• Laurent 2

IV) γ-convergence, Mosco convergence:

• **Def:** let $(\Omega_n)_n \subseteq D$; we say that
$$\Omega_n \xrightarrow{\gamma} \Omega \quad \text{if} \quad \forall f \in H^1(D)$$
$$u^f_{\Omega_n} \xrightarrow{H^1_0(D)} u^f_\Omega$$

• Note: • $f = 1$ is enough
• also L^2-convergence is enough
• this is because there is ~~no~~ a good definition of convergence ensuring convergence of solution.

• **Def:** let $(A_n)_n \subseteq X$ closed convex, X normed space.
We say $A_n \xrightarrow{M} A$ if:

(M1) $\forall x \in A \; \exists x_n \in A_n$ s.t. $x_n \to x$,

(M2) $\forall y_{n_k} \in A_{n_k}$ s.t. $y_{n_k} \xrightarrow{w} y \Rightarrow y \in A$.

- **Proposition:**

The following are equivalent:

i) $\Omega_n \xrightarrow{\gamma} \Omega$,

ii) $H_0^1(\Omega_n) \xrightarrow{M} H_0^1(\Omega)$.

$\rightarrow$ That is: γ-convergence does not depend on $-\Delta$, but it is **OK** $\forall$ elliptic operator in divergence form with bounded coefficients [also for some higher-dimensional operator]

Proof:

- ii) $\Rightarrow$ i): let $f \in H^{-1}(D)$, let $u_n := u^f_{\Omega_n}$;

we know that, up to sub-seq., $u_n \xrightarrow{w\text{-}H_0^1} u^* \in H_0^1(D)$.

$\overset{(M2)}{\Longrightarrow} u^* \in H_0^1(\Omega)$.

Let us take a test function $\varphi \in H_0^1(\Omega)$;

by (M1) $\exists\, \varphi_n \in H_0^1(\Omega_n)$, $\varphi_n \xrightarrow{H_0^1(D)} \varphi$.

Then u^* solves the limiting eq. $\Rightarrow u^* = u^f_\Omega$.

- i) $\Rightarrow$ ii): (M1) let $\varphi \in H_0^1(\Omega)$; call $f := -\Delta\varphi \in H^{-1}(D)$

and $\varphi = u^f_\Omega$; let $\varphi_n := u^f_{\Omega_n} \in H_0^1(\Omega_n)$

$\overset{i)}{\Longrightarrow} \varphi_n \xrightarrow{H_0^1(D)} \varphi$

(M2) let $\varphi_k \in H_0^1(\Omega_{n_k})$, $\varphi_k \xrightarrow{w\text{-}H_0^1(D)} \varphi$;

? $\varphi \in H_0^1(\Omega)$?

Call $f := -\Delta\varphi$, $u_K := u^f_{\Omega_{n_K}}$;

by γ-convergence, $u_K \xrightarrow{s\text{-}H^1_0(D)} u := u^f_\Omega \in H^1_0(\Omega)$.

? $u=\varphi$? $\rightsquigarrow$

$$\int_D \nabla(u_K - \varphi_K)\cdot\nabla u_K\,dx$$

$$\underset{u_K\in H^1_0(\Omega_{n_K})}{=} \int_{\Omega_{n_K}} \nabla(u_K-\varphi_K)\cdot\nabla u_K\,dx =$$

$$\underset{\text{equation of } u_K}{=} \int_{\Omega_{n_K}} (u_K-\varphi_K)\cdot f$$

$$= \int_D (u_K - \varphi_K)\cdot f$$

u-s limit $\to$

$$\int_D \nabla(u-\varphi)\cdot\nabla u\,dx = \int_D (u-\varphi) f$$

$$\underset{\text{eq. of } u}{=} \int_D \nabla(u-\varphi)\cdot\nabla\varphi$$

$$\Downarrow$$

$$\int_D |\nabla(u-\varphi)|^2\,dx = 0 \Rightarrow u=\varphi \quad [\text{same Dirichlet condition}]$$

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→ The Mosco convergence is a statement about the capacity of $\Omega_m \setminus \Omega$

- Result of Dal Maso–Garroni:

  - Thm:

    Let A be an elliptic operator in divergence form with bounded coeff.
    For any sequence $(\Omega_n)_n \subseteq \Omega$. $\exists$ subsequence Ω_n,
    μ: positive Borel measure not charging polar set. s.t. $\forall f \in H^{-1}(D)$,

    $$u^f_{\Omega_n} \xrightarrow{H^1_0(D)} u \in H^1_0(D) \cap L^2_\mu(D)$$

    where:

    $$\langle Au, v\rangle + \int_D uv\, d\mu = \langle f, v\rangle$$

    $\forall v \in H^1_0(D) \cap L^2_\mu(D)$.

