpoonup x$: $F(x) \le \liminf_n F_n(x_n)$
  ii) $\forall x$ $\exists x_n \rightharpoonup x$ s.t. $F(x) = \liminf_n F_n(x_n)$

4) Γ-convergence and K-convergence:

• Def: let $(E_m)_m$ be a sequence of sets in X.
We define:

if X satisfies the first axiom of countability: limit pts

• $K\text{-}\liminf_m E_m := \{x \in X : \forall U \in \mathcal{N}(x)\ \exists k \in \mathbb{N}$ s.t. $\forall h \geq k\ U \cap E_h \neq \emptyset\}$,

cluster pts of sequences $x_m \in E_m$

• $K\text{-}\limsup_m E_m := \{x \in X : \forall U \in \mathcal{N}(x)\ \forall k \in \mathbb{N}\ \exists h \geq k$ s.t. $U \cap E_h \neq \emptyset\}$.

In case they coincide, we say that $E_m \xrightarrow{K} E$.

• Def: let $F : X \to \overline{\mathbb{R}}$; we define the epigraph of F

$$\mathrm{epi}(F) := \{(x,t) \in X \times \mathbb{R} : F(x) \leq t\}.$$

• Thm:

$$E_m \xrightarrow{K} E \iff \chi_{E_m} \xrightarrow{\Gamma} \chi_E, \text{ where } \chi_E(x) := \begin{cases} 0 & x \in E, \\ +\infty & x \notin E. \end{cases}$$

• Thm:

$$F_m \xrightarrow{\Gamma} F \iff \mathrm{epi}(F_m) \xrightarrow{K} \mathrm{epi}(F).$$

5) Γ- convergence & pointwise- convergence:

It holds:

- $\begin{cases} F_m \xrightarrow{\Gamma} F \\ F_m \to G \end{cases} \Rightarrow F \leq G$

- $F_m \to F$ uniformly $\Rightarrow F_m \xrightarrow{\Gamma} F$

- $F_m \nearrow \Rightarrow \quad \Gamma\text{-}\lim_m F_m = \sup_m \overline{F_m}$

- $F_m \downarrow F \Rightarrow \quad F_m \xrightarrow{\Gamma} \overline{F}$

In general, Γ- limit & pt-wise limit are independent, as the following example shows:

$$X = \mathbb{R}, \quad (q_m)_m = \mathbb{Q}, \quad F_m(x) := \begin{cases} 0 & \text{if } x = q_k \quad k \geq m, \\ 1 & \text{otherwise.} \end{cases}$$

Then:
- $F_m \to 1$
- $F_m \xrightarrow{\Gamma} 0$

Nevertheless there are some interesting cases where the two limits commute.

- Def: we say that $(F_m)_m$ is equi- l.s.c. at $x \in X$ if $\forall \varepsilon > 0 \quad \exists\, U \in \mathcal{N}(x)$ s.t. $F_m(y) \geq F_m(x) - \varepsilon \quad \forall y \in U, \forall m$.

- **Thm:**

Assume $(F_n)_n$ equi-l.s.c. $\rightarrow$ $F_n(x) - \varepsilon \le \inf_{y \in U} F_n$ to have the same set $U \in \mathcal{N}(x)$ and thus writing:

Then:

$$F_n \xrightarrow{\Gamma} F \iff F_n \to F \text{ pointwise}$$

- **Thm:**

Assume $(F_n)_n$ are convex & equi-bounded. $\Rightarrow (F_n)_n$ equi-continuous

Then

$$F_n \xrightarrow{\Gamma} F \iff F_n \to F \text{ ptwise.}$$

6) Properties of Γ-limits:

It holds:

a) $\Gamma\text{-}\liminf_n F_n$, $\Gamma\text{-}\limsup_n F_n$ are l.s.c.

b) $\Gamma\text{-}\liminf_n F_n = \Gamma\text{-}\liminf_n \overline{F_n}$, $\Gamma\text{-}\limsup_n F_n = \Gamma\text{-}\limsup_n \overline{F_n}$

c) $\begin{cases} F_n \le G_n \\ F_n \xrightarrow{\Gamma} F \\ G_n \xrightarrow{\Gamma} G \end{cases} \Rightarrow F \le G$

d) If σ is weaker than τ, then

U σ-open

$\Downarrow$

U τ-open $\Rightarrow$ the sup increases

$$\begin{cases} F_n \xrightarrow{\Gamma\text{-}\sigma} F_\sigma \\ F_n \xrightarrow{\Gamma\text{-}\tau} F_\tau \end{cases} \Rightarrow F_\sigma \le F_\tau.$$

