H-Convergence

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Foreword to the English Translation

The article is the translation of notes originally written in French that were intended as a first draft for a joint book which has yet to be written. These notes presented part of the material that Luc Tartar taught in his *Cours Peccot* at the Collège de France in March 1977 and were also based on a series of lectures given by François Murat at Algiers University in March 1978. They were subsequently reproduced by mimeograph in the *Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger 1977/78* under the signature of only François Murat. We have chosen to return to our original project by cosigning the present translation.

We would like to note that a small change in the definition of the set $M(\alpha, \beta, \Omega)$, which is introduced and used in the following, would result in an improvement of the presentation of these notes. Indeed, define $M'(\alpha, \gamma, \Omega)$ as the set of those matrices $A \in [L^\infty(\Omega)]^{N^2}$ which are such that $(A(x)\lambda, \lambda) \geq \alpha \mid \lambda \mid^2$ and $((A)^{-1}(x)\lambda, \lambda) \geq \gamma \mid \lambda \mid^2$ for any λ in $\mathbf{R}^N$ and a.e. x in Ω. A proof similar to that presented hereafter implies that the H-limit of a sequence of matrices of $M'(\alpha, \gamma, \Omega)$ also belongs to $M'(\alpha, \gamma, \Omega)$, whereas the H-limit of a sequence of matrices of $M(\alpha, \beta, \Omega)$ only belongs to $M(\alpha, (\beta^2/\alpha), \Omega)$ when the matrices are not symmetric.

1 Notation

Ω is an open subset of $\mathbf{R}^N$.
$\omega \subset\subset \Omega$ denotes a bounded open subset ω of Ω such that $\overline{\omega} \subset \Omega$.
$\alpha, \beta, \alpha', \beta'$ are strictly positive real numbers satisfying

$$0 < \alpha < \beta < +\infty ,$$

$$0 < \alpha' < \beta' < +\infty.$$

$(\cdot, \cdot)$ and $\mid \cdot \mid$ respectively denote the euclidean inner product and norm on $\mathbf{R}^N$.
$(e_1, \ldots, e_N)$ is the canonical basis of $\mathbf{R}^N$.
$E = \{\epsilon = 1/n \,:\, n \in \mathbf{Z}^+ - \{0\}\}$.
$E', E'', \ldots$ are infinite subsets of E (subsequences).

$$M(\alpha, \beta, \Omega) = \{A \in [L^\infty(\Omega)]^{N^2} \,:\, (A(x)\lambda, \lambda) \geq \alpha \mid \lambda \mid^2, \mid A(x)\lambda \mid \leq \beta \mid \lambda \mid \text{ for any } \lambda \in \mathbf{R}^N \text{ and a.e. } x \text{ in } \Omega\}.$$

A. Cherkaev et al. (eds.), *Topics in Mathematic Modelling of Composite Materials*

If A is an element of $M(\alpha, \beta, \Omega)$ and u is an element of $H_0^1(\Omega)(= W_0^{1,2}(\Omega))$,

$$-div\,(A\,grad\,u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\sum_{j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j}).$$

2 Introductory Remarks

Let $A^\epsilon, \epsilon \in E$, be a sequence of elements of $M(\alpha, \beta, \Omega)$. Then, for any ϵ, any bounded open set Ω, and any f in $H^{-1}(\Omega)$, there exists a unique solution of

$$\begin{cases} -div\,(A^\epsilon grad\,u^\epsilon) = f & \text{in } \Omega, \\ u^\epsilon \in H_0^1(\Omega). \end{cases}$$

Furthermore one has

$$\alpha \parallel u^\epsilon \parallel_{H_0^1(\Omega)} \leq \parallel f \parallel_{H^{-1}(\Omega)} ,$$

which implies the existence of a subsequence E' such that, for ϵ in E',

$$u^\epsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega).$$

The following question is raised: does u^0 satisfy an equation of the same type as that satisfied by u^ϵ?

Whenever the matrices A^ϵ converge almost everywhere to a matrix A^0, A^ϵ converges to A^0 in $[L^p(\Omega)]^{N^2}$ for any finite p, and the weak limit of $A^\epsilon grad\,u^\epsilon$ in $[L^2(\Omega)]^N$ is $A^0 grad\,u^0$ (for ϵ in E'). Therefore u^0 is the solution of

$$\begin{cases} -div\,(A^0 grad\,u^0) = f & \text{in } \Omega, \\ u^0 \in H_0^1(\Omega). \end{cases}$$

Note that the uniqueness of u^0 is ensured because the pointwise limit A^ϵ of A^0 belongs to $M(\alpha, \beta, \Omega)$.

In the absence of pointwise convergence of the matrices A^ϵ the setting is drastically different, as illustrated by the one-dimensional case.

3 The One-Dimensional Case

Set $\Omega = (0, 1)$, take f in $L^2(\Omega)$ and A^ϵ in $M(\alpha, \beta, \Omega)$, which is here just $M(\alpha, \beta, \Omega) = \{A^\epsilon \in L^\infty(\Omega) \ : \alpha \leq A^\epsilon(x) \leq \beta \ \text{ a.e. in } \Omega\}$.

Define u^ϵ as the unique solution of

$$\begin{cases} -\frac{d}{dx}(A^\epsilon \frac{du^\epsilon}{dx}) = f & \text{in } \Omega, \\ u^\epsilon \in H_0^1(\Omega). \end{cases}$$

Since $\alpha \parallel u^\epsilon \parallel_{H_0^1(\Omega)} \leq \parallel f \parallel_{H^{-1}(\Omega)}$, a subsequence E' of E is such that

$$u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)\ ,\ \epsilon \in E'.$$

Set $\xi^\epsilon = A^\epsilon \dfrac{du^\epsilon}{dx}$. The function ξ^ϵ is bounded in $H^1(\Omega)$ because

$$\| \xi^\epsilon \|_{L^2(\Omega)} \leq \frac{\beta}{\alpha} \| f \|_{H^{-1}(\Omega)} \quad \text{and} \quad \frac{d\xi^\epsilon}{dx} = -f \quad \text{in } \Omega.$$

Hence a subsequence E'' of E' is such that

$$\xi^\epsilon \to \xi^0 \quad \text{strongly in } L^2(\Omega),\ \epsilon \in E''.$$

Since A^ϵ belongs to $M(\alpha, \beta, \Omega)$,

$$\frac{1}{\beta} \leq \frac{1}{A^\epsilon(x)} \leq \frac{1}{\alpha} \quad \text{a.e. in } \Omega,$$

and a subsequence E''' of E is such that

$$\frac{1}{A^\epsilon} \rightharpoonup \frac{1}{A^0} \quad \text{weak-* in } L^\infty(\Omega), \epsilon \in E'''.$$

Furthermore A^0 belongs to $M(\alpha, \beta, \Omega)$.

The limit of each side of the equality

$$\frac{1}{A^\epsilon}\xi^\epsilon = \frac{du^\epsilon}{dx},\ \epsilon \in E''' ,$$

is computable and it yields

$$\frac{1}{A^0}\xi^0 = \frac{du^0}{dx}.$$

Since $\dfrac{d\xi^0}{dx} = -f$, u^0 is a solution of

$$\begin{cases} -\dfrac{d}{dx}(A^0 \dfrac{du^0}{dx}) = f & \text{in } \Omega, \\ u^0 \in H_0^1(\Omega), \end{cases}$$

and it is unique because A^0 belongs to $M(\alpha, \beta, \Omega)$.

