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J. Differential Equations 188 (2003) 591–624

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Dirichlet problems on varying domains

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Received October 26, 2001; revised May 6, 2002

Abstract

The aim of the paper is to characterise sequences of domains for which solutions to an elliptic equation with Dirichlet boundary conditions converge to a solution of the corresponding problem on a limit domain. Necessary and sufficient conditions are discussed for strong and uniform convergence for the corresponding resolvent operators. Examples are given to illustrate that most results are optimal.

MSC: 35B20; 35B25; 35J25

Keywords: Boundary value problems for second-order elliptic equations; Domain convergence; Singular perturbation of domain

1. Introduction

The purpose of this paper is to discuss conditions on sequences of domains $\Omega_n \subset \mathbb{R}^N$ $(N \geqslant 2)$ such that solutions of the elliptic boundary value problems

$$\begin{aligned} \mathscr{A}u + \lambda u &= f_n \quad \text{in } \Omega_n, \\ u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{1.1}$$

converge to a solution of the corresponding problem

$$\begin{aligned} \mathscr{A}u + \lambda u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

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PII: S0022-0396(02)00105-5

on a limit domain Ω as $n \to \infty$. The motivation to look at such problems comes from variational inequalities (see [35]), numerical analysis (see [27,37,41–44]), potential and scattering theory (see [4,38,40,46]), control and optimisation (see [12,13,30,45]), Γ-convergence (see [9,15]) and solution structures of non-linear elliptic equations (see [16–18,24]). We do not attempt here to give a complete bibliography, but make a rather arbitrary choice of references. As the framework, motivation and notation used in the literature vary enormously, it can be difficult to compare results.

We start our analysis with two conditions naturally coming up when trying to prove convergence of solutions of (1.1) to a solution of (1.2) (see proof of Theorem 3.1). They are

$$\text{the weak limit points of every sequence } u_n \in H_0^1(\Omega_n),\ n \in \mathbb{N}, \text{ in } H^1(\mathbb{R}^N) \text{ are in } H_0^1(\Omega), \tag{1.3}$$

$$\text{for every } \varphi \in H_0^1(\Omega) \text{ there exist } \varphi_n \in H_0^1(\Omega_n) \text{ such that } \varphi_n \to \varphi \text{ in } H^1(\mathbb{R}^N). \tag{1.4}$$

Here $H_0^1(\Omega)$ and $H^1(\mathbb{R}^N)$ denote the usual Sobolev spaces of functions vanishing on $\partial\Omega$. Extending functions in $H_0^1(\Omega)$ by zero outside Ω we may consider $H_0^1(\Omega)$ as a closed subspace of $H^1(\mathbb{R}^N)$, so (1.3) and (1.4) make sense. It turns out that the two conditions are not only sufficient but also necessary for convergence, which is known for some classes of operators (see for instance [14]). For this reason we make the following definition.

Definition 1.1. If $\Omega_n, \Omega \subset \mathbb{R}^N$ are such that (1.3) and (1.4) are satisfied we write $\Omega_n \to \Omega$.

It is often said that $\Omega_n \to \Omega$ in the sense of Mosco (as this is equivalent to $H_0^1(\Omega_n) \to H_0^1(\Omega)$ in the sense of Mosco [35, Section 1]). The conditions also appear in a more disguised form in [41], and explicitly in [44]. Necessary and sufficient conditions in terms of capacity for (1.3) and (1.4) are discussed in [10] in case Ω_n is contained in a fixed bounded set for all $n \in \mathbb{N}$.

In this paper we improve previous results in several directions. First of all we work with necessary and sufficient conditions for convergence. We allow unbounded domains with infinite measure and non-self-adjoint operators. Many papers allow one or the other, but not simultaneously. Also, we look at convergence in L_p-norms, $p \in (1, \infty)$. Finally, we characterise under what conditions the resolvents converge uniformly, that is, in the operator norm of $\mathscr{L}(L_2(\mathbb{R}^N))$. Note that the methods could be used to treat some other, even non-linear or parabolic operators. We refrain from doing so and restrict ourselves to one class of operators allowing quite elementary proofs, not involving the theory of Γ-convergence.

An outline of the paper is as follows. In Section 2 we fix the assumptions, notation and framework used throughout the paper. Section 3 is concerned with basic convergence results. In particular, we prove that convergence holds for all operators of the class under consideration if and only if it holds for one particular operator. In Section 4 we discuss conditions for uniform convergence of the resolvents. As a consequence we get continuity properties of the spectrum with respect to the domain not true in general. Section 5 deals with convergence in L_p-spaces. A good theory in L_p-spaces is important when dealing with non-linear problems such as those in [16]. In Section 6 we establish some necessary conditions for convergence. Conditions (1.3) and (1.4) are not always easy to verify. We discuss some sufficient conditions which are easy to check in Section 7. Examples showing that most results are optimal are given in Section 8. We conclude with an appendix containing some auxiliary abstract results.

2. Assumptions and framework

The purpose of this section is to introduce the framework we need for a precise formulation of our results. We will always assume that Ω_n, Ω are open (possibly unbounded and disconnected) sets in $\mathbb{R}^N$, $N \geqslant 2$. The Lebesgue measure of a set $S \subset \mathbb{R}^N$ we denote by $|S|$. If Ω is an open set we denote by $H_0^1(\Omega)$ the closure of the set of test functions $C_c^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$. The norm we use is always $\|u\|_{H^1} = (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}$, where $\|u\|_p$ is the L_p-norm. Extending elements of $C_c^\infty(\Omega)$ by zero outside Ω we may consider $C_c^\infty(\Omega)$ in a natural way as a subspace of $C_c^\infty(\mathbb{R}^N)$. Hence, taking closures we may identify $H_0^1(\Omega)$ with a closed subspace of $H_0^1(\mathbb{R}^N) = H^1(\mathbb{R}^N)$, and we will do so henceforth.

The operator $\mathscr{A}$ is always of the form

$$\mathscr{A}u \coloneqq -\sum_{i=1}^{N} \partial_i \left(\left(\sum_{j=1}^{N} a_{i,j} \partial_j u \right) + a_i u \right) + \sum_{i=1}^{N} b_i \partial_i u + c_0 u, \tag{2.1}$$

where $a_{i,j}, a_i, b_i, c_0 \in L_\infty(\mathbb{R}^N)$ for all $i, j = 1, \ldots, N$. Moreover, we assume that there exists a constant $\alpha > 0$, called the ellipticity constant, such that

$$\sum_{j=1}^{N} \sum_{i=1}^{N} a_{i,j} \xi_i \xi_j \geqslant \alpha |\xi|^2 \tag{2.2}$$

for almost all $x \in \mathbb{R}^N$ and all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$. The simplest case is the Laplace operator $-\Delta$. We define the form, $a(\cdot, \cdot)$, associated with $\mathscr{A}$ by

$$a(u,v) \coloneqq \int_{\mathbb{R}^N} \sum_{i=1}^{N} \left(\left(\sum_{j=1}^{N} a_{i,j} \partial_j u \right) + a_i u \right) \partial_i v + \left(\sum_{i=1}^{N} b_i \partial_i u + c_0 u \right) v \, dx \tag{2.3}$$

for all $u, v \in H^1(\mathbb{R}^N)$. It is easy to check that $a(\cdot\,,\cdot)$ is a bounded bilinear form on $H^1(\mathbb{R}^N)$ (and thus on $H^1_0(\Omega)$ for every open set $\Omega \subset \mathbb{R}^N$). If $u, v: \Omega \to \mathbb{R}$ are two measurable functions we set

$$\langle u, v \rangle := \int_\Omega uv\, dx$$

if the integral exists. By the Riesz representation theorem we can identify $L_2(\Omega)$ with its dual. If we do that then $H^1_0(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the topological dual of $H^1_0(\Omega)$ equipped with the dual norm. Duality between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ we also denote by $\langle\,\cdot\,,\cdot\rangle$. Given $f \in H^{-1}(\Omega)$, we call u a weak solution of (1.2) if $u \in H^1_0(\Omega)$, and

$$a(u, v) + \lambda \langle u, v \rangle = \langle f, v \rangle \tag{2.4}$$

for all $v \in H^1_0(\Omega)$. If we set

$$\lambda_0 := \|c_0^-\|_\infty + \frac{1}{2\alpha} \sum_{i=1}^N \|a_i + b_i\|_\infty^2, \tag{2.5}$$

then standard arguments show that

$$\frac{\alpha}{2} \|u\|^2_{H^1(\mathbb{R}^N)} \leqslant a(u, u) + \lambda \|u\|_2^2 \tag{2.6}$$

for all $u \in H^1(\mathbb{R}^N)$ and $\lambda \geqslant \lambda_0$, where $c_0^- := \max(-c_0, 0)$ is the negative part of c_0. The Lax–Milgram Theorem [47, Section III.7] then ensures the existence of a unique weak solution $u \in H^1_0(\Omega)$ of (1.2) for all $f \in H^{-1}(\Omega)$ whenever $\lambda \geqslant \lambda_0$. Moreover, that solution satisfies the a priori estimate

$$\|u\|_{H^1_0(\Omega)} \leqslant \frac{2}{\alpha} \|f\|_{H^{-1}(\Omega)}. \tag{2.7}$$