$\rightarrow$ The inequality can be strict, as the following example shows:

$(a_n)_n \subset L^\infty(\Omega)$ with:

- $0 < c_1 \leq a_n(x) \leq c_2 < +\infty \quad \forall x \in \Omega$
- $a_n \xrightarrow{w^*-L^\infty} a$
- $\frac{1}{a_n} \xrightarrow{w^*-L^\infty} b$

Consider the functionals:

$$F_n(u) := \int_\Omega a_n u^2 dx.$$

Then:

- $F_n \xrightarrow{\Gamma-L^2} F \quad , \quad F(u) := \int_\Omega a u^2 \,,$
- $F_n \xrightarrow{\Gamma-w L^2} G \quad , \quad G(u) := \int_\Omega b u^2 .$

Indeed:

- $F_n \to F$ pointwise, F_n convex, $F_n(u) \leq c_2 \int_\Omega u^2 \Rightarrow F_n \xrightarrow{\Gamma-L^2} F$
- since L^2 is reflexive & $F_n(u) \geq c_1 \int_\Omega u^2$, we can use the sequential characterization to prove the weak-Γ-limit.

So: fix $u \in L^2(\Omega)$

- let $u_n := \frac{b}{a_n} u$; then:
  - $u_n \xrightarrow{w\text{-}L^2} u$
  - $F_n(u_n) \longrightarrow G(u)$
- let $v_n \xrightarrow{w\text{-}L^2} u$; write:

$$\int_\Omega a_n v_n^2 = \int_\Omega a_n \big((v_n - u_n) + u_n\big)^2$$

$$\geq \int_\Omega 2 a_n u_n (v_n - u_n) + a_n u_n^2$$

$$= -\int_\Omega a_n u_n^2 + \int_\Omega 2 b u v_n$$

$$\underset{n \text{ large}}{\geq} -\int_\Omega a_n u_n^2 + \int_\Omega 2 b u^2 - \varepsilon$$

$$\longrightarrow G(u) - \varepsilon.$$

$\square$

e) <u>Γ- convergence of a sum:</u>

- $\begin{cases} F_n \xrightarrow{\Gamma} F \\ G_n \xrightarrow{\Gamma} G \\ F_n + G_n \xrightarrow{\Gamma} H \end{cases} \Longrightarrow F + G \leq H$

$\liminf (a_n + b_n) \geq \liminf_n a_n + \liminf_n b_n$

can be strict:

$F_n(x) := -\sin(nx)$

$G_n(x) := \sin(nx)$

$\Longrightarrow F_n \xrightarrow{\Gamma} -1$

$G_n \xrightarrow{\Gamma} -1$

$F_n + G_n \xrightarrow{\Gamma} 0$

this is necessary:

$F_n(x) := \sin(nx)$

$G_n(x) := (-1)^n \sin(nx)$

$\Longrightarrow F_n \xrightarrow{\Gamma} -1$

$G_n \xrightarrow{\Gamma} -1$

$\Longrightarrow F_n + G_n \not\xrightarrow{\Gamma}$, Γ-liminf $= -2$

Nevertheless, under some additional hypothesis, we have equality:

- $$\begin{cases} F_n \xrightarrow{\Gamma} F \\ G_n \to G \text{ uniformly} \\ G, G_n \text{ everywhere } \underline{\text{finite}} \end{cases} \Longrightarrow F_n + G_n \xrightarrow{\Gamma} F + G$$

to have everything well-defined

- $$\begin{cases} F_n \xrightarrow{\Gamma} F \\ G \ \underline{\text{continuous}},\ \underline{\text{finite}} \end{cases} \Longrightarrow F_n + G \xrightarrow{\Gamma} F + G$$

<u>essential:</u> $F_n(x) := \arctan(nx + (-1)^n n) \xrightarrow{\Gamma} \begin{cases} -\pi/2 & x \le 0 \\ \pi/2 & x > 0 \end{cases}$

$G(x) := \begin{cases} \pi & x < 0 \\ 0 & x \ge 0 \end{cases}$

$$\Gamma\text{-}\liminf_n (F_n + G) = \begin{cases} -\pi/4 & x = 0, \\ \pi/2 & x \neq 0, \end{cases} \quad \Gamma\text{-}\limsup_n (F_n + G) = \begin{cases} \pi/4 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

- $$\begin{cases} F_n \xrightarrow{\Gamma} F, & F_n \to F \text{ ptwise} \\ G_n \xrightarrow{\Gamma} G, & G_n \to G \text{ ptwise} \end{cases} \Longrightarrow F_n + G_n \xrightarrow{\Gamma} F + G$$

F) <u>compactness</u>:

- if X has a <u>countable base</u>, then every sequence $(F_n)_n$ has a Γ-convergent subsequence.
- if X is a Banach space with a <u>separable dual</u>. Let $(F_n)_n$ s.t. $F_n \ge \psi$ with $\lim_{\|x\| \to +\infty} \psi(x) = +\infty$. Then there exists a subsequence of $(F_n)_n$ which Γ-converges in the weak topology.