Note that if B^0 is the weak-* limit in $L^\infty(\Omega)$ of A^ϵ for a subsequence E'''' of E''', then A^0 is generally different from B^0 as easily seen upon consideration of the following example:

$$\begin{cases} A^\epsilon(x) = \alpha \quad , & k\epsilon \leq x < (k + \frac{1}{2})\epsilon, \\ A^\epsilon(x) = \beta \quad , & (k + \frac{1}{2})\epsilon \leq x < (k+1)\epsilon, \end{cases}$$

with $k \in \mathbf{Z}^+$, in which case

$$\begin{cases} \dfrac{1}{A^0} = \dfrac{1}{2}(\dfrac{1}{\alpha} + \dfrac{1}{\beta}), \\ B^0 = \dfrac{1}{2}(\alpha + \beta). \end{cases}$$

The reader should, however, refrain from drawing the hasty conclusion that weak-* convergence in $[L^\infty(\Omega)]^{N^2}$ of the inverse matrices $(A^\epsilon)^{-1}$ of A^ϵ is the key to the understanding of the problem in the $N-$dimensional case. Consider, for example, the following setting.

4 Layering

A sequence $A^\epsilon, \epsilon \in E$, of elements of $M(\alpha, \beta, \Omega)$ such that $A^\epsilon(x) = A^\epsilon(x_1)$ is investigated. Since it satisfies

$$A^\epsilon_{11}(x) = (A^\epsilon(x)e_1, e_1) \geq \alpha \mid e_1 \mid^2 = \alpha,$$

a subsequence E' of E is such that

$$\begin{cases} \dfrac{1}{A^\epsilon_{11}} \rightharpoonup \dfrac{1}{A^0_{11}}, \\ \dfrac{A^\epsilon_{i1}}{A^\epsilon_{11}} \rightharpoonup \dfrac{A^0_{i1}}{A^0_{11}}, \quad i > 1, \\ \dfrac{A^\epsilon_{1j}}{A^\epsilon_{11}} \rightharpoonup \dfrac{A^0_{1j}}{A^0_{11}}, \quad j > 1, \\ A^\epsilon_{ij} - \dfrac{A^\epsilon_{i1}A^\epsilon_{1j}}{A^\epsilon_{11}} \rightharpoonup A^0_{ij} - \dfrac{A^0_{i1}A^0_{1j}}{A^0_{11}}, \quad i > 1,\ j > 1, \end{cases} \tag{1}$$

for $\epsilon \in E'$. The convergences in equation (1) are to be understood as weak-* convergences in $L^\infty(\Omega)$.

If Ω is bounded and f is an element of $L^2(\Omega)$, the solution u^ϵ of

$$\begin{cases} -div\,(A^\epsilon grad\, u^\epsilon) = f \quad \text{in } \Omega, \\ u^\epsilon \in H^1_0(\Omega)\,, \end{cases}$$

is such that, for a subsequence E'' of E',

$$u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H^1_0(\Omega)\,, \epsilon \in E''.$$

Let us prove that u^0 is the solution of

$$\begin{cases} -div\,(A^0 grad\, u^0) = f & \text{in } \Omega, \\ u^0 \in H_0^1(\Omega)\,, \end{cases} \tag{2}$$

with A^0 defined through (1).

Let $\omega = \prod_{i=1}^{N} (a_i, b_i)$ be a rectangle such that $\omega \subset \Omega$. Set $\omega' = \prod_{i=2}^{N} (a_i, b_i)$ and

$$\xi_i^\epsilon = \sum_{j=1}^{N} A_{ij}^\epsilon \frac{\partial u^\epsilon}{\partial x_j}, \quad 1 \le i \le N.$$

Each of the ξ_i^ϵ's is bounded in $L^2(\omega)$ and

$$-\frac{\partial \xi_1^\epsilon}{\partial x_1} = f + \sum_{i=2}^{N} \frac{\partial \xi_i^\epsilon}{\partial x_i}\,.$$

Thus ξ_1^ϵ is bounded in $H^1((a_1, b_1); H^{-1}(\omega'))$.

The identity mapping from $L^2(\omega')$ into $H^{-1}(\omega')$ is compact, which implies, by virtue of Aubin's compactness lemma, that

$$\Xi = \{\xi \in L^2((a_1, b_1)); L^2(\omega')) \,:\, \frac{\partial \xi}{\partial x_1} \in L^2((a_1, b_1)); H^{-1}(\omega'))\}$$

is compactly embedded in $L^2((a_1, b_1); H^{-1}(\omega'))$. Thus, at the expense of extracting a subsequence E''' of E'', we are at liberty to assume that

$$\begin{cases} \xi_i^\epsilon \rightharpoonup \xi_i^0 & \text{weakly in } L^2(\omega)\,, \\ \xi_1^\epsilon \rightarrow \xi_1^0 & \text{strongly in } L^2((a_1, b_1)); H^{-1}(\omega'))\,, \\ u^\epsilon \rightharpoonup u^0 & \text{strongly in } L^2(\omega)\,, \end{cases} \tag{3}$$

for ϵ in E'''.

But A_{ij}^ϵ is a function of x_1 and only x_1, thus

$$\frac{\partial u^\epsilon}{\partial x_1} + \sum_{j=2}^{N} \frac{\partial}{\partial x_j}(\frac{A_{1j}^\epsilon}{A_{11}^\epsilon} u^\epsilon) = \frac{1}{A_{11}^\epsilon} \xi_1^\epsilon,$$

$$\xi_i^\epsilon = \frac{A_{i1}^\epsilon}{A_{11}^\epsilon} \xi_1^\epsilon + \sum_{j=2}^{N} \frac{\partial}{\partial x_j}((A_{ij}^\epsilon - \frac{A_{i1}^\epsilon A_{1j}^\epsilon}{A_{11}^\epsilon}) u^\epsilon), \quad i > 1.$$

The limit of every single term in the preceding equalities is immediately computable upon recalling equations (1) and (3). For example, if φ is an arbitrary element of $\mathcal{C}_0^\infty(\omega)$,

$$\int_\omega \frac{1}{A_{11}^\epsilon}\xi_1^\epsilon \varphi dx =< \xi_1^\epsilon, \frac{1}{A_{11}^\epsilon}\varphi > ,$$

where $< \, , \, >$ stands for the duality bracket between $L^2((a_1,b_1);H^{-1}(\omega'))$ and $L^2((a_1,b_1);H_0^1(\omega'))$. We finally obtain

$$\xi_i^0 = \sum_{j=1}^N A_{ij}^0 \frac{\partial u^0}{\partial x_j}, \quad i \geq 1,$$

which yields equation (2) because $-div\, \xi^0 = f$ in ω , and $\omega \subset \Omega$ is arbitrary.

