· Peter - Nonlinear stochastic homogenization

Consider: $(1)_\varepsilon \begin{cases} F(D^2u, Du, u, x, \frac{x}{\varepsilon}, \omega) = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

→ there exists a solution $u_\varepsilon(\cdot,\omega)$ (in the viscosity sense)

· **Question:** $u_\varepsilon(\cdot,\omega) \overset{?}{\to} u_0$, u_0 deterministic?

We will see that that is the case, with uniform convergence, where u_0 solves:

$$\begin{cases} \bar{F}(D^2u, Du, u, x) = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

→ See papers:

· Hypothesis on F:

· $(\Omega, \mathcal{F}, P)$; probability space

· $(T_x)_{x \in \mathbb{R}^d} : \Omega \to \Omega$ s.t.
- group · $T_x T_y = T_{x+y}$
- measure pres. · $P(T_x A) = P(A)$
- ergodic · if $T_x A = A$ $\forall x \in \mathbb{R}^d$ $\Rightarrow P(A) \in \{0, 1\}$.

· $F : \mathbb{R}^{d \times d}_{sym} \times \mathbb{R}^d \times \mathbb{R} \times \bar{U} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be s.t.

1) $F(M, p, u, x, y+h, \omega) = F(M, p, u, x, y, T_h\omega)$ $\forall h$

[ergodicity]

2) • $\forall N \geq M \quad F(N, p, u, x, y, \omega) \leq F(M, p, u, x, y, \omega) - \lambda |N - M|$

[ellipticity]

• if $u \leq v$, then $F(M, p, u, x, y, \omega) \leq F(M, p, v, x, y, \omega)$

3) F is uniformly Lipschitz in M, p and equicontinuous in u, x, y. [a strong hypothesis]

• Examples:

i) the Monge-Ampère eq.:

$$\begin{cases} -\det D^2 u + \dfrac{f(x,\omega)}{g(Du(x),\omega)} = 0 \\ u \text{ convex} \end{cases}$$

ii) minimal surface eq.:

$$\min \int a(x,\omega) \sqrt{|Du|^2 + 1}$$

$$\rightsquigarrow \quad -\operatorname{div}\left(a(x,\omega) \frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$$

- For simplicity, remove u, x, w and let us put ourselves in 1D: $(0,R) = U$

$$F(u'', y) = -u'' - a(y)$$

$$\Big\Downarrow$$

$$\begin{cases} -u'' - a\left(\frac{x}{\varepsilon}\right) = 0 \\ u = 0 \end{cases} \rightsquigarrow u(x) \approx \frac{a}{2}(x^2 - R^2) + \varepsilon^2 \underbrace{\bar{u}\left(\frac{x}{\varepsilon}\right)}_{\text{corrector function}}$$

$\lim_{R \to \infty} \int_0^R a(y)\,dy$

$$\Big\Downarrow$$

$$\bar{F}(u'') = -u'' - a$$

$\rightsquigarrow$ another way to look at it is the following:

> $\bar{F}(M)$ is the unique λ s.t.
> the solution to $\begin{cases} F\left(u'', \frac{x}{\varepsilon}\right) = \lambda & \text{in } (0,R), \\ u = 0 & \text{at } \pm R, \end{cases}$
> converges uniformly to $\frac{M}{2}(|x|^2 - R^2)$.

$\longrightarrow$ the above is difficult to prove.

So, instead, we consider another pb.

- Obstacle problem:

Solve:

$$(O)_{F,V,\omega} \begin{cases} \min\left(F(D^2u, Du, y, \omega), u\right) = 0 & \text{in } V, \\ u = 0 & \text{on } \partial V. \end{cases}$$

$\rightarrow$ there exists a viscosity solution w defined as:

$$w_{F,V,\omega}(x) := \inf\left\{ u(x) : u \geq 0 \text{ in } \bar{V}, \; F(D^2u, Du, y, \omega) \geq 0 \text{ in } V \right\}$$

[good if V has a Lipschitz boundary]