- idea: get rid of $H^1_0(\Omega_m)$ & $H^1_0(\Omega)$ & use a measure, by a relaxation of the Dirichlet pb.:

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{in } \Omega_n$$

$$\Downarrow$$

$$\langle Au, v\rangle + \int_D uv\, d\mu = \langle f, v\rangle$$

$\forall v \in H^1_0(D) \cap L^2_\mu(D)$

where: if E is closed $\left[\overset{\text{think it as}}{A_{\Omega_h}} \right]$, pick

$$\mu(B) := \infty_E(B) := \begin{cases} 0 & \text{if } \mathrm{cap}(B \cap E, D) = 0, \\ +\infty & \text{otherwise}, \end{cases}$$

then $u = 0$ q.e. in E and u solves

in $D \setminus E$: $\langle Au, v \rangle = \langle f, v \rangle$

→ the thm they have is:

$\forall$ sequence of equations: $\langle Au_n, v \rangle + \int_D u_n v \, d\mu_n = \langle f, v \rangle$

$\mu_n \in M_0(D) :=$ positive Borel meas. s.t. • $\mu(B) = 0$ $\forall$ B Borel, $\mathrm{cap}(B) = 0$ • $\mu(B) = \inf\{\mu(A) : A \supseteq B,\ A \text{ quasi-open}\}$

$\exists$ subsequence s.t. $u_n \xrightarrow{H_0^1(D)} u$, where

$$\langle Au, v \rangle + \int_D uv \, d\mu = \langle f, v \rangle,$$

where μ is characterized as follows:

adjoint

consider the dual pb: when $f = 1$ (recall Varak): $\langle A^* w_n^*, v \rangle + \int_D w_n^* v \, d\mu_n = \int_D v \, dx$

$\underset{\text{by ellipticity}}{\Longrightarrow}$ $w_n^* \xrightarrow{w\text{-}H_0^1(D)} w^*$; let $\nu^* := 1 - A^* w^*$

Then: ν^* is a Radon positive measure in $H^{-1}(D)$

and
$$\mu(B) := \begin{cases} \displaystyle\int_B \frac{d\nu^*}{w^*} & \text{if } \mathrm{cap}(B \cap \{w^* = 0\}, D) = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

- $\underline{\gamma^A\text{-convergence}}$:

Def: let $(\mu_n)_n \subseteq M_0(D)$, we say that $\mu_n \xrightarrow{\gamma^A} \gamma$
if $\forall f \in H^{-1}(D)$ the sol. $u_n \in H_0^1(D) \cap L^2_{\mu_n}(D)$
of
$$\langle Au_n, v\rangle + \int_D u_n v \, d\mu_n = \langle f, v\rangle$$
are s.t.
$$u_n \xrightarrow{w-H_0^1(D)} u \in H_0^1(D) \cap L^2_\mu(D),$$
where
$$\langle Au, v\rangle + \int_D uv \, d\mu = \langle f, v\rangle.$$

- $\underline{\text{Thm}}$:
$$\mu_n \xrightarrow{\gamma^A} \mu \iff \mu_n \xrightarrow{\gamma^{A^*}} \mu \iff A \text{ with } f \equiv 1$$
$$\iff A^* \text{ with } f \equiv 1$$

→ Does it depend on the operator A?

- If $A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = A^s + A^a$

if $A^a \in C^0$ then: $\gamma^A \iff \gamma^{A^s}$

if not it is false: $\Omega := (-1,1)^N$, $\Gamma := \{x_N = 0\}$

⇝ $\Omega_n = B_{R+\varepsilon_n}$ $\quad \varepsilon_n \to 0$ $\quad$ what about this case?