To prove (2.7) note that by (2.4) and (2.6)

$$\frac{\alpha}{2} \|u\|^2_{H^1_0(\Omega)} \leqslant a(u, u) + \langle \lambda u, u \rangle = \langle f, u \rangle \leqslant \|u\|_{H^1_0(\Omega)} \|f\|_{H^{-1}(\Omega)}.$$

Dividing by $\|u\|_{H^1_0(\Omega)}$ the required estimate follows. It is often convenient to write (1.2) in an abstract form. To do so recall that $a(\cdot\,,\cdot)$ is a bounded bilinear form on $H^1_0(\Omega)$. Therefore, there exists $A_\Omega \in \mathscr{L}(H^1_0(\Omega), H^{-1}(\Omega))$ such that

$$a(u, v) = \langle A_\Omega u, v \rangle \tag{2.8}$$

for all $u, v \in H_0^1(\Omega)$. We call A_Ω the operator induced by $\mathscr{A}$ on Ω. From the definition of A_Ω it is quite obvious that $u \in H_0^1(\Omega)$ is a weak solution of (1.2) if and only if u is a solution of $(\lambda + A_\Omega)u = f$ in $H^{-1}(\Omega)$. It is sometimes useful to consider A_Ω as an operator on $H^{-1}(\Omega)$ with domain $H_0^1(\Omega)$. As we know that $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$ it follows from (2.6) that A_Ω is a closed densely defined operator on $H^{-1}(\Omega)$. We denote by $\varrho(A_\Omega)$ and $\sigma(A_\Omega)$ the resolvent set and the spectrum of A_Ω, respectively. By the previous consideration and (2.7)

$$[\lambda_0, \infty) \subset \varrho(-A_\Omega) \quad \text{for every open set } \Omega \subset \mathbb{R}^N. \tag{2.9}$$

As we are working with varying domains we want a family of operators with domain and range independent of Ω_n and $\Omega \subset \mathbb{R}^N$. To do so denote by $i_\Omega \in \mathscr{L}(H_0^1(\Omega), H^1(\mathbb{R}^N))$ the operator extending functions in $H_0^1(\Omega_n)$ by zero outside Ω. Moreover, denote by $r_\Omega \in \mathscr{L}(H^{-1}(\mathbb{R}^N), H^{-1}(\Omega))$ the operator restricting functionals $f \in H^{-1}(\mathbb{R}^N)$ to $H_0^1(\Omega)$. Obviously $\langle f, i_{\Omega_n}(u) \rangle = \langle r_{\Omega_n}(f), u \rangle$ for all $u \in H_0^1(\Omega_n)$ and $f \in H^{-1}(\mathbb{R}^N)$, so

$$i'_\Omega = r_\Omega \quad \text{and} \quad r'_\Omega = i_\Omega. \tag{2.10}$$

The following lemma relates A_Ω to $A := A_{\mathbb{R}^N}$.

Lemma 2.1. *For every open set $\Omega \subset \mathbb{R}^N$*

$$A_\Omega + \lambda = r_\Omega \circ (A + \lambda) \circ i_\Omega$$

and $\|A_\Omega + \lambda\|_{\mathscr{L}(H^1(\Omega), H^{-1}(\Omega))} \leqslant \|A + \lambda\|_{\mathscr{L}(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))}$. Moreover, if $u \in H_0^1(\Omega)$ and $g := (\lambda + A) \circ i_\Omega(u)$ then u is a weak solution of (1.2) *with $f := r_\Omega(g)$.* (*In our exposition we always identified f with $r_\Omega(g)$.*)

Proof. By (2.8) and (2.10) we have

$$\langle A_\Omega u, v \rangle = a(u, v) = a(i_\Omega u, i_\Omega v) = \langle A \circ i_\Omega u, i_\Omega v \rangle = \langle r_\Omega \circ A \circ i_\Omega u, v \rangle$$

for all $u, v \in H_0^1(\Omega)$. Hence the first assertion of the lemma follows. The estimate follow as $\|i_\Omega\|, \|r_\Omega\| \leqslant 1$. The last assertion follows from the first as u is a weak solution of (1.2) if and only if $(\lambda + A_\Omega)u = f$ (if we identify f with $= r_\Omega(f)$ as usual). □

Given open sets $\Omega_n, \Omega \subset \mathbb{R}^N$ we set

$$\mathscr{R}_n(\lambda) := i_{\Omega_n} \circ (\lambda + A_{\Omega_n})^{-1} \circ r_{\Omega_n} \quad \text{and} \quad \mathscr{R}(\lambda) := i_\Omega \circ (\lambda + A_\Omega)^{-1} \circ r_\Omega \tag{2.11}$$

whenever the operators are defined. By looking at elements of $H^{-1}(\mathbb{R}^N)$ we do not lose anything as by the Hahn–Banach Theorem, every functional in $H^{-1}(\Omega)$ can be extended to a functional in $H^{-1}(\mathbb{R}^N)$ with equal norm (see [47, Section IV.5]). If there is no confusion likely we identify $u \in H_0^1(\Omega)$ with $i_\Omega(u)$ and $f \in H^{-1}(\mathbb{R}^N)$ with $r_\Omega(f)$. To prove our results we will often work with the adjoint form $a^\sharp(\cdot\,,\cdot)$ defined by $a^\sharp(u,v) := a(v,u)$ for all $u, v \in H_0^1(\mathbb{R}^N)$. This is the form associated with the formally adjoint operator, $\mathscr{A}^\sharp$, of $\mathscr{A}$ given by

$$\mathscr{A}^\sharp u := -\sum_{i=1}^{N} \partial_i\left(\left(\sum_{j=1}^{N} a_{j,i}\partial_j u\right) + b_i u\right) + \sum_{i=1}^{N} a_i \partial_i u + c_0 u. \tag{2.12}$$

If we denote by $A_\Omega^\sharp \in \mathscr{L}(H_0^1(\Omega), H^{-1}(\Omega))$ the operator induced by $a^\sharp(\cdot\,,\cdot)$ then clearly

$$A'_\Omega = A_\Omega^\sharp \quad \text{and} \quad (A_\Omega^\sharp)' = A_\Omega. \tag{2.13}$$

Further, recall that an operator and its dual have the same spectrum. Hence, by (2.13) we can define

$$\mathscr{R}_n^\sharp(\lambda) := i_{\Omega_n} \circ (\lambda + A_{\Omega_n}^\sharp)^{-1} \circ r_{\Omega_n} \quad \text{and} \quad \mathscr{R}^\sharp(\lambda) := i_\Omega \circ (\lambda + A_\Omega^\sharp)^{-1} \circ r_\Omega \tag{2.14}$$

whenever $\mathscr{R}_n(\lambda)$ and $\mathscr{R}(\lambda)$ exist. Using (2.10) and (2.13) we also see that

$$(\mathscr{R}_n^\sharp(\lambda))' = \mathscr{R}_n(\lambda) \quad \text{and} \quad (\mathscr{R}^\sharp(\lambda))' = \mathscr{R}(\lambda). \tag{2.15}$$

Note that $\mathscr{R}(\lambda)$ is not a resolvent, but only a pseudo-resolvent, that is, a family of operators satisfying the resolvent identity. For completeness we include the following standard lemma on positivity of solutions.

Lemma 2.2. *Suppose that u is the solution of* (1.2), *that* $\lambda \geqslant \lambda_0$, *and that* $f \in L_2(\mathbb{R}^N)$ *is non-negative. Then u is non-negative.*

Proof. It follows from [26, Lemma 7.6] that $u^- := \max\{-u, 0\} \in H_0^1(\Omega)$, that $\nabla u^- = -\nabla u$ if $u > 0$, and that $\nabla u^- = 0$ otherwise. As u is a weak solution of (1.2) it follows from (2.6) that

$$\langle f, u^- \rangle = a(u, u^-) + \lambda \langle u, u^- \rangle = -a(u^-, u^-) - \lambda \|u^-\|_2^2 \leqslant -\frac{\alpha}{2}\|u^-\|_{H^1(\mathbb{R}^N)}^2 \leqslant 0.$$

As $\langle f, u^- \rangle \geqslant 0$, we have $\|u^-\|_{H^1} = 0$, that is, $u \geqslant 0$. □

3. Basic convergence results

In this section we discuss some basic convergence results. Throughout we will use the assumptions and notation from Section 2. Note that in the whole paper we could replace the operator $\mathscr{A}$ by a sequence of operators $\mathscr{A}_n$ whose coefficients converge in a sufficiently strong way as done for instance in [18]. We refrain from doing so to keep the notation and statement of results as simple as possible.

The first result does not require uniqueness of solutions of (1.1) or the limit problem (1.2).

Theorem 3.1. *Suppose that* $u_n \in H_0^1(\Omega_n)$ *are weak solutions of* (1.1) *for all* $n \in \mathbb{N}$. *If* (1.4) *holds then every weak limit point of* $(u_n)_{n\in\mathbb{N}}$ *lying in* $H_0^1(\Omega)$ *is a weak solution of* (1.2) *for some* $f \in H^{-1}(\Omega)$.