5 Definition of the *H*-Convergence

Definition 1 *A sequence $A^\epsilon, \epsilon \in E$, of elements of $M(\alpha,\beta,\Omega)$ $H-$converges to an element A^0 of $M(\alpha',\beta',\Omega)$ $(A^\epsilon \overset{H}{\rightharpoonup} A^0)$ if and only if, for any $\omega \subset\subset \Omega$ and any f in $H^{-1}(\omega)$, the solution u^ϵ of*

$$\begin{cases} -div\,(A^\epsilon grad\, u^\epsilon) = f & in\ \omega, \\ u^\epsilon \in H_0^1(\omega)\ , \end{cases} \tag{4}$$

is such that

$$\begin{cases} u^\epsilon \rightharpoonup u^0 & weakly\ in\ H_0^1(\omega)\ , \\ A^\epsilon grad\, u^\epsilon \rightharpoonup A^0 grad\, u^0 & weakly\ in\ [L^2(\omega)]^N\ , \end{cases} \tag{5}$$

for $\epsilon \in E$, where u^0 is the solution of

$$\begin{cases} -div\,(A^0 grad\, u^0) = f & in\ \omega\ , \\ u^0 \in H_0^1(\omega). \end{cases}$$

Remarks

1. According to the results obtained in Sections 2, 3, and 4, the following results hold true:
   (i) If A^ϵ converges to A^0 a.e. in Ω, then $A^\epsilon \overset{H}{\rightharpoonup} A^0$.
   (ii) If $N=1$, $A^\epsilon \overset{H}{\rightharpoonup} A^0$ if and only if $\frac{1}{A^\epsilon} \rightharpoonup \frac{1}{A^0}$ weak-* in $L^\infty(\Omega)$, as easily seen upon approximation in $H^{-1}(\Omega)$ of f by functions of $L^2(\Omega)$ (see Section 3).
   (iii) If $A^\epsilon(x) = A^\epsilon(x_1)$, and if $A^\epsilon \overset{H}{\rightharpoonup} A^0$, equation (1) is satisfied. Conversely if (1) is satisfied, then it can be shown that A^0 is coercive and Section 4 implies that $A^\epsilon \overset{H}{\rightharpoonup} A^0$.

2. If equation (4) is interpreted as the equation for the electrostatic potential u^ϵ, A^ϵ as the tensor of dielectric permittivity , $E^\epsilon = grad\, u^\epsilon$ as the electric field, and $D^\epsilon = A^\epsilon grad\, u^\epsilon$ as the polarization field, then convergence (5) is a statement about the weak convergence of the fields E^ϵ and D^ϵ. It is shown later on that the electrostatic energy $e^\epsilon = (D^\epsilon, E^\epsilon) = (A^\epsilon grad\, u^\epsilon, grad\, u^\epsilon)$ is also a weakly converging quantity.

3. The concept of H-convergence generalizes that of Gconvergence introduced by Spagnolo (see, for example, Spagnolo [5] and De Giorgi and Spagnolo [2]). Furthermore, the theory of periodic homogenization, as developed in A. Bensoussan et al. [1], may be construed as a systematic study of the H-convergence in a periodic framework. The latter reference offers a thorough bibliography as well as a wealth of open problems.

6 Locality

In essence, H-convergence amounts to a statement of convergence of the inverse operators $[-div\,(A^\epsilon grad)]^{-1}$, which are bounded linear mappings from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$, when both spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ are endowed with their weak topologies. The underlying topology satisfies the property of uniqueness of the H-limit, and the H-limit is local as demonstrated in the following proposition:

Proposition 1 *(i) A sequence A^ϵ, $\epsilon \in E$, of elements of $M(\alpha, \beta, \Omega)$ has at most one H-limit.*

(ii) Let A^ϵ and $B^\epsilon, \epsilon \in E$, be two sequences in $M(\alpha, \beta, \Omega)$ that satisfy

$$\begin{cases} A^\epsilon \overset{H}{\rightharpoonup} A^0 \,, \\ \\ B^\epsilon \overset{H}{\rightharpoonup} B^0 \,. \end{cases}$$

and are such that $A^\epsilon = B^\epsilon$ on an open set $\omega \subset \Omega$. Then $A^0 = B^0$ on ω.

Proof:

Let A^0 be an H-limit of $A^\epsilon, \epsilon \in E$. Consider $\omega \subset\subset \omega_1 \subset \Omega$, $\varphi \in C_0^\infty(\omega_1)$ with $\varphi = 1$ on ω, and define, for any λ in $\mathbf{R}^N$,

$$f_\lambda = -div\,(A^0(x) grad\,((\lambda, x)\varphi(x))).$$

Then u_λ^ϵ, defined as the solution of

$$\begin{cases} -div\,(A^\epsilon grad\, u^\epsilon_\lambda) = f_\lambda \quad \text{in } \omega_1, \\ u^\epsilon_\lambda \in H^1_0(\omega_1)\ , \end{cases}$$

for $\epsilon \in E$ and $\epsilon = 0$, is such that

$$\begin{cases} u^0_\lambda(x) = (\lambda, x)\varphi(x), \\ u^\epsilon_\lambda \rightharpoonup u^0_\lambda \quad \text{weakly in } H^1_0(\omega_1), \\ A^\epsilon grad\, u^\epsilon_\lambda \rightharpoonup A^0 grad\, u^0_\lambda \quad \text{weakly in } [L^2(\omega_1)]^N. \end{cases}$$

If B^0 is another H-limit for A^ϵ, then

$$A^\epsilon grad\, u^\epsilon_\lambda \rightharpoonup B^0 grad\, u^0_\lambda \quad \text{weakly in } [L^2(\omega_1)]^N.$$

Thus $A^0 grad\, u^0_\lambda = B^0 grad\, u^0_\lambda$ and, since $grad\, u^0_\lambda = \lambda$ in ω, $A^0 = B^0$ in ω, which proves (i). The proof of (ii) is immediate in view of (i) together with the definition of H-convergence. ■

7 Two Fundamental Lemmata

Lemma 1 *Let Ω be an open subset of $\mathbf{R}^N$ and $\xi^\epsilon, v^\epsilon, \epsilon \in E$, be such that*

$$\begin{cases} \xi^\epsilon \in [L^2(\Omega)]^N\ , \\ \xi^\epsilon \rightharpoonup \xi^0 \quad \textit{weakly in } [L^2(\Omega)]^N\ , \\ div\, \xi^\epsilon \to div\, \xi^0 \quad \textit{strongly in } H^{-1}(\Omega), \end{cases}$$

$$\begin{cases} v^\epsilon \in H^1(\Omega)\ , \\ v^\epsilon \rightharpoonup v^0 \quad \textit{weakly in } H^1(\Omega). \end{cases}$$

Then

$$(\xi^\epsilon, grad\, v^\epsilon) \rightharpoonup (\xi^0, grad\, v^0) \quad \textit{weakly-* in } \mathcal{D}'(\Omega).$$

Remarks

1. The product $(\xi^\epsilon, grad\, v^\epsilon)$ is that of two weakly and not strongly converging sequences; thus it is a miracle that the limit of the product should be equal to the product of the limits. This phenomenon is known as compensated compactness (see Murat [4] and Tartar [7]).

2. The product $(\xi^\epsilon, grad\, v^\epsilon)$ is bounded in $L^1(\Omega)$ independently of ϵ. Thus it actually converges vaguely to a measure. However, it does not in general converge weakly in $L^1(\Omega)$ (see Murat [6] for a counterexample).