6) Urysohn property:

- if X satisfies the first axiom of countability, then
  $F_n \xrightarrow{\Gamma} F \iff$ every subsequence of $(F_n)_n$ has a further subsequence Γ-converging to F

- if X is a Banach space that is either reflexive or with a separable dual, and $F_n \geq \psi$, $\lim_{\|x\| \to +\infty} \psi(x) = +\infty$, then:

  $F_n \xrightarrow{\Gamma - w} F \iff$ every subsequence of $(F_n)_n$ has a further subsequence that Γ-converges in the weak topology to F.

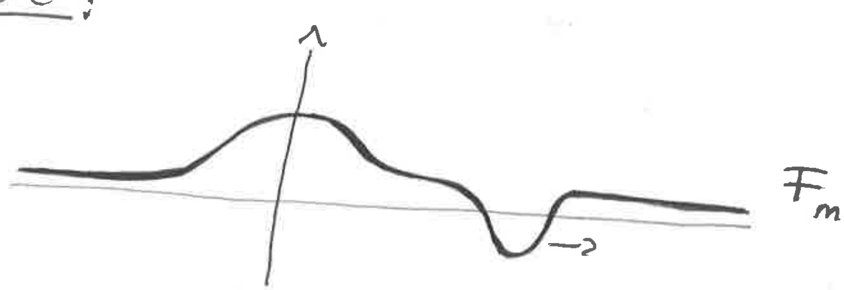
7) Convergence of minima and minimizers:

- Thm:

  Assume $(F_n)_n$ is equi-coercive. Then:

$$F_n \xrightarrow{\Gamma} F \implies \begin{cases} \cdot\ F \text{ coercive} \\ \cdot\ \min_X F = \lim_n \inf_X F_n . \end{cases}$$

- Example:



- Def: Let $F : X \to \overline{\mathbb{R}}$, $\varepsilon > 0$.
  We say that $x \in X$ is an <u>ε-minimizer</u> of F if

$$F(x) \le \left(\inf_X F + \varepsilon\right) \vee \underbrace{\left(-\frac{1}{\varepsilon}\right)}_{\text{To treat the case } \inf_X F = -\infty}$$

- <u>Thm</u>:

Assume $(F_n)_n$ is <u>equi-coercive</u>, and that $F_n \xrightarrow{\Gamma} F \not\equiv +\infty$.
Then:

- if x_n is an ε_n-minimizer, $\varepsilon_n \to 0$ and $x_n \to x$, then x is a minimizer of F
- if x is a minimizer of F, then $\exists (x_n)_n$, x_n <u>ε_n-minimizer</u>, $\varepsilon_n \to 0$, s.t. $x_n \to x$.

8) Integral Functionals:

Consider, for $0 < c_0 \leq c_1$, $p > 1$, the class $\mathcal{F}$ of all functionals $F: L^p(\Omega) \to [0,+\infty]$ that can be written as:

$$F(u) = \int_\Omega f(x, Du)\,dx,$$

where $f: \Omega \times L^p(\Omega) \to [0,+\infty]$ is a Borel function s.t.

$$c_0|\xi|^p \leq f(x,\xi) \leq c_1(1+|\xi|^p),$$

for all $\xi \in \mathbb{R}^N$, $x \in \Omega$.

- Thm: [Γ-compactness of $\mathcal{F}$]

  Let $(F_n)_n$ be a sequence of functionals in $\mathcal{F}$. Then, there exists a subsequence $(F_{n_k})_k$ and a functional $F \in \mathcal{F}$ s.t. $F_{n_k} \xrightarrow{\Gamma\text{-}L^p} F$.

- Thm:

  Assume $f_n(\cdot,\xi) \to f(\cdot,\xi)$ a.e. in Ω for all $\xi \in \mathbb{R}^N$. Then $F_n \xrightarrow{\Gamma\text{-}L^p} F$.