Note that if (1.3) holds then every weak limit point of $(u_n)_{n\in\mathbb{N}}$ is in $H_0^1(\Omega)$.

Proof. Suppose that $v \in H_0^1(\Omega)$ is a limit point of $(u_n)_{n\in\mathbb{N}}$, which means that there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ with $u_{n_k} \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$ as $k \to \infty$. As u_n is a weak solution of (1.1) and $a(\cdot\,,\cdot)$ is a bounded bilinear form on $H^1(\mathbb{R}^N)$ there exists $M \geqslant 1$ independent of $n \in \mathbb{N}$ such that

$$|\langle f_n, \varphi \rangle| = |a(u_n, \varphi) + \lambda \langle u_n, \varphi \rangle| \leqslant M \|u_n\|_{H^1} \|\varphi\|_{H^1}$$

for all $\varphi \in H_0^1(\Omega_n)$. By definition of the dual norm $\|f_n\|_{H^{-1}(\Omega_n)} \leqslant M\|u_n\|_{H^1}$. By the Hahn–Banach Theorem (see [47, Section IV.5]) it is possible to extend f_n to $\tilde{f}_n \in H^{-1}(\mathbb{R}^N)$ such that $\|\tilde{f}_n\|_{H^{-1}(\mathbb{R}^N)} = \|f_n\|_{H^{-1}(\Omega_n)} \leqslant M\|u_n\|_{H^1}$. As every weakly convergent sequence is bounded it follows that $(\tilde{f}_{n_k})_{k\in\mathbb{N}}$ is bounded in $H^{-1}(\mathbb{R}^N)$. Using that every bounded sequence in a Hilbert space has a convergent subsequence we can, after possibly passing to another subsequence, assume that $f_{n_k} \rightharpoonup \tilde{f}$ in $H^{-1}(\mathbb{R}^N)$ for some $\tilde{f} \in H^{-1}(\mathbb{R}^N)$. We now show that $v \in H_0^1(\Omega)$ is a weak solution of (1.2) for $f := r_\Omega(\tilde{f}) \in H^{-1}(\Omega)$. To do so fix $\varphi \in C_c^\infty(\Omega)$. By assumption (1.4) there exist $\varphi_n \in H_0^1(\Omega_n)$ such that $\varphi_k \to \varphi$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Using that u_n is a weak solution of (1.1)

$$a(u_{n_k}, \varphi_{n_k}) + \lambda \langle u_{n_k}, \varphi_{n_k} \rangle = \langle f_{n_k}, \varphi_{n_k} \rangle \tag{3.1}$$

for all $k \in \mathbb{N}$. As $u_{n_k} \rightharpoonup v$ weakly and $\varphi_{n_k} \to \varphi$ strongly in $H^1(\mathbb{R}^N)$ we conclude that $a(u, \varphi) + \lambda \langle v, \varphi \rangle = \langle f, \varphi \rangle$ by letting k go to infinity in (3.1). Because $v \in H_0^1(\Omega)$, and $\varphi \in C_c^\infty(\Omega)$ was arbitrary, v is a weak solution of (1.2). □

Corollary 3.2. *If in addition to the assumptions of Theorem* 3.1 *we suppose that* (1.3) *holds, that* $f_n \rightharpoonup f$ *weakly in* $H^{-1}(\mathbb{R}^N)$, *that* (1.2) *has unique solution and that* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $H^1(\mathbb{R}^N)$ *then* $u_n \rightharpoonup u = \mathscr{R}(\lambda)f$ *weakly in* $H^1(\mathbb{R}^N)$.

Proof. By Theorem 3.1 and (1.3) every weak limit point of (u_n) is a solution. By uniqueness of solutions of (1.2) and since $f_n \rightharpoonup f$ the only possible weak limit point of (u_n) is $u = \mathscr{R}(\lambda)f$. As bounded sequences in a Hilbert space are sequentially weakly compact it follows that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. □

The second theorem shows that (1.3) and (1.4) are necessary and sufficient for convergence. In particular, it shows that convergence is independent of the particular operator $\mathscr{A}$, hence it generalises a result in [6, Section 5], where equivalence was only shown for self-adjoint operators, and if Ω_n is contained in a fixed bounded set for all $n \in \mathbb{N}$. Recall that A_{Ω_n} is the operator induced by $\mathscr{A}$ on Ω_n defined in (2.1) and $\mathscr{R}_n(\lambda), \mathscr{R}(\lambda)$ are given by (2.11).

Theorem 3.3. *Suppose that $\Omega, \Omega_n \subset \mathbb{R}^N$ are open sets, and that $\lambda \in \varrho(-A_{\Omega_n}) \cap \varrho(-A_\Omega)$ for all $n \in \mathbb{N}$. Then the following assertions are equivalent.*

(1) *$\Omega_n \to \Omega$ in the sense of Definition* 1.1 *and*

$$\limsup_{n\to\infty} \|(\lambda + A_{\Omega_n})^{-1}\|_{\mathscr{L}(L_2(\Omega_n))} < \infty; \tag{3.2}$$

(2) *$\mathscr{R}(\lambda)f_n \rightharpoonup \mathscr{R}(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$;*

(3) *$\mathscr{R}_n(\lambda)f_n \to \mathscr{R}(\lambda)f$ converges in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$.*
Let λ_0 be given by (2.5). *If $\lambda \geqslant \lambda_0$ then the following is equivalent to the above.*

(4) *$\mathscr{R}_n(\lambda)f \rightharpoonup \mathscr{R}(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$ for f in a dense subset of $H^{-1}(\mathbb{R}^N)$.*
Finally, if $\lambda \geqslant \lambda_0$ and $\sup_{n\in\mathbb{N}} |\Omega_n| < \infty$ then also the following is equivalent to the above.

(5) *$\mathscr{R}_n(\lambda)1 \rightharpoonup \mathscr{R}(\lambda)1$ weakly in $H^1(\mathbb{R}^N)$.*

The proof of the above theorem will be given in Section 9. Note that (3.2) is necessary in Theorem 3.3 as Example 8.2 shows.

Remark 3.4. If $f_n \rightharpoonup f$ weakly in $H^1(\mathbb{R}^N)$ we do not have strong convergence of $\mathscr{R}(\lambda)f_n$ in $H^1(\mathbb{R}^N)$ or $L_2(\mathbb{R}^N)$. If (2) of Theorem 3.3 holds and $f_n \rightharpoonup f$ weakly in $H^1(\mathbb{R}^N)$ then by Rellich's Theorem $\mathscr{R}_n(\lambda)f_n \to \mathscr{R}(\lambda)f$ in $L_{p,\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2, 2N(N-2)^{-1})$. If $\lambda \geqslant \lambda_0$ and $f_n \to f$ in $H^{-1}_{\mathrm{loc}}(B)$ for some open set $B \subset \mathbb{R}^N$ then $\mathscr{R}_n(\lambda)f_n \to \mathscr{R}(\lambda)f$ in $H^1_{\mathrm{loc}}(B)$. In particular, convergence takes place in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ if $f_n \to f$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^N)$. For a proof of these facts see Lemma 9.1.

Remark 3.5. In (5) we need to be careful what we mean by 1 as $1 \notin H^{-1}(\mathbb{R}^N)$. As Ω_n has finite measure it is clear that $1 \in H^{-1}(\Omega_n)$. We define $f_n \in L_2(\mathbb{R}^N)$ by $f_n = 1$ on Ω_n, and $f_n = 0$ outside Ω_n. Then $f_n \in H^{-1}(\mathbb{R}^N)$ and $\|f_n\|_{H^{-1}(\mathbb{R}^N)} \leqslant |\Omega_n|^{1/2}$. If Ω_n has uniformly bounded measure then f_n is bounded, showing that (5) makes only sense for sequences $(\Omega_n)_{n\in\mathbb{N}}$ with uniformly bounded measure. Also note that we cannot expect $\mathscr{R}_n(\lambda)1$ to converge strongly in $L_2(\mathbb{R}^N)$ in general, as f_n does not in general converge strongly. In fact, assuming strong convergence Theorem 4.4 below shows that $\mathscr{R}_n(\lambda)$ converges uniformly, that is, in $\mathscr{L}(L_2(\mathbb{R}^N))$.

4. Uniform convergence and continuity of the spectrum

For applications such as those in [16] it is important to know how the spectrum of (1.1) and (1.2) relate to each other if $\Omega_n \to \Omega$. In the general framework considered in Section 3 we cannot expect continuity of the spectrum as the results in Theorem 3.3 only show that $\mathscr{R}_n(\lambda)$ converges strongly in $\mathscr{L}(L_2(\mathbb{R}^N))$, that is pointwise. Under suitable assumptions on c_0 one can prove continuity of part of the spectrum. We will not pursue this further but refer to [37] or [46]. We only discuss continuity of the spectrum in case of uniform convergence. It is convenient here to look at the complexification of the problem as usual in spectral theory.