Proof of Lemma 1:

Let φ be an element of $\mathcal{C}_0^\infty(\Omega)$. Then

$$\int_\Omega (\xi^\epsilon, grad\, v^\epsilon)\varphi dx = - < div\, \xi^\epsilon, \varphi v^\epsilon >_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega (\xi^\epsilon, grad\, \varphi) v^\epsilon dx.$$

Passing to the limit in each term of the right-hand side is easy (use Rellich's theorem in the second term). Integration by parts of the resulting expression yields the desired result. ■

Lemma 2 *Let Ω be an open subset of $\mathbf{R}^N$. Let A^ϵ belong to $M(\alpha,\beta,\Omega)$ for $\epsilon \in E$. Assume that, for $\epsilon \in E$,*

$$\begin{cases} u^\epsilon \in H^1(\Omega), \\ u^\epsilon \rightharpoonup u^0 \quad weakly\ in\ H^1(\Omega), \\ \xi^\epsilon = A^\epsilon grad\, u^\epsilon \rightharpoonup \xi^0 \quad weakly\ in\ [L^2(\Omega)]^N, \\ -div\, (A^\epsilon grad\, u^\epsilon) \rightarrow -div\, \xi^0 \quad strongly\ in\ H^{-1}(\Omega), \end{cases} \tag{6}$$

$$\begin{cases} v^\epsilon \in H^1(\Omega), \\ v^\epsilon \rightharpoonup v^0 \quad weakly\ in\ H^1(\Omega), \\ \eta^\epsilon = {}^tA^\epsilon grad\, v^\epsilon \rightharpoonup \eta^0 \quad weakly\ in\ [L^2(\Omega)]^N, \\ -div\, ({}^tA^\epsilon grad\, v^\epsilon) \rightarrow -div\, \eta^0 \quad strongly\ in\ H^{-1}(\Omega). \end{cases} \tag{7}$$

Then

$$(\xi^0, grad\, v^0) = (grad\, u^0, \eta^0) \quad a.e.\ in\ \Omega. \tag{8}$$

Proof:

The proof is immediate upon observing that

$$(\xi^\epsilon, grad\, v^\epsilon) = (A^\epsilon grad\, u^\epsilon, grad\, v^\epsilon) = (grad\, u^\epsilon, {}^tA^\epsilon grad\, v^\epsilon) = (grad\, u^\epsilon, \eta^\epsilon),$$

and through application of Lemma 1.

Note that equality (8) is a pointwise equality, which is a much stronger statement than an integral equality. ■

8 Irrelevance of the Boundary Conditions. Convergence of the Energy

Proposition 2 *If $A^\epsilon, \epsilon \in E$, belongs to $M(\alpha,\beta,\Omega)$ and H-converges to A^0 which belongs to $M(\alpha',\beta',\Omega)$, then ${}^tA^\epsilon, \epsilon \in E, H$-converges to ${}^tA^0$.*

Proof:

Let $\omega \subset\subset \Omega$ and g be an element of $H^1(\omega)$. Let v^ϵ be the solution of

$$\begin{cases} -div\,({}^tA^\epsilon grad\, v^\epsilon) = g & \text{in } \omega, \\ v^\epsilon \in H_0^1(\omega)\,, \end{cases}$$

for $\epsilon \in E$. Our task is to show that

$$\begin{cases} v^\epsilon \rightharpoonup v^0 \quad \text{weakly in } H_0^1(\omega), \\ {}^tA^\epsilon grad\, v^\epsilon \rightharpoonup {}^tA^0 grad\, v^0 \quad \text{weakly in } [L^2(\omega)]^N\,, \end{cases}$$

for $\epsilon \in E$, where v^0 is the solution of

$$\begin{cases} -div\,({}^tA^0 grad\, v^0) = g & \text{in } \omega, \\ v^0 \in H_0^1(\omega)\,. \end{cases}$$

Because $v^\epsilon, \epsilon \in E$ is bounded in $H_0^1(\omega)$, a subsequence E' of E is such that

$$\begin{cases} v^\epsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\omega), \\ {}^tA^\epsilon grad\, v^\epsilon \rightharpoonup \eta \quad \text{weakly in } [L^2(\omega)]^N\,, \end{cases}$$

for ϵ in E'. Furthermore, $-div\, \eta = g$ in ω.

For any f in $H^{-1}(\omega)$, u^ϵ defined as the solution of

$$\begin{cases} -div\,(A^\epsilon grad\, u^\epsilon) = f & \text{in } \omega, \\ u^\epsilon \in H_0^1(\omega)\,, \end{cases}$$

for $\epsilon \in E$ and $\epsilon = 0$, is such that

$$\begin{cases} u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\omega), \\ A^\epsilon grad\, u^\epsilon \rightharpoonup A^0 grad\, u^0 \quad \text{weakly in } [L^2(\omega)]^N\,, \end{cases}$$

for $\epsilon \in E$, because A^ϵ H-converges to A^0 for $\epsilon \in E$.

Application of Lemma 2 yields

$$(A^0 grad\, u^0, grad\, v) = (grad\, u^0, \eta) \quad \text{a.e. in } \omega.$$

As f spans $H^{-1}(\omega)$, u^0 spans $H_0^1(\omega)$; thus, if $\omega_1 \subset\subset \omega$, $grad\, u^0$ can be taken to be any $\lambda \in \mathbf{R}^N$ on ω_1 and we obtain

$$(A^0\lambda, grad\, v) = (\lambda, \eta) \quad \text{a.e. in } \omega_1 \text{ and for any } \lambda \in \mathbf{R}^N,$$

which implies that

$$\eta = {}^tA^0 \, grad \, v \qquad \text{a.e. in } \omega.$$

Since $-div\, \eta = g$, we conclude that $v = v^0$ and $\eta = {}^tA^0 grad\, v^0$.

Since ${}^tA^0$ is unique, v^0 is unique and the whole sequence $\epsilon \in E$ (and not only the subsequence $\epsilon \in E'$) is found to converge. ∎

Theorem 1 *Assume that* $A^\epsilon, \epsilon \in E$, *belongs to* $M(\alpha, \beta, \Omega)$ *and* H*-converges to* $A^0 \in M(\alpha', \beta', \Omega)$. *Assume that*

$$\begin{cases} u^\epsilon \in H^1(\Omega) \,, \\ f^\epsilon \in H^{-1}(\Omega) \,, \\ -div\,(A^\epsilon grad\, u^\epsilon) = f^\epsilon \quad in\ \Omega \,, \\ u^\epsilon \rightharpoonup u^0 \quad weakly\ in\ H^1(\Omega) \,, \\ f^\epsilon \to f^0 \quad strongly\ in\ H^{-1}(\Omega) \,, \end{cases}$$

for $\epsilon \in E$. *Then*

$$A^\epsilon grad\, u^\epsilon \rightharpoonup A^0 grad\, u^0 \qquad weakly\ in\ [L^2(\Omega)]^N \,,$$

$$(A^\epsilon grad\, u^\epsilon, grad\, u^\epsilon) \rightharpoonup (A^0 grad\, u^0, grad\, u^0) \qquad weakly\text{-}*\ in\ \mathcal{D}'(\Omega).$$

The proof of Theorem 1 is analogous to that of Proposition 2: it merely uses Proposition 2 and Lemmata 1 and 2.

It can be further proved, with the help of Meyers' regularity theorem (see Meyers [3]), that the energy $(A^\epsilon grad\, u^\epsilon, grad\, u^\epsilon)$ actually converges weakly in $L^1_{loc}(\Omega)$.

9 Sequential Compactness of $M(\alpha, \beta, \Omega)$ for the Topology Induced by H-convergence

The notion of H-convergence finds its *raison d'être* in the following theorem.

Theorem 2 *Let* $A^\epsilon, \epsilon \in E$ *belong to* $M(\alpha, \beta, \Omega)$. *There exists a subsequence* E' *of* E *and a matrix* A^0 *in* $M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ *such that* A^ϵ H*-converges to* A^0 *for* $\epsilon \in E'$.

Proof:

The proof of Theorem 2 consists of the following steps.