Properties:

i) $F \geq G \Rightarrow w_{F,V,\omega} \leq w_{G,V,\omega}$

ii) if $V \subseteq W$, then $w_{F,V,\omega} \leq w_{F,W,\omega}$ in $\bar{V}$

- Let $m(F, V, \omega) := \left| \{ x \in V : w_{F,V,\omega} = 0 \} \right|$.

iii) $F \geq G \Rightarrow m(F, V, \omega) \geq m(G, V, \omega)$

iv) if $U_1, \dots, U_n, V \in \mathcal{L}$, $|V \setminus U_1 \cup \dots \cup U_n| = 0$,

$\mathcal{L}$ Lipschitz bounded sets

then

$$m(F, V, \omega) \leq \sum_{i=1}^{n} m(F, U_i, \omega).$$ [subadditivity]

v) m is stationary & ergodic

We recall the following result:

- <u>Thm</u>:

  If $\mu : \Omega \times \mathcal{L} \to \mathbb{R}$ is subadditive and stationary, then $\exists\ \bar{\mu} : \Omega \to \mathbb{R}$ s.t.

$$\lim_{t \to +\infty} \frac{\mu(tV, \omega)}{|tV|} = \bar{\mu}(\omega),$$

  for a.e. $\omega \in \Omega$, every $V \in \mathcal{L}$.

  Moreover, if μ is ergodic, then $\bar{\mu}$ is constant.

→ The idea is to apply the above result to the function $m(F, \cdot, \cdot) \rightsquigarrow \bar{m}(F, \cdot)$

→ How to find the limiting operator:

define

$$F_{M,p}(N, Y, \omega) := F(M+N, p, Y, \omega).$$

and consider the function:

$$\alpha \longmapsto \bar{m}(F_{M,p} - \alpha).$$

That function is decreasing, and will eventually go to zero.

We thus set:

$$\overline{F}(M,p) := \sup\{\alpha : \overline{m}(F_{M,p}-\alpha) > 0\}.$$

It turns out that $\overline{F}$ is strongly elliptic and Lipschitz in M, p.

- We want to sketch the proof of the homogenization:

fix $\omega \in \Omega_0$; take $\overline{u}(x,\omega) := \limsup_{\varepsilon\to 0} u_\varepsilon(x,\omega)$.

Show that: $\overline{F}(D^2\overline{u}, D\overline{u}) \le 0$ in the viscosity sense.

So, fix $x_0 \in U$, take $\varphi \in C^2(\mathbb{R}^d)$ s.t. $\varphi - \overline{u}$ has a strict local minimum at x_0.

Assume $\overline{F}(D^2\varphi, D\varphi) = \theta > 0$.

For simplicity, assume $x_0 = 0$.

Take $\varphi_\varepsilon(x) := \varphi(x) + \varepsilon^2 w_{F_{M,p}-\frac{\theta}{2}, B_{\frac{r}{\varepsilon}}, \omega}\left(\frac{x}{\varepsilon}\right)$, where

$M := D^2\varphi(x_0)$, $p = D\varphi(x_0)$, $r \ll 1$.

<u>Fact</u> [not exactly true]: $F(D^2\varphi_\varepsilon, D\varphi_\varepsilon, \frac{x}{\varepsilon}, \omega) \ge \frac{\theta}{4}$

[True if we consider the Yosida transform]

Use that $F(D^2u_\varepsilon, Du_\varepsilon, \frac{x}{\varepsilon}, \omega) = 0$.

For $\varepsilon << 1$ it holds:

$$\varphi_\varepsilon(x_0) - u_\varepsilon(x_0) < \inf_{\partial B(x_0,\varepsilon)} (\varphi_\varepsilon - u_\varepsilon) =: c$$

$\Rightarrow \varphi_\varepsilon(x_0) < u_\varepsilon(x_0) + c$

$\rightarrow \varphi_\varepsilon(x) > u_\varepsilon(x) + c$ for $x \in \partial B(x_0, \varepsilon)$

$$F(D^2\varphi_\varepsilon, D\varphi_\varepsilon, \tfrac{x}{\varepsilon}, \omega) \geq F(D^2(u_\varepsilon + c), D(u_\varepsilon + c), \tfrac{x}{\varepsilon}, \omega)$$

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$\square$