Theorem 4.1. *Suppose that $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ in $\mathscr{L}(L_2(\mathbb{C}^{\mathbb{N}}))$ for some $\lambda \in \mathbb{C}$. Then, for every $\mu \in \varrho(-A_\Omega)$ we have $\mu \in \varrho(-A_{\Omega_n})$ for $n \in \mathbb{N}$ large enough, and $\mathscr{R}_n(\mu) \to \mathscr{R}(\mu)$ in $\mathscr{L}(L_2(\mathbb{C}^N))$.*

Proof. Suppose that $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ in $\mathscr{L}(L_2(\mathbb{C}^N))$ for some $\lambda \in \mathbb{R}$, and that $\mu \in \varrho(-A_\Omega)$. It follows from the Proposition A.1 that $(\mu - \lambda)^{-1} \in \varrho(-\mathscr{R}(\lambda))$. But then by Kato [31, Theorem IV.2.25] we have $(\mu - \lambda)^{-1} \in \varrho(-\mathscr{R}_n(\lambda))$ if only n is large enough, and by the resolvent identity

$$\lim_{n\to\infty} ((\mu - \lambda)^{-1} + \mathscr{R}_n(\lambda))^{-1} = ((\mu - \lambda)^{-1} + \mathscr{R}(\lambda))^{-1}$$

in $\mathscr{L}(L_2(\mathbb{C}^N))$. Applying Proposition A.1 again we see that $\mu \in \varrho(-A_{\Omega_n})$ if n is large enough, and that

$$\begin{aligned}\lim_{n\to\infty} \mathscr{R}_n(\mu) &= \lim_{n\to\infty} (\mu - \lambda)^{-1}\mathscr{R}_n(\lambda)((\mu - \lambda)^{-1} + \mathscr{R}_n(\lambda))^{-1} \\ &= (\mu - \lambda)^{-1}\mathscr{R}(\lambda)((\mu - \lambda)^{-1} + \mathscr{R}(\lambda))^{-1} = \mathscr{R}(\mu)\end{aligned}$$

in $\mathscr{L}(L_2(\mathbb{C}^N))$. This completes the proof of the theorem. □

As a consequence we get the upper semi-continuity of separated parts of the spectrum, and the continuity of every finite system of eigenvalues. Recall that a spectral set is a subset of the spectrum which is open and closed in the spectrum. To every spectral set we can consider the corresponding spectral projection (see [31, Section III.6.4]).

Corollary 4.2. *Suppose that $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ in $\mathscr{L}(L_2(\mathbb{C}^N))$ for some $\lambda \in \mathbb{C}$, that $\Sigma \subset \sigma(-A_\Omega) \subset \mathbb{C}$ is a compact spectral set, and that Γ is a rectifiable closed simple curve enclosing Σ, separating it from the rest of the spectrum. Then, for n sufficiently large $\sigma(-A_{\Omega_n})$ is separated by Γ into a compact spectral set Σ_n and the rest of the spectrum. Denote by P and P_n the corresponding spectral projections. Then the dimension of the images of P and P_n are the same, and P_n converges to P in norm.*

Proof. The assertions follow from [31, Theorem IV.3.16] and Proposition A.1 in Appendix A. □

Remark 4.3. As a consequence (see [31, Section IV.3.5]) we get the continuity of every finite system of eigenvalues (counting multiplicity) and of the corresponding spectral projection if we have uniform convergence. In particular, we get the continuity of an isolated eigenvalue of algebraic multiplicity one and its eigenfunction when normalised suitably.

We next give necessary and sufficient conditions for uniform convergence in the special case $(\lambda + A_\Omega)^{-1}$ is compact as an operator on $L_2(\Omega)$. Note that this is equivalent for $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ to be compact. By Rellich's Theorem we have always compactness if Ω is bounded. Conditions for compactness to occur for unbounded domains are discussed in [1, Chapter 6] or [23, Section VIII.3]. Recall that the spectral bound of $-\Delta$ on the open set $U \subset \mathbb{R}^N$ with Dirichlet boundary conditions is given by

$$\lambda_1(U) = \inf_{\substack{u \in C_c^\infty(U) \\ u \neq 0}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} = \inf_{\substack{u \in H_0^1(U) \\ u \neq 0}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}. \tag{4.1}$$

For consistency we set $\lambda(\emptyset) := \infty$. Assuming that the limit problem (1.2) has compact resolvent on $L_2(\Omega)$ we have the following characterisation of uniform convergence. Note that the implication (5) $\Rightarrow$ (1) is proved in [11] for $\mathscr{A} = -\Delta$ using Γ-convergence.

Theorem 4.4. *Suppose that $\Omega, \Omega_n \subset \mathbb{R}^N$ are open sets with $\Omega_n \to \Omega$ and that $\lambda \in \varrho(-A_{\Omega_n}) \cap \varrho(-A_\Omega)$ for all $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) *$\mathscr{R}(\lambda)$ is compact and $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ in $\mathscr{L}(L_2(\mathbb{R}^N))$;*
(2) *$\mathscr{R}_n(\lambda) f_n \to \mathscr{R}(\lambda) f$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$.*
(3) *$\mathscr{R}_n(\lambda) f_n \to 0$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup 0$ weakly in $L_2(\mathbb{R}^N)$.*

If Ω is bounded then the above is equivalent to the following:

(4) *Eq. (3.2) holds and $\lambda_1(\Omega_n \cap \complement \bar{\Omega}) \to \infty$ as $n \to \infty$.*

Let λ_0 be given by (2.5). If $\lambda \geqslant \lambda_0$ and $\sup_{n \in \mathbb{N}} |\Omega_n| < \infty$ then also the following is equivalent to the above:

(5) *$\mathscr{R}_n(\lambda) 1 \to \mathscr{R}(\lambda) 1$ in $L_2(\mathbb{R}^N)$.*

Proof. First note that (1) and (2) imply (3.2). We show that (3) also implies (3.2). Assume to the contrary that (3.2) does not hold. Then for every $k \in \mathbb{N}$ there exists $f_k \in L_2(\mathbb{R}^N)$ and $n_k \in \mathbb{N}$ such that $\|f_k\|_2 = 1$ and $\|\mathscr{R}_{n_k}(\lambda) f_k\|_2 \geqslant k$. Setting $g_k := f_k / k$ we have $g_k \to 0$ in $L_2(\mathbb{R}^N)$ but $\|\mathscr{R}_{n_k}(\lambda)g_k\|_2 \geqslant 1 \nrightarrow 0$ as $k \to \infty$, contradicting (3). Hence (3.2) must be true in all cases (1)–(3). By Theorem 3.3 we have $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ strongly in $\mathscr{L}(L_2(\mathbb{R}^N))$. Now the equivalence of (1)–(3) immediately follows from Proposition B.1 in Appendix B. To show that (2) $\Rightarrow$ (4) we prove the contrapositive. Hence assume that either (3.2) does not hold or $\lambda_1(\Omega_n \cap \complement\bar{\Omega}) \nrightarrow \infty$. If (3.2) does not hold then by the uniform boundedness principle (2) cannot be true. If $\lambda_1(\Omega_n \cap \complement\bar{\Omega}) \nrightarrow \infty$ then there exist $c>0$ and $u_{n_k} \in C_c^\infty(\Omega_n)$ such that $\|u_{n_k}\|_2 = 1$ and $\|\nabla u_{n_k}\|_2 \leqslant c$. In particular, $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. By Lemma 2.1 the sequence $f_{n_k} := (\lambda + A)u_{n_k}$ is bounded in $H^{-1}(\mathbb{R}^N)$, and $u_{n_k} = \mathscr{R}_n(\lambda)f_n$. As bounded sequences in a Hilbert space are weakly sequentially compact we can, after possibly passing to another subsequence, assume that $f_{n_k} \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. Therefore, by Theorem 3.3, $u_{n_k} \rightharpoonup u := \mathscr{R}(\lambda)f$ weakly in $L_2(\mathbb{R}^N)$. As $\operatorname{supp} u_n \subset \complement\bar{\Omega}$ for all $n\in\mathbb{N}$ we have $u = 0$. Because $\|u_n\|_2 = 1$ for all $n\in\mathbb{N}$ it is not possible that $u_{n_k} \to 0$ in $L_2(\mathbb{R}^N)$, showing that (2) does not hold. Hence (2) $\Rightarrow$ (4). Note that we did not use that Ω is bounded here. Assuming that Ω is bounded we now show that (4) $\Rightarrow$ (3). Suppose that $f_n \rightharpoonup 0$ weakly in $H^{-1}(\mathbb{R}^N)$ and set $u_n := \mathscr{R}_n(\lambda)f_n$. From Theorem 3.3 we know that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Hence by Rellich's Theorem $u_n \to 0$ in $L_2(B)$ for every open bounded set B containing $\bar{\Omega}$. To show that $u_n \to 0$ in $L_2(\mathbb{R}^N \backslash B)$ we choose a smooth function $\psi \in C^\infty(\mathbb{R}^N)$ with $0 \leqslant \psi \leqslant 1$, $\psi = 0$ on $\bar{\Omega}$, and $\psi = 1$ on a neighbourhood of $\complement B$. Then $\psi u_n \in H_0^1(\mathbb{R}^N \backslash \bar{\Omega})$ and by (4.1)