Step 1:

Proposition 3 *Let F and G be two Banach spaces, with F separable and G reflexive. Let $T^\epsilon, \epsilon \in E$ be elements of $\mathcal{L}(F,G)$ satisfying*

$$|||\, T^\epsilon \,|||_{\mathcal{L}} \leq C\,.$$

Then there exist a subsequence E' of E and an element T^0 of $\mathcal{L}(F,G)$ such that, for any element f of F, $T^\epsilon f \rightharpoonup T^0 f$ weakly in $G, \epsilon \in E'$.

Proof:

Take X to be a countable dense subset of F. A diagonal process ensures the existence of a subsequence E' of E such that $T^\epsilon x$ has a weak limit in G denoted by $T^0 x$ for $\epsilon \in E'$ and $x \in X$.

Fix f in F and g' in G' and approximate f by elements $x \in X$. This allows one to prove that $< T^\epsilon f, g' >_{G,G'}$ is a Cauchy sequence for $\epsilon \in E'$. Denote the corresponding limit by $< T^0 f, g' >_{G,G'}$; then T^0 is linear and bounded. Specifically,

$$\|\, T^0 f \,\|_G \leq \liminf_{\epsilon \in E'} \|\, T^\epsilon f \,\|_G \leq C \|\, f \,\|_F.$$

■

Step 2:

Proposition 4 *Let V be a reflexive separable Banach space and $T^\epsilon, \epsilon \in E$ be elements of $\mathcal{L}(V,V')$ such that*

$$\begin{cases} |||\, T^\epsilon \,|||_{\mathcal{L}} & \leq \beta\,, \\ < T^\epsilon v, v >_{V',V} & \geq \alpha \,\|\, v \,\|_V^2\,,\ v \in V. \end{cases}$$

Then there exist a subsequence E' of E and an element T^0 in $\mathcal{L}(V,V')$ such that

$$\begin{cases} |||\, T^0 \,|||_{\mathcal{L}} & \leq \beta^2/\alpha\,, \\ < T^0 v, v >_{V',V} & \geq \alpha \,\|\, v \,\|_V^2\,,\ v \in V, \end{cases}$$

which satisfy for any f in V',

$$(T^\epsilon)^{-1} f \rightharpoonup (T^0)^{-1} f \quad \textit{weakly in } V, \epsilon \in E'.$$

Proof:

By virtue of Lax–Milgram's lemma, T^ϵ has an inverse $(T^\epsilon)^{-1}$ that satisfies $|||\ (T^\epsilon)^{-1}\ |||_{\mathcal{L}} \leq 1/\alpha$. Application of Proposition 3 yields a subsequence E' of E and an element S in $\mathcal{L}(V', V)$ such that, for any element f of V',

$$(T^\epsilon)^{-1} f \rightharpoonup Sf \quad \text{weakly in } V,\ \epsilon \in E'.$$

Since

$$\begin{aligned} < (T^\epsilon)^{-1} f, f >_{V,V'} &= < (T^\epsilon)^{-1} f,\ T^\epsilon (T^\epsilon)^{-1} f >_{V,V'} \\ \geq \alpha \parallel (T^\epsilon)^{-1} f \parallel_V^2 &\geq \frac{\alpha}{\beta^2} \ |||\ T^\epsilon\ |||_{\mathcal{L}}^2 \parallel (T^\epsilon)^{-1} f \parallel_V^2 \geq \frac{\alpha}{\beta^2} \parallel f \parallel_{V'}^2, \end{aligned}$$

we obtain

$$< Sf, f >_{V,V'} \geq \frac{\alpha}{\beta^2} \parallel f \parallel_{V'}^2 .$$

Thus S, being coercive, is invertible. Denote by $T^0 \in \mathcal{L}(V, V')$ its inverse. It satisfies, for any element v of V,

$$\frac{\alpha}{\beta^2} \parallel T^0 v \parallel_V^2 \leq < ST^0 v, T^0 v >_{V,V'} \leq \parallel v \parallel_V \|T^0 v\|_{V'}.$$

Hence $|||\ T^0\ |||_{\mathcal{L}} \leq \beta^2/\alpha$.

Since

$$\begin{aligned} \alpha \parallel (T^\epsilon)^{-1} f \parallel_V^2 &\leq < T^\epsilon (T^\epsilon)^{-1} f, (T^\epsilon)^{-1} f >_{V',V} \\ &= < f, (T^\epsilon)^{-1} f >_{V',V}, \end{aligned}$$

the sequential weak lower semicontinuity of $\parallel\ \parallel_V$ implies

$$\alpha \parallel Sf \parallel_V^2 \leq < f, Sf >_{V',V},$$

and the choice of $f = Tv^0, v \in V$, finally yields

$$\alpha \parallel v \parallel_V^2 \leq < T^0 v, v >_{V',V} .$$

■

Step 3:

For the remainder of the proof of Theorem 2 it will be assumed that Ω is bounded. If such was not the case the argument would be applied to $\Omega \cap \{x \in \mathbf{R}^N : |\ x\ | \leq p\}$ with $p \in \mathbf{Z}^+$ and a diagonalization argument would permit us to conclude.

We propose to manufacture a sequence of test functions to be later inserted into Lemma 2. To this effect a bounded open set Ω' of $\mathbf{R}^N$ with

$\Omega \subset \Omega'$ is considered. We define B^ϵ to be an element of $M(\alpha, \beta, \Omega')$ such that

$$B^\epsilon = {}^t A^\epsilon \text{ in } \Omega.$$

(Take for example $B^\epsilon = \alpha I$ in $\Omega' \setminus \Omega$.)

Set

$$\mathcal{B}^\epsilon = -div\,(B^\epsilon grad) \in \mathcal{L}(H_0^1(\Omega'); H^{-1}(\Omega')).$$

Proposition 4 implies the existence of a subsequence E' of E and of an element $\mathcal{B}^0 \in \mathcal{L}(H_0^1(\Omega');\ H^{-1}(\Omega'))$ such that, for any element g in $H^{-1}(\Omega')$,

$$(\mathcal{B}^\epsilon)^{-1} g \rightharpoonup (\mathcal{B}^0)^{-1} g \text{ weakly in } H_0^1(\Omega),$$

when $\epsilon \in E'$. Let φ be an element of $C_0^\infty(\Omega')$ such that $\varphi = 1$ on Ω and, for any $i \in \{1, \ldots, N\}$, set

$$g_i = \mathcal{B}^0(x_i \varphi(x)) \in H^{-1}(\Omega').$$

Define $v_i^\epsilon, \epsilon \in E', i \in \{1, \ldots, N\}$, as

$$v_i^\epsilon = (\mathcal{B}^\epsilon)^{-1} g_i.$$

The restriction of v_i^ϵ to Ω belongs to $H^1(\Omega)$ and satisfies

$$\begin{cases} v_i^\epsilon \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega)\ , \\ -div\,({}^t A^\epsilon grad\, v_i^\epsilon) = g_i \quad \text{in } \Omega. \end{cases}$$

At the possible expense of the extraction of a subsequence E'' of E', we are at liberty to further assume that

$${}^t A^\epsilon grad\, v_i^\epsilon \rightharpoonup \eta_i \quad \text{weakly in } [L^2(\Omega)]^N\ ,$$

when $\epsilon \in E'', i \in \{1, \ldots, N\}$.

Note that $-div\,\eta_i = g_i$ in Ω and that the functions $v_i^\epsilon, \epsilon \in E''$ satisfy equation (7) in Lemma 2 with $v^0 = x_i$.

