• Matteo 2 – **H**-convergence part 2

Proof of thm 3.1 :

By thm 1 : $F_\varepsilon \xrightarrow{\Gamma\text{-}d} F_M$, with M having the same blocks of σ_ε. Let $f, g, u_\varepsilon, v_\varepsilon, a_\varepsilon, b_\varepsilon$ as before.

Then:

- $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon$ bdd in H_0^1,
- $(a_\varepsilon)_\varepsilon, (b_\varepsilon)_\varepsilon$ bdd in $(L^2)^N$,

so up to sub-sequences, they weakly converge.

• **step 1** : • $a_0 = \sigma_0 \nabla u_0$, for some σ_0

• $-\operatorname{div}(\sigma_0 \nabla u_0) = f_0$

By (CG) we have: $(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon)$ are minimizers of $F_\varepsilon^{f+g, f-g}$, and $F_\varepsilon^{f+g,f-g} \xrightarrow{\Gamma\text{-}(d)} F_M^{f+g,f-g}$.

Thus:

$$\lim_{\varepsilon \to 0} F_\varepsilon^{f+g,f-g}(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon) = \min F_M^{f+g,f-g}.$$

$\Downarrow$ result by Ansini – Dal Maso – Zeppieri

$$\text{I)} \quad \operatorname{grad} Q_\varepsilon(a_\varepsilon + b_\varepsilon, \nabla u_\varepsilon - \nabla v_\varepsilon) \xrightarrow{w\text{-}(L^2)^N \times (L^2)^N} \operatorname{grad} Q_M(a_0 + b_0, \nabla u_0 - \nabla v_0)$$

by applying the same argument above to $(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon)$

$$\text{II)} \quad \operatorname{grad} Q_\varepsilon(a_\varepsilon - b_\varepsilon, \nabla u_\varepsilon + \nabla v_\varepsilon) \xrightarrow{w\text{-}L^2} \operatorname{grad} Q_M(a_0 - b_0, \nabla u_0 + \nabla v_0)$$

We now compute the gradients:

I): $A_\varepsilon(a_\varepsilon+b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon-\nabla v_\varepsilon)=\nabla u_\varepsilon+\nabla v_\varepsilon$

II) = $\nabla u_\varepsilon - \nabla v_\varepsilon$

$$\Downarrow$$

$$\begin{cases} \nabla u_\varepsilon+\nabla v_\varepsilon \rightharpoonup A(a_0+b_0)+B(\nabla u_0-\nabla v_0) \\ \nabla u_\varepsilon-\nabla v_\varepsilon \rightharpoonup A(a_0-b_0)+B(\nabla u_0+\nabla v_0) \end{cases}$$

$\Downarrow$ by summing them

$$\nabla u_\varepsilon \rightharpoonup A a_0 + B\nabla u_0$$

But we know: $\nabla u_\varepsilon \rightharpoonup \nabla u_0 \implies a_0 = \sigma_0 \nabla u_0$,
where:

$$\sigma_0 := A^{-1} - A^{-1}B.$$

Now notice that, by convergence: $-\operatorname{div} a_0 = f$.

• <u>step 2</u>: we want to show that $\sigma_0 \in \mathcal{M}(c_0, c_1, \Omega)$.

<u>steps</u>: • let $u \in H_0^1(\Omega)$ and let $f := -\operatorname{div}(\sigma_0 \nabla u)$
$\implies u_0 = u$

• use φu_ε as a test function ($\varphi \in C_c^\infty$) to show that:

$$\lim_{\varepsilon\to 0}\int_\Omega (\sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon)\varphi = \int_\Omega (\sigma_0\nabla u_0\cdot\nabla u_0)\varphi.$$

• use $\psi \geq 0$ to prove the lower bound

$\rightsquigarrow$ similar argument for σ_ε^T.

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• **Heuristics for the Σ:**

Let us consider the pb:

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = f \\ -\operatorname{div}(\sigma^T \nabla v) = g \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} -\operatorname{div} J_u = f \\ -\operatorname{div} J_v = g \end{cases} \qquad \text{where} \begin{cases} J_u := \sigma \nabla u \\ J_v := \sigma^T \nabla v \end{cases}$$

write: $\sigma = \sigma^s + \sigma^a$;

$$\Leftrightarrow \quad \begin{cases} (\sigma^s + \sigma^a)\nabla u = J_u \\ \sigma^s - \sigma^a)\nabla v = J_v \end{cases}$$

Let $\varphi := u + v$, $\psi := u - v$. Then:

$$(**) \quad \begin{cases} \sigma^s \nabla\varphi + \sigma^a \nabla\psi = J_u + J_v \\ -\sigma^a \nabla\varphi - \sigma^s \nabla\psi = J_v - J_u \end{cases}$$

$$\Updownarrow$$

$$X \begin{pmatrix} \nabla\varphi \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} J_u + J_v \\ J_v - J_u \end{pmatrix}$$

where:

$$X := \begin{pmatrix} \sigma^s & \sigma^a \\ -\sigma^a & -\sigma^s \end{pmatrix}$$

$\to$ <u>Pb of X</u>: not positive definite!

$\rightsquigarrow$ if we solve ~~(**)~~ w.r.t. $\nabla\varphi$ and $J_u - J_v$ (the <u>partial Legendre transform</u>), getting:

$$\begin{cases} \nabla\varphi = (\sigma^s)^{-1}(J_u + J_v) - (\sigma^s)^{-1}\sigma^a \nabla\psi \\ J_u - J_v = \sigma^a(\sigma^s)^{-1}(J_u + J_v) + \left(\left(\sigma^s - \sigma^a\right)\sigma^s\right)^{-1}\sigma^a\,\nabla\psi \end{cases}$$

$$\Sigma \begin{pmatrix} J_u + J_v \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} \nabla\varphi \\ J_u - J_v \end{pmatrix}$$

$\rightsquigarrow$ and this is where Σ comes from!