$$\begin{aligned}\|u_n\|_{2,\complement B}^2 &\leqslant \|\psi u_n\|_2^2 \leqslant \lambda_1(\Omega_n \cap \complement\bar{\Omega})^{-1}\|\nabla(\psi u_n)\|_2^2 \\ &\leqslant \lambda_1(\Omega_n \cap \complement\bar{\Omega})^{-1}(\|\psi\|_\infty^2 + \|\nabla\psi\|_\infty^2)\|u_n\|_{H^1}^2\end{aligned}$$

for all $n\in\mathbb{N}$. As $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ it follows from (4) that $u_n \to 0$ in $L_2(\mathbb{R}^N \backslash B)$, proving (3). Finally, assume that $|\Omega_n|$ is uniformly bounded. To prove that (1) $\Rightarrow$ (5) note that

$$\begin{aligned}\|\mathscr{R}_n(\lambda)1 - \mathscr{R}(\lambda)1\|_2 &\leqslant \|\mathscr{R}_n(\lambda)1 - \mathscr{R}(\lambda)1\|_{2,\Omega} + \|\mathscr{R}_n(\lambda)1 - \mathscr{R}(\lambda)1\|_{2,\Omega_n} \\ &\leqslant \|\mathscr{R}_n(\lambda) - \mathscr{R}(\lambda)\|_{\mathscr{L}(L_2(\mathbb{R}^N))}(|\Omega|^{1/2} + |\Omega_n|^{1/2})\end{aligned}$$

showing (5). It remains to show that (5) $\Rightarrow$ (1). As $\Omega_n \to \Omega$, Theorem 3.3 implies that $\mathscr{R}_n^\sharp(\lambda) \to \mathscr{R}^\sharp(\lambda)$ strongly in $\mathscr{L}(L_2(\mathbb{R}^N))$, where $\mathscr{R}^\sharp(\lambda)$ is given by (2.14). Hence, by Theorem 3.3, $v_n := \mathscr{R}^\sharp(\lambda)f_n \rightharpoonup 0$ weakly in $L_2(\mathbb{R}^N)$ if $f_n \rightharpoonup 0$ weakly in $L_2(\mathbb{R}^N)$. Now by (2.15) and our assumption

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} v_n\, dx = \lim_{n\to\infty} \langle \mathscr{R}_n(\lambda)1, f_n \rangle = \langle \mathscr{R}(\lambda)1, 0 \rangle = 0.$$

By splitting f_n into positive and negative parts we can assume without loss of generality that f_n is non-negative. As $\lambda \geqslant \lambda_0$ it follows from Lemma 2.2 that v_n is non-negative, so $v_n \to 0$ in $L_1(\mathbb{R}^N)$. From (2.7) we know that v_n is bounded in $H^1(\mathbb{R}^N)$, and so by the Sobolev inequality bounded in $L_q(\mathbb{R}^N)$ for some $q>2$. Thus $v_n \to 0$ in $L_2(\mathbb{R}^N)$ by an interpolation inequality (see [26, inequality (7.9)]). Hence (3) holds for the formally adjoint problem, and thus $\mathscr{R}_n^\sharp(\lambda) \to \mathscr{R}^\sharp(\lambda)$ in $\mathscr{L}(L_2(\mathbb{R}^N))$. By (2.15) and the fact that an operator and its dual have the same norm (1) follows. $\square$

Remark 4.5. In the above theorem we only assume that the limit problem has compact resolvent. The problems on Ω_n do not need to have compact resolvent, and hence the family of resolvents is not necessarily collectively compact in the sense of [3]. As an example consider the sequence of sun-like domains in Example 8.4.

Remark 4.6. The above proof shows that uniform convergence always implies that $\lambda_1(\Omega \cap \complement \bar{\Omega}) \to \infty$, no matter what the limit domain Ω is. It would be interesting to know whether $\Omega_n \to \Omega$ and (4) imply uniform convergence for arbitrary limit domains Ω.

Corollary 4.7. *Suppose that Ω_n, Ω are contained in a fixed bounded set and that $\Omega_n \to \Omega$. If $\lambda \in \varrho(-A_\Omega)$ then $\lambda \in \varrho(-A_{\Omega_n})$ for n large enough, and $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ in $\mathscr{L}(L_2(\mathbb{R}^N))$.*

Proof. Fix a bounded set $B \subset \mathbb{R}^N$ such that $\Omega_n \subset B$ for all $n \in \mathbb{N}$. First assume that $\lambda \geqslant \lambda_0$ and $f_n \to f$ weakly in $L_2(B)$. Then by Theorem 3.3 $\mathscr{R}_n(\lambda) f_n \to \mathscr{R}(\lambda) f$ weakly in $H_0^1(B) \subset H^1(\mathbb{R}^N)$. By Rellich's Theorem convergence is strong in $L_2(B)$, so by the above theorem $\mathscr{R}_n(\lambda)$ converges uniformly. For general $\lambda \in \varrho(-A_\Omega)$ the assertions of the corollary then follow from Theorem 4.1. $\square$

Remark 4.8. Note that all results in [18] concerning parabolic problems remain true if we assume that $\Omega_n \to \Omega$, Ω bounded and (4) of Theorem 4.4 holds. We only need to modify the proof of [18, Theorem 3.1] in quite an obvious way. Also the results in [16] remain true whenever the resolvents converge uniformly.

5. Convergence in higher norms

When looking at non-linear problems on varying domains such as in [16] it is important to be able to get a good perturbation theory in L_p-spaces for $p>2$. The reason is that, in general, a non-linearity does not map L_2 into L_2 without severe restrictions on its growth. We want to show here how to get convergence in L_p for $p>2$. Suppose that Ω is an arbitrary open set, and let $A := A_\Omega$ the operator defined by (2.1). Moreover, let A_2 denote the part of A in L_2 given by $D(A_2) := \{u \in H_0^1(\Omega) : Au \in L_2(\Omega)\}$ and $A_2 u := Au$ for $u \in D(A_2)$. Then it is well known that $-A_2$ is the generator of a strongly continuous analytic semigroup on $L_2(\Omega)$ (see [22, Proposition XVII.6/3]). It is also well known that $T_2(t) := e^{-tA_2}$ has an integral kernel satisfying pointwise Gaussian estimates (see [5] or [20]) and thus interpolates to L_p for all $p \in (1, \infty)$. Denote by $-A_p$ its infinitesimal generator. We then look at solutions to the abstract equation

$$(A_p + \lambda)u = f \tag{5.1}$$

with $f \in L_p(\Omega)$. We call such a solution a generalised solution of (1.2) in $L_p(\Omega)$. The first difficulty is whether the spectrum of $-A_p$ is independent of $p \in (1, \infty)$. It indeed follows from the above and [33, Theorem 1.1] that $\sigma(A_p) = \sigma(A_2) = \sigma(A)$ for all $p \in (1, \infty)$. Let us note that the results in the present section can be obtained in a much easier way if we do not allow unbounded domains!

Proposition 5.1. *Problem* (5.1) *is solvable with bounded resolvent operator if and only if the same is true for* $(A_2 + \lambda)u = f$. *Moreover, for all* $\lambda \in \varrho(-A)$ *we have* $(\lambda + A_2)^{-1}|_{L_2 \cap L_p} = (\lambda + A_p)^{-1}|_{L_2 \cap L_p}$ *for all* $p \in (1, \infty)$.

To prove a convergence result we will need a priori estimates independent of the choice of Ω. If we set

$$m(p) := \begin{cases} Np(N - 2p)^{-1} & \text{if } p \in (1, N/2), \\ \infty & \text{if } p > N/2, \end{cases}$$

then the following estimates hold.

Proposition 5.2. *Suppose that* $\lambda \in \varrho(-A_2)$ *and that* $p \in (1, \infty)$. *Then* $(\lambda + A_p)^{-1} L_p \subset L_{m(p)} \cap L_p$. *Moreover, there exist constants* $C > 0$ *and* $\omega \in \mathbb{R}$ *only depending on* N, p *the ellipticity constant and the* L_∞*-norm of the coefficients of* $\mathscr{A}$ *such that*

$$\max\{\|(\lambda + A_p)^{-1}\|_{\mathscr{L}(L_p)}, \|(\lambda + A_p)^{-1}\|_{\mathscr{L}(L_p, L_{m(p)})}\} \leqslant C \tag{5.2}$$

whenever $\lambda > \omega$.