We now define a matrix $A^0 \in [L^2(\Omega)]^{N^2}$ by

$$(A^0)_{ij} = (\eta_i)_j \in L^2(\Omega), \quad i, j \in \{1, \ldots, N\}.$$

The matrices $A^\epsilon, \epsilon \in E''$, are shown to H-converge to A^0.

Step 4:

Let $\omega \subset\subset \Omega$. Define the isomorphism $\mathcal{A}^\epsilon$ by

$$\mathcal{A}^\epsilon = -div\,(A^\epsilon grad) \in \mathcal{L}(H_0^1(\omega); H^{-1}(\omega)),$$

and set

$$\mathcal{C}^\epsilon = A^\epsilon grad\,((\mathcal{A}^\epsilon)^{-1}) \in \mathcal{L}(H^{-1}(\omega); [L^2(\omega)]^N).$$

Then, for any element f in $H^{-1}(\omega)$,

$$\| \mathcal{C}^\epsilon f \|_{[L^2(\omega)]^N} \leq \beta \| (\mathcal{A}^\epsilon)^{-1} f \|_{H_0^1(\omega)} \leq \frac{\beta}{\alpha} \| f \|_{H^{-1}(\omega)} \,.$$

Direct applications of Proposition 3 to $\mathcal{C}^\epsilon$ and of Proposition 4 to $\mathcal{A}^\epsilon$, imply the existence of a subsequence E_ω of E'', of $\mathcal{C}^0 \in \mathcal{L}(H^{-1}(\omega); [L^2(\omega)]^N)$, and of an isomorphism $\mathcal{A}^0 \in \mathcal{L}(H_0^1(\omega); H^{-1}(\omega))$, such that, for any element f in $H^{-1}(\omega)$,

$$\begin{cases} (\mathcal{A}^\epsilon)^{-1} f \rightharpoonup (\mathcal{A}^0)^{-1} f \quad \text{weakly in } H_0^1(\omega)\,, \\ \mathcal{C}^\epsilon f \rightharpoonup \mathcal{C}^0 f \quad \text{weakly in } [L^2(\omega)]^N. \end{cases}$$

Note that E_ω depends upon the choice of ω.

The sequence $u^\epsilon = (\mathcal{A}^\epsilon)^{-1} f$, $\epsilon \in E''$, satisfies

$$\begin{cases} u^\epsilon \rightharpoonup u^0 = (\mathcal{A}^0)^{-1} f \quad \text{weakly in } H_0^1(\omega), \\ A^\epsilon grad\, u^\epsilon \rightharpoonup \mathcal{C}^0 f = \xi^0 \quad \text{weakly in } [L^2(\omega)]^N, \\ -div\,(A^\epsilon grad\, u^\epsilon) = f \quad \text{in } \omega, \end{cases}$$

which is precisely equation (6) of Lemma 2.

Thus application of Lemma 2 to u^ϵ and v_i^ϵ yields, for $i \in \{1, \dots, N\}$,

$$(\xi^0, grad\, x_i) = (grad\; u^0, \eta_i) \quad \text{a.e. in } \omega\,,$$

which, in view of the definition of A^0, is precisely

$$\mathcal{C}^0 f = \xi^0 = A^0 grad\, u^0 \quad \text{a.e. in } \omega.$$

Step 5:

The matrix A^0, which is by definition an element of $[L^2(\omega)]^{N^2}$, is such that $A^0 grad\, u^0$ belongs to $[L^2(\omega)]^N$ for any u^0 in $H_0^1(\omega)$. We prove that A^0 belongs to $M(\alpha, \frac{\beta^2}{\alpha}, \omega)$. Indeed, application of Lemma 1 to $A^\epsilon grad\, u^\epsilon$ and u^ϵ, $\epsilon \in E_\omega$ yields

$$(A^\epsilon grad\, u^\epsilon, grad\, u^\epsilon) \rightharpoonup (A^0 grad\, u^0, grad\, u^0) \quad \text{weakly-* in } \mathcal{D}'(\omega).$$

Let φ be an arbitrary nonnegative element of $C_0^\infty(\omega)$. The inequality

$$\int_\omega \varphi\, (A^\epsilon grad\, u^\epsilon, grad\, u^\epsilon)\, dx \geq \alpha \int_\omega \varphi \mid grad\, u^\epsilon \mid^2 dx$$

implies

$$\int_\omega \varphi\,(A^0 grad\,u^0, grad\,u^0)\,dx \ge \alpha \int_\omega \varphi \mid grad\,u^0 \mid^2 dx.$$

Since the preceding result holds true for any u^0 in $H^1_0(\omega)$, taking $u^0(x) = (\lambda, x)$ on the support of φ yields

$$(A^0(x)\lambda, \lambda) \ge \alpha \mid \lambda \mid^2, \quad \lambda \in \mathbf{R}^N, \text{ a.e. } x \in \omega.$$

On the other hand, for any $\mu \in \mathbf{R}^N$ (see the beginning of the proof of Proposition 4),

$$((A^\epsilon)^{-1}(x)\mu, \mu) \ge \frac{\alpha}{\beta^2} \mid \mu \mid^2, \quad \text{a.e. } x \in \omega.$$

Let φ be an arbitrary nonnegative element of $C^\infty_0(\omega)$. The inequality

$$\int_\omega \varphi\,(grad\,u^\epsilon, A^\epsilon grad\,u^\epsilon)\,dx \ge \frac{\alpha}{\beta^2} \int_\omega \varphi \mid A^\epsilon grad\,u^\epsilon \mid^2 dx$$

implies

$$\int_\omega \varphi\,(grad\,u^0, A^0 grad\,u^0)\,dx \ge \frac{\alpha}{\beta^2} \int_\omega \varphi \mid A^0 grad\,u^0 \mid^2 dx.$$

Since the preceding result holds true for any u^0 in $H^1_0(\omega)$, taking $u^0(x) = (\lambda, x)$ on the support of φ yields

$$(\lambda, A^0(x)\lambda) \ge \frac{\alpha}{\beta^2} \mid A^0(x)\lambda \mid^2, \quad \lambda \in \mathbf{R}^N, \text{ a.e. } x \in \omega,$$

and thus

$$\mid A^0(x)\lambda \mid \le \frac{\beta^2}{\alpha} \mid \lambda \mid.$$

We have proved that A^0 belongs to $M(\alpha, \frac{\beta^2}{\alpha}, \omega)$.

Step 6:

Because $A^0 \in M(\alpha, \frac{\beta^2}{\alpha}, \omega)$, the limit u^0 of $u^\epsilon, \epsilon \in E_\omega$ is uniquely defined, independently of E_ω, through

$$\begin{cases} -div\,(A^0 grad\,u^0) = f & \text{in } \omega, \\ u^0 \in H^1_0(\omega). \end{cases}$$

Thus there is no need to extract E_ω from E'' and the sequences u^ϵ and $A^\epsilon grad\,u^\epsilon$ converge for $\epsilon \in E''$. But E'' is independent of ω. Thus $A^\epsilon, \epsilon \in E''$, H-converges to A^0. ∎

10 Definition of the Corrector Matrix P^ϵ

Let $A^\epsilon, \epsilon \in E$, be a sequence of elements of $M(\alpha, \beta, \Omega)$ that H-converges to $A^0 \in M(\alpha, \beta', \Omega)$. Consider $\omega \subset\subset \Omega$, $\lambda \in \mathbf{R}^N$, and $\epsilon \in E$ and define w^ϵ_λ such that

$$\begin{cases} w^\epsilon_\lambda \in H^1(\omega), \\ w^\epsilon_\lambda \rightharpoonup (\lambda, x) \qquad \text{weakly in } H^1(\omega), \\ -div\,(A^\epsilon\, grad\, w^\epsilon_\lambda) \to -div\,(A^0\lambda) \qquad \text{strongly in } H^{-1}(\omega). \end{cases} \tag{9}$$

The existence of w^ϵ_λ is readily asserted upon solving

$$\begin{cases} -div\,(A^\epsilon\, grad\, w^\epsilon_\lambda) = -div\,(A^0\, grad\,((\lambda, x)\varphi(x))) \qquad \text{in } \omega_1, \\ w^\epsilon_\lambda \in H^1_0(\omega_1), \end{cases}$$

with $\omega \subset\subset \omega_1 \subset\subset \Omega$ and φ an element of $C^\infty_0(\omega_1)$ such that $\varphi = 1$ on ω.

