• Janusz – Periodic homogenization of certain fully nonlinear PDEs

⤳ based on the article of Evans, '92.

Consider:

$$(P)\quad \begin{cases} F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}) = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense, where $\Omega \subseteq \mathbb{R}^N$ is a nice set, and F is periodic in the last variable.

→ is that an effective limit eq.?

→ if F linear ⤳ cell problem

→ if F nonlinear ⤳ similar idea!

To be more precise:

let

$$F: Sym \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times \mathbb{R}^n \to \mathbb{R}$$

s.t.

i) $y \mapsto F(R, P, u, x, y)$ periodic

ii) $\exists\ \theta > 0$ s.t. $\theta |\xi|^2 \leq -\frac{\partial F}{\partial R_{ij}} \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^n$

$\Updownarrow$

ii') $F(R+S, p, u, x, y) + \text{trace}(S) \leq F(R, p, u, x, y) \quad \forall S \geq 0$

iii) $\exists \mu > 0$ s.t. $u \mapsto F(R, p, u, x, y) - \mu u$

is non-decreasing

iv) F is Lipschitz

- It is possible to prove, under these hypothesis, that $\exists \gamma > 0$ s.t.

$$\sup_{\varepsilon > 0} \| u_\varepsilon \|_{C^{0,\gamma}} < +\infty.$$

Then, up to subsequence, $u_\varepsilon \to u_0$ uniformly.

$\to$ Does the u_0 satisfy a limiting equation?

In order to understand it, let us recall the linear case:

$$F(R, p, u, x, y) = a_{ij}(x, y) R_{ij} - f(x).$$

$\big\}$ by a formal asympotic expansion

$$\overline{a}_{ij}(x) u_{x_i x_j} - f(x) = 0$$

where:
$$\overline{a_{ij}}(x) := \int_{\mathbb{T}^n} a_{ij}(x, y) m(y) \, dy,$$

where m solves:

$$\begin{cases} (a_{ij} m)_{x_i x_j} = 0, \\ \int_{\mathbb{T}^n} m(y) = 1. \end{cases}$$ $\rightsquigarrow$ the adjoint problem for a

→ | can we write the cell problem in a way that it involves $\bar{F}$?

It holds that:

$$(*)\quad \begin{cases} F(R+D^2_{yy}v, p, u, x, y) = \lambda \\ v \text{ periodic} \end{cases} \quad \rightsquigarrow \left[\begin{array}{l}\text{this is the} \\ \text{analogous of} \\ \text{the cell problem}\end{array}\right.$$

has a solution $\iff$ $\lambda = \bar{a}_{ij}(x) R_{ij} - f(x)$.

[the argument is by Fredholm alternative:

$(*)$ has a solution $\iff$ $0 = \int_{T^n} [a_{ij}(x,y) R_{ij} - f(x) - \lambda] m$

Thus from this heuristic perspective, we guess that

$$\bar{F}(R, p, u, x) = \lambda(R, p, u, x).$$

s.t. $(*)$ has a solution.

- Results:

- Thm 1:

$\forall R \in Sym, p \in \mathbb{R}^n, u \in \mathbb{R}, x \in \bar{\Omega}$, $\exists ! \lambda \in \mathbb{R}$ s.t. $(*)$ has a solution.

- Thm 2:

$\bar{F}$ satisfies i), iii), iv).

· Thm 3:

u_0 is a solution of $\begin{cases} \overline{F}(D^2 u_0, Du_0, u_0, x) = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$

Let us take a look at the proof of this last thm:

Take a test function $\varphi \in C^\infty$ s.t.

$$(u-\varphi)(x_0) = 0$$

is a strict local maximum for $u-\varphi$.

claim: $\overline{F}(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq 0.$

Assume not. $= \theta > 0$.

Let v be a solution of:

$$\begin{cases} F(D^2\varphi(x_0) + D_y^2 v, D\varphi(x_0), u(x_0), x_0, y) = \theta \\ v \text{ periodic.} \end{cases}$$

Define: $\varphi_\varepsilon(x) := \varphi(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right).$

Then: (at least heuristically),

$$F\left(D^2\varphi_\varepsilon(x), D\varphi_\varepsilon(x), \varphi_\varepsilon(x), x, \frac{x}{\varepsilon}\right) =$$

$$= F\left(D^2\varphi(x) + D_y^2 v\left(\frac{x}{\varepsilon}\right), D\varphi(x) + \varepsilon Dv\left(\frac{x}{\varepsilon}\right), \varphi(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right), x, \frac{x}{\varepsilon}\right)$$

$$\overset{x \sim x_0}{\sim} F\left(D^2\varphi(x_0) + D_y^2 v\left(\frac{x}{\varepsilon}\right), D\varphi(x_0), \varphi(x_0), x_0, \frac{x}{\varepsilon}\right)$$

$$= \theta > 0.$$

$\Rightarrow \exists r > 0$ s.t. $\forall x \in B_r(x_0)$

$$F\left(D^2\varphi_\varepsilon(x), D\varphi_\varepsilon(x), \varphi_\varepsilon(x), x, \frac{x}{\varepsilon}\right) \geq \frac{\theta}{2} > 0$$

$\downarrow$ by maximum principle

Also: $F\left(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x, \frac{x}{\varepsilon}\right) = 0$

$$\Rightarrow \quad \underbrace{(u_\varepsilon - \varphi_\varepsilon)(x_0)}_{\to u - \varphi \text{ uniformly}} \leq \max_{x \in \partial B_r(x_0)} \underbrace{(u_\varepsilon - \varphi_\varepsilon)(x)}_{\to u - \varphi \text{ uniformly}}$$

$\Rightarrow$ $\lightning$ in contradiction with the hypothesis that x_0 was a point of strict local maximum.

$\to$ in the same way, it's possible to prove the opposite inequality.

$\boxed{\checkmark}$

Let us consider the proof of Thm 1:

- let $\delta > 0$ and let w_δ be the periodic solution of

$$\delta w_\delta + F(D_y^2 w_\delta + R, p, u, x, y) = 0.$$

[it exists by using iii) & Perron's method]

It holds:

- $\| \delta w_\delta \|_{L^\infty} \leq \| F(R, p, u, \cdot, \cdot) \|_{L^\infty}$,

- $\sup_{0 < \delta < 1} [w_\delta]_{C^{0,\alpha}} < +\infty.$

Let $v_\delta := w_\delta - \min w_\delta$.

Then, up to a subsequence:

- $\delta \to 0$
- $v_\delta \to v$
- $\delta w_\delta \to \lambda \in \mathbb{R}$

and v solves:

$$-\lambda + F(D_y^2 v + R, p, u, x, y) = 0.$$

- For uniqueness, assume you have $\bar{\lambda} > \lambda$ and solutions $\bar{v}$ and v of the corresponding pbs. Up to adding a constant, we may assume that $v > \bar{v}$. Moreover, for ε small enough, we have

$$\varepsilon \bar{v} + \underbrace{F(D_y^2 \bar{v} + R, p, u, x, y)}_{= \bar{\lambda}} > \varepsilon v + \underbrace{F(D_y^2 v + R, p, u, x, y)}_{= \lambda}$$

But now we can apply the comparison principle, to get that $\bar{v} \geq v$.