Proof. From Proposition 5.1 it follows that $\lambda \in \varrho(-A_p)$ if and only if $\lambda \in \varrho(-A_2)$. Then the first assertion follows from [21, Theorem 4.5]. To prove (5.2) we need to use that $-A_p$ generates a semigroup on L_p. It follows from [20, Theorem 6.1 and Corollary 7.2] that there exist constants $C_1 > 0$ and $\omega_1 \in \mathbb{R}$ depending only on the quantities listed in the proposition such that $\|e^{-tA_\Omega}\|_{\mathscr{L}(L_p)} \leqslant C_1 e^{\omega_1 t}$ for all $p \in [1, \infty]$. As $-A_\Omega$ generates a strongly continuous semigroup we have $(\lambda + A_p)^{-1} = \int_0^\infty e^{-tA_p} e^{-\lambda t}\, dt$ for $\lambda > \omega_1$ (see [47, Section IX.4]). It therefore follows that $\|(\lambda + A_p)^{-1}\|_{\mathscr{L}(L_p)} \leqslant C_1(\lambda - \omega_1)^{-1}$. By interpolation the first of (5.2) follows if we set $\omega := \omega_1 + 1$ and $C := C_1/\omega$. By a density argument the second inequality in (5.2) now follows from [21, Theorem 4.5], if we choose C appropriately. $\square$

From the above it is clear that we may consider $\mathscr{R}_n(\lambda), \mathscr{R}(\lambda)$ as operators in $\mathscr{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ whenever $p \in (1, \infty)$, and $q \in [p, m(p))$. The following is our main theorem.

Theorem 5.3. *Suppose that $\Omega_n \to \Omega$, and that ω is as in Proposition 5.2. Moreover suppose that $\lambda \geqslant \omega$, and that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$ for some $p \in (1, \infty)$, $p \neq N/2$. Then $\mathscr{R}_n(\lambda) f_n \rightharpoonup \mathscr{R}(\lambda) f$ weakly in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$. If convergence of f_n is strong in $L_p(\mathbb{R}^N)$ then $\mathscr{R}_n(\lambda) f_n$ converges strongly in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$.*

Proof. We first suppose that $f \in C_c^\infty(\mathbb{R}^N)$. By Theorem 3.3 we have $\mathscr{R}_n(\lambda) f \to \mathscr{R}(\lambda) f$ in $L_2(\mathbb{R}^N)$ as $f \in L_p(\mathbb{R}^N) \cap L_2(\mathbb{R}^N)$. Moreover, by Proposition 5.2 the sequence $(\mathscr{R}_n(\lambda) f)_{n \in \mathbb{N}}$ is bounded in $L_\infty(\mathbb{R}^N) \cap L_s(\mathbb{R}^N)$ for all $s > 1$. If $p \in (2, \infty)$ then by a well-known interpolation inequality

$$\|\mathscr{R}_n(\lambda) f - \mathscr{R}(\lambda) f\|_p \leqslant \|\mathscr{R}_n(\lambda) f - \mathscr{R}(\lambda) f\|_2^\theta \|\mathscr{R}_n(\lambda) f - \mathscr{R}(\lambda) f\|_\infty^{1-\theta}$$

for some $\theta \in (0, 1)$ (see [26, inequality (7.9)]). As one factor is bounded and the other converges to zero $\mathscr{R}_n(\lambda) f_n \to \mathscr{R}(\lambda) f$ in $\mathscr{L}(L_p(\mathbb{R}^N))$. If $p \in (1, 2)$ we use a similar argument, replacing the L_∞-bound by the L_s-bound with $1 < s < p$. We next assume that $f_n \to f$ in $L_p(\mathbb{R}^N)$ is arbitrary. Fix $\varepsilon > 0$, and choose $g \in C_c^\infty(\mathbb{R}^N)$ such that $\|f_n - g\|_p \leqslant \varepsilon$ for large $n \in \mathbb{N}$. This is possible as $C_c^\infty(\mathbb{R}^N)$ is dense in $L_p(\mathbb{R}^N)$ if $p \in (1, \infty)$. Taking into account Proposition 5.2

$$\begin{aligned}
&\|\mathscr{R}_n(\lambda) f_n - \mathscr{R}(\lambda) f\|_p \\
&\quad \leqslant \|\mathscr{R}_n(\lambda)(f_n - g)\|_p + \|\mathscr{R}_n(\lambda) g - \mathscr{R}(\lambda) g\|_p + \|\mathscr{R}(\lambda)(g - f)\|_p \\
&\quad \leqslant 4C\|f_n - g\|_p + \|\mathscr{R}_n(\lambda) g - \mathscr{R}(\lambda) g\|_p \leqslant 4C\varepsilon + \|\mathscr{R}_n(\lambda) g - \mathscr{R}(\lambda) g\|_p
\end{aligned}$$

for all $n \in \mathbb{N}$. As $\mathscr{R}_n(\lambda) g \to \mathscr{R}(\lambda) g$ in $L_p(\mathbb{R}^N)$ and $\varepsilon > 0$ was arbitrary it follows that $\mathscr{R}_n(\lambda) f_n \to \mathscr{R}(\lambda) f$ in $L_p(\mathbb{R}^N)$. Using again interpolation and the uniform bound from Proposition 5.2, convergence takes place in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$. This proves the second assertion of the theorem. To prove the first we use duality. As the formally adjoint operator $\mathscr{A}^\sharp$ given by (2.12) has the same structure as $\mathscr{A}$ we can define $A_p^\sharp$ as before for $p \in (1, \infty)$. We know from (2.13) that $(A_2^\sharp)' = A_2$. It therefore follows that $(e^{-tA_2^\sharp})' = e^{-tA_2}$, and thus $(e^{-A_p^\sharp})' = e^{-tA_p}$, implying that $(A_p^\sharp)' = A_{p'}$ (see [36, Corollary 1.10.6]). Here p' is the dual exponent to p defined by $1/p + 1/p' = 1$. Also note that the constants C, ω in Proposition 5.2 are the same for $\mathscr{A}$ and $\mathscr{A}^\sharp$. Suppose now that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$, and that $g \in L_{q'}(\mathbb{R}^N)$ for some $q \in [p, m(p))$. Then $p' \in [q', m(q'))$, and by our previous result

$$\lim_{n \to \infty} \langle \mathscr{R}_n(\lambda) f_n, g \rangle = \lim_{n \to \infty} \langle f_n, \mathscr{R}_n^\sharp(\lambda) g \rangle = \langle f, \mathscr{R}^\sharp(\lambda) g \rangle = \langle \mathscr{R}(\lambda) f, g \rangle,$$

showing that $\mathscr{R}_n(\lambda)f_n \rightharpoonup \mathscr{R}(\lambda)f$ weakly in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$. This concludes the proof of the theorem. □

Let us finally consider the case of uniform convergence.

Theorem 5.4. *Suppose that* $\mathscr{R}_n(\lambda) \to \mathscr{R}(\lambda)$ *in* $\mathscr{L}(L_2(\mathbb{R}^N))$ *for some* $\lambda \in \mathbb{R}$. *Then convergence takes place in* $\mathscr{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ *for all* $p \in (1, \infty)$ *and* $q \in [p, m(p))$.

Proof. The assertion directly follows from Theorem 4.1, Propositions 5.1 and 5.2, and the Riesz–Thorin interpolation theorem (see [8]). □

As in Corollary 4.2 we get the upper semi-continuity of the L_p-spectrum with respect to the domains.

6. Necessary conditions for convergence

In this section we discuss some conditions which are necessary for convergence. One obvious necessary condition is that the support of the weak limit of every convergent subsequence of solutions of (1.1) be in $\bar{\Omega}$. We will characterise this by looking at the spectral bound of $-\Delta$ on bounded sets outside $\bar{\Omega}$. Recall that for an arbitrary nonempty open set $U \subset \mathbb{R}^N$ the spectral bound, $\lambda_1(U)$, of $-\Delta$ subject to Dirichlet boundary conditions is given by (4.1). We will write $S \subset\subset T$ if $\bar{S}$ is compact and contained in the interior of T.

Theorem 6.1. *Suppose that* $\Omega_n, \Omega \subset \mathbb{R}^N$ *are open sets. Then the following assertions are equivalent*:

(1) *The weak limit points of every sequence* $u_n \in H_0^1(\Omega_n)$, $n \in \mathbb{N}$, *in* $H^1(\mathbb{R}^N)$ *have support in* $\bar{\Omega}$.

(2) *For all open sets* $B \subset\subset \mathbb{R}^N \setminus \bar{\Omega}$ (*Note B is bounded as* $\bar{B}$ *is compact*)

$$\lim_{n\to\infty} \lambda_1(\Omega_n \cap B) = \infty. \tag{6.1}$$

(3) *There exists an open covering* $\mathcal{O}$ *of* $\mathbb{R}^N \setminus \bar{\Omega}$ *such that* (6.1) *holds for all* $B \in \mathcal{O}$.

If (1.3) *is satisfied then* (6.1) *holds for all open bounded sets* $B \subset \mathbb{R}^N \setminus \bar{\Omega}$.

