• Matteo - H-convergence $\rightarrow$ compactness
$\searrow$ relation with Γ-convergence

$\rightsquigarrow$ $\boxed{\text{connection between the two def. ???}}$

• **1. Setup**: recall that if $(\sigma_\varepsilon)_\varepsilon$ symmetric $\rightsquigarrow$ relation between G-convergence & minimizers of some energies [De Giorgi - Spagnolo]

$\rightarrow$ the goal here is to have a similar parallelism, i.e., $\sigma_\varepsilon \longleftrightarrow$ functionals F_ε, F_0 s.t.

$$F_\varepsilon \longrightarrow F_0 \iff \sigma_\varepsilon \xrightarrow{H} \sigma$$

• let $\Omega \subseteq \mathbb{R}^N$ open, $\boxed{\text{bdd}}$ [and if Ω unbounded ???]

$A \rightsquigarrow A^s, A^a$; let $\mathcal{M}(c_0, c_1, \Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{R}^{N\times N}),\ \sigma(x)\xi\cdot\xi \geq c_0|\xi|^2,\ \sigma^{-1}(x)\xi\cdot\xi \geq c_1|\xi|^2\}$

given σ, we define:

$$\Sigma := \begin{pmatrix} (\sigma^s)^{-1} & -(\sigma^s)^{-1}\sigma^a \\ \sigma^a(\sigma^s)^{-1} & \sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a \end{pmatrix}$$

symmetric, $2N \times 2N$
$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$

$\rightsquigarrow \Sigma_\varepsilon$ if $\sigma \rightsquigarrow \sigma_\varepsilon$

We also define the functionals:

- $$Q_\varepsilon : L^2(\Omega;\mathbb{R}^N)\times L^2(\Omega;\mathbb{R}^N)\longrightarrow [0,+\infty),$$

by

$$Q_\varepsilon(a,b) := \int_\Omega \Sigma_\varepsilon \begin{pmatrix} a \\ b\end{pmatrix}\cdot\begin{pmatrix} a \\ b\end{pmatrix} dx.$$

Then:

$$\operatorname{grad} Q_\varepsilon(a,b) = \left(A_\varepsilon a + B_\varepsilon b,\ B_\varepsilon^T a + C_\varepsilon b\right),$$ $A_\varepsilon, B_\varepsilon, C_\varepsilon$ are the blocks in Σ_ε.

- $$F_\varepsilon : L^2(\Omega;\mathbb{R}^N)\times H^1_0(\Omega;\mathbb{R}^N)\longrightarrow [0,+\infty),$$

by

$$F_\varepsilon(\alpha,\Psi) := Q_\varepsilon(\alpha,\nabla\Psi)$$

- $\forall \lambda,\mu\in H^{-1}(\Omega)$ we define

$$F_\varepsilon^{\lambda,\mu}(\alpha,\Psi) := \begin{cases} F_\varepsilon(\alpha,\Psi) - \langle \mu,\Psi\rangle & \text{if } -\operatorname{div}\alpha=\lambda, \\ +\infty & \text{otherwise.}\end{cases}$$

$\rightsquigarrow$ similarly, if M <u>symmetric</u> $\rightsquigarrow$ Q_M, $F_M^{\lambda,\mu}$

- <u>distance</u>: in $L^2(\Omega;\mathbb{R}^N)\times H^1_0(\Omega)$ defined by

$$d\left((\alpha,\varphi),(\beta,\Psi)\right) := \|\alpha-\beta\|_{H^{-1}} + \|\operatorname{div}(\alpha-\beta)\|_{H^{-1}} + \|\varphi-\Psi\|_{L^2}$$