Definition 2 *Let $A^\epsilon, \epsilon \in E$ be a sequence of elements of $M(\alpha, \beta, \Omega)$ that H-converges to $A^0 \in M(\alpha, \beta', \Omega)$. The corrector matrix $P^\epsilon \in [L^2(\omega)]^{N^2}$ is defined by*

$$P^\epsilon\lambda = grad\, w^\epsilon_\lambda, \quad \lambda \in \mathbf{R}^N, \;\; \epsilon \in E, \tag{10}$$

where the sequence w^ϵ_λ satisfies (9).

Remarks

1. It can easily be shown from equation (9) that the matrix P^ϵ is "unique" to the extent that if P^ϵ and $\tilde{P}^\epsilon$, $\epsilon \in E$, are two such sequences, then

$$P^\epsilon - \tilde{P}^\epsilon \to 0 \quad \text{strongly in } [L^2_{loc}(\omega)]^{N^2}.$$

2. The sequence P^ϵ is bounded in $[L^2(\omega)]^{N^2}$ independently of ϵ. Bounds for this sequence in $[L^q(\omega)]^{N^2}$, $q > 2$ can be achieved through application of Meyers' regularity result (see Meyers [3]).

3. In the case of layers where $A^\epsilon(x) = A^\epsilon(x_1)$ (see Step 4), the functions w^ϵ_λ are of the form

$$w^\epsilon_\lambda(x) = (\lambda, x) + z^\epsilon_\lambda(x_1),$$

and it is easily proved that P^ϵ can be defined by

$$\left\{\begin{array}{l} P^\epsilon_{11} = \dfrac{A^0_{11}}{A^\epsilon_{11}}, \\ \\ P^\epsilon_{1j} = \dfrac{A^0_{1j} - A^\epsilon_{1j}}{A^\epsilon_{11}}, \quad j > 1, \\ \\ P^\epsilon_{ii} = 1, \quad i > 1, \\ \\ P^\epsilon_{ij} = 0, \quad i, j > 1, \ i \neq j. \end{array}\right. \tag{11}$$

4. Note that the previous remark immediately demonstrates that a sequence Q^ϵ associated with ${}^tA^\epsilon$ through Definition 2 does not generally coincide with ${}^tP^\epsilon$. Indeed, in the case of layers, both P^ϵ and Q^ϵ given by (11) have nonzero terms only on the diagonal and in the first line.

Proposition 5 *Let P^ϵ be the sequence of corrector matrices defined through Definition 2. Then, as $\epsilon \in E$,*

$$\begin{array}{rcll} P^\epsilon & \rightharpoonup & I & \textit{weakly in } [L^2(\omega)]^{N^2}, \\ A^\epsilon P^\epsilon & \rightharpoonup & A^0 & \textit{weakly in } [L^2(\omega)]^{N^2}, \\ {}^tP^\epsilon A^\epsilon P^\epsilon & \rightharpoonup & A^0 & \textit{weakly-* in } [\mathcal{D}'(\omega)]^{N^2}. \end{array}$$

Proof:

The sequence P^ϵ is bounded in $[L^2(\omega)]^{N^2}$. If φ is an arbitrary element of $[\mathcal{C}_0^\infty(\omega)]^N$, that is, if

$$\varphi = \sum_{i=1}^N \varphi_i e_i, \quad \varphi_i \in \mathcal{C}_0^\infty(\omega),$$

one has

$$\left\{\begin{array}{l} \displaystyle\int_\omega P^\epsilon \varphi dx = \int_\omega \sum_{i=1}^N \varphi_i P^\epsilon e_i \, dx = \int_\omega \sum_{i=1}^N \varphi_i \, grad \, w^\epsilon_{e_i} \, dx \\ \\ \displaystyle\to \int_\omega \sum_{i=1}^N \varphi_i e_i dx = \int_\omega \varphi \, dx. \end{array}\right.$$

Thus P^ϵ converges weakly to I in $[L^2(\omega)]^N$. The remaining statements of convergence are obtained in a similar way with the help of Theorem 1 and Lemma 1. ■

11 Strong Approximation of $grad\, u^\epsilon$. Correctors

Theorem 3 *Assume that $A^\epsilon, \epsilon \in E$ belongs to $M(\alpha,\beta,\Omega)$ and H-converges to $A^0 \in M(\alpha,\beta',\Omega)$. Assume that*

$$\begin{cases} u^\epsilon \in H^1(\omega), \\ f^\epsilon \in H^{-1}(\omega), \\ -div\,(A^\epsilon\, grad\, u^\epsilon) = f^\epsilon \quad in\ \omega, \\ u^\epsilon \rightharpoonup u^0 \quad weakly\ in\ H^1(\omega), \\ f^\epsilon \to f^0 \quad strongly\ in\ H^{-1}(\omega), \end{cases} \tag{12}$$

where ω is such that $\omega \subset\subset \Omega$. Let P^ϵ be the corrector matrix introduced in Definition 2. Then one has for $\epsilon \in E$:

$$\begin{cases} grad\, u^\epsilon = P^\epsilon\, grad\, u^0 + z^\epsilon, \\ z^\epsilon \to 0 \quad strongly\ in\ [L^1_{loc}(\omega)]^N. \end{cases} \tag{13}$$

Further, if

$$\begin{cases} P^\epsilon \in [L^q(\omega)]^{N^2}, \quad \|P^\epsilon\|_{[L^q(\omega)]^{N^2}} \le C, \quad 2 \le q \le +\infty, \\ grad\ u^0 \in [L^p(\omega)]^N, \quad 2 \le p < +\infty, \end{cases} \tag{14}$$

then

$$z^\epsilon \to 0 \quad strongly\ in\ [L^r_{loc}(\omega)]^N, \tag{15}$$

with

$$\frac{1}{r} = max(\frac{1}{2}, \frac{1}{p} + \frac{1}{q}).$$

Finally, if

$$\int_\omega (A^\epsilon\, grad\, u^\epsilon,\ grad\, u^\epsilon)\, dx \to \int_\omega (A^0\, grad\, u^0,\ grad\, u^0)\, dx, \tag{16}$$

then

$$z^\epsilon \to 0 \quad strongly\ in\ [L^r(\omega)]^N. \tag{17}$$

Remarks

1. Theorem 3 provides a "good" approximation for $grad\, u^\epsilon$ in the strong topology of L^1_{loc}, L^r_{loc}, or even L^r. Such an approximation is a useful tool in the study of the limit of non linear functions of $grad\, u^\epsilon$.