Proof. Suppose that (1) holds, and that $B \subset\subset \mathbb{R}^N \setminus \bar{\Omega}$ is open and set $\lambda_n := \lambda_1(\Omega_n \cap B)$. Then, by (4.1) for every $n \in \mathbb{N}$ there exists $v_n \in C_c^\infty(\Omega_n \cap B)$ such that

$$(\lambda_n + 1)\|v_n\|_2^2 \geqslant \|\nabla v_n\|_2^2 = 1. \tag{6.2}$$

As $B \subset\subset \mathbb{R}^N \backslash \bar{\Omega}$, in particular B is bounded. Hence by the Sobolev inequality v_n is bounded in $H^1(\mathbb{R}^N)$ and therefore has a weak limit point $v \in H_0^1(\Omega_n \cap B)$. Suppose that v is such a weak limit point, and that $v_{n_k} \rightharpoonup v$ weakly as $k \to \infty$. By assumption $\operatorname{supp}(v) \subset \bar{B} \subset\subset \mathbb{R}^N \backslash \bar{\Omega}$, so by (1) it follows that $v = 0$. As B is bounded Rellich's Theorem shows that $v_{n_k} \to 0$ in $L_2(\mathbb{R}^N)$. Hence, (6.2) can only be true if $\lambda_{n_k} - 1 \to \infty$, implying that $\lambda_{n_k} \to \infty$ as $k \to \infty$. The above arguments apply to every weak limit point, so (1) implies (2). If (1.3) is satisfied then every limit point is in $v \in H_0^1(\Omega) \cap H_0^1(B)$ even if we only assume that $\bar{\Omega} \cap B = 0$. Hence $v = 0$, and the above argument again shows that $\lambda_{n_k} \to \infty$. This proves the last statement of the theorem. Choosing $\mathcal{O}$ to be the class of all open sets $B \subset\subset \mathbb{R}^N \backslash \bar{\Omega}$ assertion (3) immediately follows from (2). We now prove that (3) implies (1). Suppose that $u_n \in H_0^1(\Omega_n)$, and that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Further, suppose that $\mathcal{O}$ is an open covering of $\mathbb{R}^N \backslash \bar{\Omega}$ and fix $B \subset \mathcal{O}$. If $\varphi \in C_c^\infty(B)$ then $\varphi u_n \in H_0^1(\Omega_n \cap \complement\bar{\Omega} \cap B)$. Multiplication with φ is a bounded linear map on $H^1(\mathbb{R}^N)$, and thus it is weakly continuous, so $\varphi u_{n_k} \rightharpoonup \varphi u$ weakly in $H^1(\mathbb{R}^N)$. As $u_n \varphi$ has support in a fixed bounded set for all $n \in \mathbb{N}$ by Rellich's theorem $\varphi u_{n_k} \to \varphi u$ strongly in $L_2(\mathbb{R}^N)$ as $k \to \infty$. Moreover, by (6.1) we have

$$\|\varphi u\|_2^2 = \lim_{k \to \infty} \|\varphi u_{n_k}\|_2^2 \leqslant \lim_{k \to \infty} \lambda_1(\Omega_n \cap B)^{-1} \|\nabla \varphi u_{n_k}\|_2^2 = 0.$$

Hence $\varphi u = 0$ almost everywhere for all $\varphi \in C_c^\infty(B)$, so $u = 0$ almost everywhere in B. As $\mathcal{O}$ is a covering of $\mathbb{R}^N \backslash \bar{\Omega}$ it follows that $\operatorname{supp} u \subset \bar{\Omega}$, proving (1). $\square$

Remark 6.2. In general, it is not true that $\lambda_1(\Omega_n \cap \complement\bar{\Omega}) \to \infty$ (see Example 8.1 below). We showed in Theorem 3.3 that $\lambda_1(\Omega_n \cap \complement\bar{\Omega}) \to \infty$ implies uniform convergence if Ω is bounded. We also pointed out in the proof of Theorem 3.3 there that $\lambda_1(\Omega_n \cap \complement\bar{\Omega}) \to \infty$ always if convergence is uniform.

So far we discussed necessary conditions on Ω_n outside $\bar{\Omega}$. Next we want to derive a necessary condition on the part of Ω_n inside Ω. Recall that the capacity (or more precisely $(1,2)$-capacity) of a set $E \subset \mathbb{R}^N$ is given by

$$\operatorname{cap}(E) := \inf\{\|u\|_{H^1}^2 \colon u \in H_0^1(\mathbb{R}^N) \text{ and } u \geqslant 1 \text{ in a neighbourhood of } E\}$$

(see [29, Section 2.35]). Next, we characterise (1.4) in terms of capacity. A variant appears in [38, Proposition 4.1]. A proof is given in [27, p. 75] or [44, p. 24], but for completeness we include one in our framework.

Proposition 6.3. *Suppose that $\Omega, \Omega_n \subset \mathbb{R}^N$ are open sets. Then condition* (1.4) *holds if and only if for every compact set $K \subset \Omega$*

$$\lim_{n \to \infty} \operatorname{cap}(K \cap \complement\Omega_n) = 0. \tag{6.3}$$

Proof. To prove that the condition is necessary fix a compact set $K \subset \Omega$, and let $\psi \in C_c^\infty(\Omega)$ be such that $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on a neighbourhood of K. By (1.4) there exists a sequence $\tilde{\psi}_n \in H_0^1(\Omega_n)$ such that $\tilde{\psi}_n \to \psi$ in $H^1(\mathbb{R}^N)$. By cutting $\tilde{\psi}_n$ off with an appropriate cutoff function, we can assume that $\operatorname{supp} \tilde{\psi}_n \subset B$ for all $n \in \mathbb{N}$ for some open bounded set $B \supset K$. As $C_c^\infty(\Omega_n \cap B)$ is dense in $H_0^1(\Omega_n \cap B)$ there exists $\psi_n \in C_c^\infty(\Omega_n \cap B)$ such that $\|\psi_n - \tilde{\psi}_n\|_{H^1} \leqslant 1/n$. We now set $\varphi_n := \psi - \psi_n$. Then $\varphi_n = 1$ on a neighbourhood of $K \cap \complement\Omega_n$, and

$$\operatorname{cap}(K \cap \complement\Omega_n) \leqslant \|\varphi_n\|_{H^1}^2 \leqslant \|\psi - \psi_n\|_{H^1}^2 \leqslant (\|\psi - \tilde{\psi}_n\|_{H^1}^2 + 1/n)^2$$

for all $n \in \mathbb{N}$. By choice of $\tilde{\psi}_n$ the right-hand side of the above inequality converges to zero, whence (1.4) implies (6.3). To show that the condition is sufficient note first that by density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ it is sufficient to consider $\varphi \in C_c^\infty(\Omega)$. Hence let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. By definition of capacity there exist $\psi_n \in C_c^\infty(\Omega_n)$ such that $\psi_n = 1$ in a neighbourhood of $\operatorname{supp} \varphi \cap \complement\Omega_n$ and $\|\psi_n\|_{H^1}^2 \leqslant \operatorname{cap}(\operatorname{supp} \varphi \cap \complement\Omega_n) + 1/n$ for all $n \in \mathbb{N}$. By assumption $\psi_n \to 0$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. We then define $\varphi_n := (1 - \psi_n)\varphi$. By choice of ψ_n it follows that $\varphi_n \in C_c^\infty(\Omega_n)$. Moreover,

$$\|\varphi_n - \varphi\|_{H^1} \leqslant (\|\varphi\|_\infty^2 + \|\nabla\varphi\|_\infty^2)^{1/2} \|\psi_n\|_{H^1}$$

for all $n \in \mathbb{N}$. As $\psi_n \to 0$ in $H^1(\mathbb{R}^N)$ it follows that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$, completing the proof of the proposition. □

7. Sufficient conditions for convergence

Let us first discuss two very special cases, namely monotone approximations of an open set Ω by open sets from the inside, and from the outside. The easiest case is approximation from the inside.

Proposition 7.1. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$, and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Then $\Omega_n \to \Omega$.*

Proof. As $H_0^1(\Omega)$ is weakly closed and $\Omega_n \subset \Omega$ it is obvious that (1.3) holds. Suppose that $u \in H_0^1(\Omega)$. To prove (1.4) note that by definition of $H_0^1(\Omega)$ there exist $\varphi_k \in C_c^\infty(\Omega)$ with $\varphi_k \to u$ in $H_0^1(\Omega)$ as $k \to \infty$. We can assume that $\operatorname{supp} \varphi_1 \subset \Omega_1$. By assumption $(\Omega_n)_{n \in \mathbb{N}}$ is an open covering of the compact set $\operatorname{supp}(\varphi_k)$ for all $k \in \mathbb{N}$, so for every $k \in \mathbb{N}$ there is a finite sub-covering of $\operatorname{supp}(\varphi_k)$. As $\Omega_n \subset \Omega_{n+1}$ for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\Omega_n \supset \operatorname{supp}(\varphi_k)$ for all $n \geqslant n_k$ and $n_k \to \infty$. If we set $u_n := \varphi_k$ for $n \in [n_k, n_{k+1})$ then $\operatorname{supp} u_n \subset H_0^1(\Omega_n)$ and $u_n \to u$ as required in (1.4). Hence $\Omega_n \to \Omega$ as claimed. □

The above can be used to approximate problems on non-smooth domains by a sequence of problems on smooth domains. This is a useful tool to get results for non-smooth domains, using results on smooth domains. Such techniques were for instance central in [5,19,32] or [34]. For approximations from the outside we need a weak regularity condition on the boundary of Ω, whose formulation requires some properties of functions in $H^1(\mathbb{R}^N)$. As usual we call a function quasi-continuous if it is continuous off a set of capacity zero. It can be shown (see [29, Theorem 4.4]) that for every $u \in H^1(\mathbb{R}^N)$ there exists a quasi-continuous function $\tilde{u}$ such that $u = \tilde{u}$ almost everywhere. It turns out that two such quasi-continuous functions are equal except possibly on a set of capacity zero (see [29, Theorem 4.12]). Hence, one can define traces of functions in $H^1(\mathbb{R}^N)$ on sets of non-zero capacity. If u denotes a quasi-continuous function one can show (see [1, Theorem 9.1.3] or [29, Theorem 4.5]) that for every open set $\Omega \subset \mathbb{R}^N$

$$H_0^1(\Omega) = \{u \in H_0^1(\mathbb{R}^N)\colon u \text{ quasi-contiuous and } u|_{\complement\Omega} = 0\}. \tag{7.1}$$

One can also define $H_0^1(\Omega)$ by (7.1) for arbitrary, not necessarily closed sets $\Omega \subset \mathbb{R}^N$. We make the following definition ([2, Definition 11.2.2]).