It is possible to show that the unique minimizer $(\alpha_\varepsilon, \psi_\varepsilon)$ of $F_\varepsilon^{\lambda,\mu}$ solves:

$$(**)\quad \begin{cases} -\operatorname{div} \alpha_\varepsilon = \lambda, \\ \displaystyle\int_\Omega (A_\varepsilon \alpha_\varepsilon + B_\varepsilon \nabla \psi_\varepsilon)\,\beta \, dx = 0, \qquad \forall \beta \in L^2(\Omega, \mathbb{R}^N), \operatorname{div}\beta = 0 \\ \displaystyle\int_\Omega (B_\varepsilon^T \alpha_\varepsilon + C_\varepsilon \nabla \psi_\varepsilon)\cdot \nabla \varphi = \langle \mu, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

- Thm:

If $u_\varepsilon, v_\varepsilon$ solve $(*)$, then $(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon)$ solves $(**)$, where:

$$\begin{cases} a_\varepsilon := \sigma_\varepsilon \nabla u_\varepsilon \\ b_\varepsilon := \sigma_\varepsilon^T \nabla v_\varepsilon \\ \lambda = f+g \\ \mu = f-g \end{cases}$$

Moreover, $(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon)$ minimizes $F_\varepsilon^{f-g, f+g}$.

- Thm 1:

Let $(\sigma_\varepsilon)_\varepsilon \subseteq \mathcal{M}(c_0, c_1, \Omega)$, then, up to a subsequence, $\exists\, M \in L^\infty$ symmetric s.t. $F_\varepsilon \xrightarrow{\Gamma(d)} F_M$, M is positive definite & coercive.

- Thm 2:

Let $(\sigma_\varepsilon)_\varepsilon \subseteq \mathcal{M}(c_0, c_1, \Omega)$, $M \in L^\infty$ symm., pos. def. & coercive. Assume $F_\varepsilon \xrightarrow{\Gamma(d)} F_M$, then $\forall\, \mu, \lambda \in H^{-1}(\Omega)$, we have that:

$$F_\varepsilon^{\lambda,\mu} \xrightarrow{\Gamma(w)} F_M^{\lambda,\mu}$$

$$w\text{-}(L^2 \times H_0^1)$$

- 2. Cherkaev–Gibiansky variational principle:

goal: associate to

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon \in H_0^1(\Omega) \end{cases}$$

$$(*)$$

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon^T \nabla v_\varepsilon) = g \\ v_\varepsilon \in H_0^1(\Omega) \end{cases}$$

a variational structure. [Next time!]

3) Main results:

- **Thm 3.1**:

  Let $(\sigma_\varepsilon)_\varepsilon \subseteq \mathcal{M}(c_0, c_1, \Omega)$; then, up to a subsequence, $\exists\, \sigma_0 \in \mathcal{M}(c_0, c_1, \Omega)$ s.t.
  $$\begin{cases} \sigma_\varepsilon \xrightarrow{H} \sigma_0 \\ \sigma_\varepsilon^T \xrightarrow{H} \sigma_0^T \end{cases}$$

- **Thm 3.2**:

  Let $(\sigma_\varepsilon)_\varepsilon, \sigma_0$ in $\mathcal{M}(c_0, c_1, \Omega)$.
  The following are equivalent:

  a) $\sigma_\varepsilon \xrightarrow{H} \sigma_0$

  b) $\sigma_\varepsilon^T \xrightarrow{H} \sigma_0^T$

  convergence of minimizers?

  c) $F_\varepsilon \xrightarrow{\Gamma(d)} F_0$

  d) $F_\varepsilon^{\lambda,\mu} \xrightarrow{\Gamma\text{-}w} F_0^{\lambda,\mu} \quad \forall \lambda, \mu \in H^{-1}(\Omega)$

Proof:

- a) $\Leftrightarrow$ b) : thm 3.1
- c) $\Rightarrow$ d): thm 2
- d) $\Rightarrow$ a): uses the proof of thm 3.1
- a) & b) $\Rightarrow$ c): by thm 1: $F_\varepsilon \xrightarrow{\Gamma\text{-}d} F_M$, $M \in L^\infty$, symm., pos.-def., coercive; we "only" need to show $M = \Sigma_0$.