2. When u^0 is more regular, that is, when $u^0 \in H^2(\omega)$, Theorem 3 immediately implies that

$$u^\epsilon = u^0 + \sum_{i=1}^{N} (w^\epsilon_{e_i} - x_i)\frac{\partial u^0}{\partial x_i} + r^\epsilon \quad \text{with} \quad r^\epsilon \to 0 \quad \text{strongly in } W^{1,1}_{loc}(\omega).$$

The term $\sum_{i=1}^{N} (w^\epsilon_{e_i} - x_i)\frac{\partial u^0}{\partial x_i}$ may be seen as a correcting term. In the case where $A^\epsilon(x) = A(x/\epsilon)$ with A a periodic matrix, it is precisely the term of order ϵ in the asymptotic expansion for u^ϵ (see Bensoussan et al. [1]).

3. In the absence of any hypothesis on the behavior of u^ϵ near the boundary of ω (note the absence of any kind of boundary condition on u^ϵ in (12)) the estimates (13) and (17) on $grad\, u^\epsilon - P^\epsilon grad\, u^0$ are only local estimates. Assumption (16) alleviates this latter obstacle; it is met in particular when u^ϵ is the solution of an homogeneous Dirichlet boundary value problem.

4. An approximation of $grad\, u^\epsilon$ by $P^\epsilon grad\, u$ in the strong topology of $[L^2_{loc}(\omega)]^N$ is obtained as soon as the corrector matrix P^ϵ is bounded in $[L^q(\omega)]^{N^2}$ with q large enough. Since $grad\, u^\epsilon$ is bounded in $[L^2(\omega)]^N$, such an approximation may be deemed "natural." It is unfortunately not available in general. The most pleasant setting is, of course, the case where $q = +\infty$.

5. The case where $p = +\infty$ in (14) also results in the statements (15) and (17), but its proof requires Meyers' regularity theorem to be done.

Proof of Theorem 3:

The proof consists of two steps.

Step 1:

Proposition 6 *In the setting of Theorem 3, the following convergence holds true for any φ in $[C^\infty_0(\omega)]^N$, ϕ in $C^\infty_0(\omega)$ and $\epsilon \in E$.*

$$\begin{cases} \displaystyle\int_\omega \phi\,(A^\epsilon(grad\, u^\epsilon - P^\epsilon\varphi),\ (grad\, u^\epsilon - P^\epsilon\ \varphi))\, dx \\ \displaystyle\to \int_\omega \phi\,(A^0(grad\, u^0 - \varphi),\ (grad\, u^0 - \varphi))\, dx. \end{cases} \tag{18}$$

Proof:

Set $\varphi = \sum_{i=1}^{N} \varphi_i e_i,\ \ \varphi_i \in C_0^\infty(\omega)$. Then

$$\int_\omega \phi\,(A^\epsilon(grad\, u^\epsilon - P^\epsilon\varphi),\ (grad\, u^\epsilon - P^\epsilon\ \varphi))\, dx$$

$$= \int_\omega \phi\,(A^\epsilon\, grad\, u^\epsilon,\ grad\, u^\epsilon)\, dx + \sum_{j=1}^{N} \int_\omega \phi\,(A^\epsilon\ grad\, u^\epsilon, P^\epsilon e_j)\,\varphi_j\, dx$$

$$+ \sum_{i=1}^{N} \int_\omega \phi\,(A^\epsilon P^\epsilon e_i,\ grad\, u^\epsilon)\,\varphi_i\, dx + \sum_{i,j=1}^{N} \int_\omega \phi\,(A^\epsilon P^\epsilon e_i, P^\epsilon e_j)\,\varphi_i\,\varphi_j\, dx.$$

Each term in the preceding equality passes to the limit, with the help of Theorem 1 for the first one, Lemma 1 together with the definition of P^ϵ for the second and third ones, and Proposition 5 for the last one. This proves (18).

Whenever assumption (16) is satisfied, the choice $\phi = 1$ is licit because the first term passes to the limit as well as the other terms that contain at least one φ_i which has compact support.

Step 2:

If u^0 belongs to $C_0^\infty(\omega)$, the first step permits us to conclude upon setting $\varphi = grad\, u^0$. Otherwise an approximation process is required. The regularity hypothesis (14) is assumed with no loss of generality since (13) is recovered from (15) if $p = q = 2$.

Let δ be an arbitrary (small) positive number. Choose $\varphi \in [C_0^\infty(\omega)]^N$ such that

$$\|\, grad\, u^0 - \varphi\,\|_{[L^p(\omega)]^N} \le \delta,$$

which is possible since $p < +\infty$. Then

$$\|\, P^\epsilon\, grad\, u^0 - P^\epsilon\varphi\,\|_{[L^s(\omega)]^N} \le C\delta, \quad \text{if } \frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

Take $\omega_1 \subset\subset \omega$ and $\phi \in C_0^\infty(\omega),\ \phi = 1$ on $\omega_1,\ 0 \le \phi \le 1$ in ω. Proposition 6 then yields

$$\limsup_{\epsilon\in E} \alpha \| grad\, u^\epsilon - P^\epsilon\varphi \|^2_{[L^2(\omega_1)]^N}$$

$$\leq \limsup_{\epsilon\in E} \int_\omega \phi\,(A^\epsilon(grad\, u^\epsilon - P^\epsilon\varphi), (grad\, u^\epsilon - P^\epsilon\varphi))\, dx$$

$$= \int_\omega \phi\,(A^0(grad\, u^0 - \varphi), (grad\, u^0 - \varphi))\, dx$$

$$\leq \beta' \| grad\, u^0 - \varphi \|^2_{[L^2(\omega)]^N} \leq C\delta^2.$$

The two results we have obtained imply that

$$z^\epsilon = grad\, u^\epsilon - P^\epsilon grad\, u = (grad\, u^\epsilon - P^\epsilon\varphi) - (P^\epsilon\, grad\, u^0 - P^\epsilon\varphi)$$

satisfies

$$\limsup_{\epsilon\in E} \| z^\epsilon \|_{L^r[(\omega_1)]^N} \leq C\delta,$$

with $r = \min(2, s)$. Letting δ tend to 0 yields (15).

When assumption (16) holds, the proof remains valid with the choice $\phi = 1$ and $\omega_1 = \omega$, and (17) is thus established. ■

We conclude with a straightforward application of Theorem 3.

Proposition 7 *Consider, in the setting of Theorem 3, a sequence $a^\epsilon, \epsilon \in E$, with*

$$\begin{cases} a^\epsilon \in [L^\infty(\omega)]^N, & \| a^\epsilon \|_{[L^\infty(\omega)]^N} \leq C, \\ {}^tP^\epsilon a^\epsilon \rightharpoonup a^0 & \textit{weakly in } [L^2(\omega)]^N. \end{cases}$$

Then

$$(a^\epsilon,\, grad\, u^\epsilon) \rightharpoonup (a^0,\, grad\, u^0) \quad \textit{weakly in } L^2(\omega).$$

The proof is immediate upon recalling (13). The same idea also permits, at the expense of a few technicalities, handling the case where u^ϵ converges weakly in $H^1(\Omega)$ and satisfies an equation of the type:

$$-div\,(A^\epsilon\, grad\, u^\epsilon + b^\epsilon u^\epsilon + c^\epsilon) + (d^\epsilon,\, grad\, u^\epsilon) + e^\epsilon u^\epsilon = f^\epsilon \quad \text{in } \Omega.$$

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