Definition 7.2. We say the (arbitrary) set $S \subset \mathbb{R}^N$ is *stable* if $H_0^1(S) = H_0^1(S^\circ)$, where S° denotes the interior of S.

Note that by (7.1) every open set $\Omega \subset \mathbb{R}^N$ is stable. An excellent discussion of bounded stable sets is given in [28].

Proposition 7.3. *A set $S \subset \mathbb{R}^N$ with non-empty interior is stable if one of the following conditions is satisfied*:

1. *$\partial S \cap S$ has the segment property except possibly on a set of capacity zero*;
2. *all points in $\partial S \cap S$ except possibly a set of capacity zero are Wiener regular*;
3. *for all $x \in \partial S \cap S$ except possibly a set of capacity zero*

$$\liminf_{r \to 0} \frac{\operatorname{cap}(\complement(S) \cap B(x,r))}{\operatorname{cap}(\complement(S^\circ) \cap B(x,r))} > 0,$$

where $B(x,r)$ is the ball of radius r centred at x.

The last condition is in fact necessary and sufficient for the stability of S.

Note that, if $\partial S \cap S$ is Lipschitz (or even smoother), then $\partial S \cap S$ satisfies the segment condition and $\partial S \cap S$ is therefore stable.

Proof. For a proof of (1) see [27,44, p. 77/78; Section 3.2] or [46, Satz 4.8], for (2) we refer to [25, Theorem 2.5*], and for (3) to [2, Theorem 11.4.1]. □

Proposition 7.4. *Suppose that $\Omega_n \supset \Omega_{n+1} \supset \Omega$ for all $n \in \mathbb{N}$, and that* $\operatorname{int}(\bigcap_{n\in\mathbb{N}} \Omega_n) = \Omega$. *If $\bar{\Omega}$ is stable then $\Omega_n \to \Omega$.*

Proof. Clearly (1.4) holds. As $\operatorname{int}(\bigcap_{n\in\mathbb{N}} \Omega_n) = \Omega$ it follows that all weak limit points of $u_n \in H_0^1(\Omega)$ for $n \in \mathbb{N}$ have support in $\bar{\Omega}$. Hence by definition of stability all weak limit points are in $H_0^1(\Omega)$ as required in (1.3). Hence $\Omega_n \to \Omega$. □

In [32] it is shown that bounded stable sets are those for which approximation from the outside and from the inside yields the same limit problem. We next discuss non-monotone approximations of an open set.

Theorem 7.5. *Suppose that Ω_n, Ω are open (not necessarily bounded) sets in $\mathbb{R}^N$. If the following three conditions are satisfied then $\Omega_n \to \Omega$:*

(1) $\operatorname{cap}(K \cap \complement\Omega_n) \to 0$ *as $n \to \infty$ for all compact sets $K \subset \Omega$;*
(2) *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \bar{\Omega}$ such that $\lambda_1(U \cap \Omega_n) \to \infty$ as $n \to \infty$ for all $U \in \mathcal{O}$;*
(3) *We have $H_0^1(\Omega) = H_0^1(\Omega \cup \Gamma)$, where*

$$\Gamma := \bigcap_{n\in\mathbb{N}} \left(\overline{\bigcup_{k \geqslant n} (\Omega_k \cap \partial\Omega)} \right) \subset \partial\Omega. \tag{7.2}$$

Before we give a proof let us emphasise that, by the results in Section 6, the first two conditions are necessary for convergence, and thus cannot be weakened. Note however, that the last condition is not necessary in general (see Example 8.5 below). The set (7.2) is used in [44].

Proof. From the first assumption and Proposition 6.3 we see that (1.4) holds. Moreover, by Theorem 6.1 and the second assumption every weak limit point of $u_n \in H_0^1(\Omega_n)$, $n \in \mathbb{N}$, in $H^1(\mathbb{R}^N)$ has support in $\bar{\Omega}$. It remains to show that every such limit point is in $H_0^1(\Omega)$. We assume that u is quasi-continuous and show that $u = 0$ on $\partial\Omega \setminus \Gamma$ except possibly on a set of capacity zero. It is easily seen from (7.2) that for every $x \in \partial\Omega \setminus \Gamma$ there exists a neighbourhood U such that $U \cap \Omega_n \cap \partial\Omega = \emptyset$ for all n large enough. Suppose that U is such a neighbourhood of $x \in \partial\Omega \setminus \Gamma$, and that $\psi \in C_c^\infty(U)$ is a cutoff function with $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on a neighbourhood V of x. Then $\psi u_n|_\Omega \in H_0^1(\Omega)$ for n sufficiently large by (7.1). Hence for every weak limit point of u of u_n we have $\psi u \in H_0^1(\Omega)$. In particular, by (7.1) we have $u = 0$ on $V \cap \partial\Omega$. Repeating the same argument for every $x \in \partial\Omega \setminus \Gamma$ shows that $u = 0$ on $\partial\Omega \setminus \Gamma$. As we know already that $\operatorname{supp} u \subset \bar{\Omega}$ it follows that $u \in H_0^1(\Omega \cup \Gamma)$. By condition (3) we have $u \in H_0^1(\Omega)$, proving (1.3). Hence $\Omega_n \to \Omega$ as required. □

We next discuss some sufficient conditions for (2) to be satisfied. They are by no means the best, much more general situations can occur! However, they are easy to apply.

Proposition 7.6. *Let $U_n \subset \mathbb{R}^N$ be open. Then $\lambda_1(U_n) \to \infty$ if one of the following conditions are satisfied*:

1. $|U_n| \to 0$ *as* $n \to \infty$;
2. U_n *is the union of connected components* $U_{n,k}$, *and* $\inf_k \lambda_1(U_{n,k}) \to \infty$ *as* $n \to \infty$;
3. U *is a bounded set and* $U_n \subset U \setminus K_n$, *where* K_n *is the union of n closed balls with radius* r_n, *evenly spaced in U. Moreover* $n r_n^{N-2} \to \infty$ *if* $N \geqslant 3$ *and* $n/|\log r_n| \to \infty$ *if* $N = 2$ *as* $n \to \infty$.

Proof. To prove that (1) is sufficient note that by the Faber–Krahn inequality $\lambda_1(U_n) \geqslant \lambda_1(B_n)$, where B_n is a ball of the same volume as U_n. As $|U_n| \to 0$ the radius of B_n must converge to zero. It is well known that $\lambda_1(B_n) \to \infty$ as the radius converges to zero, hence also $\lambda_1(U_n) \to \infty$ (see [7, Theorem 3.4]). To prove that (2) is sufficient simply note that the spectrum of the Dirichlet problem on U_n is the union of the spectra on the components of U_n. For a proof that (3) is sufficient we note that the spectral bound is monotone decreasing if the domain is increasing. Then use the result from [38, p. 44/45]. □

Finally, we want to give a result which can be used in certain situations to verify the uniform resolvent bound (3.2).

Proposition 7.7. *Suppose that U_n, $n \in \mathbb{N}$, are open sets with* $\inf_{n \in \mathbb{N}} \lambda_1(U_n) > 0$, *and that Ω is a bounded open set. Then* $\inf_{n \in \mathbb{N}} \lambda_1(\Omega \cup U_n) > 0$.

Proof. For $n \in \mathbb{N}$ set $\Omega_n := \Omega \cup U_n$. Assume that there exist a subsequence Ω_{n_k} such that $\lambda_1(\Omega_{n_k}) \to 0$ as $k \to \infty$. By characterisation (4.1) of $\lambda_1(\Omega_n)$ there exist $u_{n_k} \in H_0^1(\Omega_n)$ with $\|u_{n_k}\|_2 = 1$ for all $k \in \mathbb{N}$, and $\|\nabla u_{n_k}\|_2 \to 0$ as $k \to \infty$. In particular, $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. After possibly selecting another subsequence and renumbering we can therefore assume that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. We know already that $\|\nabla u_n\|_2 \to 0$, so $\nabla u = 0$. Hence u is constant, and as $u \in H^1(\mathbb{R}^N)$ we must have $u = 0$. By Rellich's Theorem $u_n \to 0$ in $L_2(B)$ for every bounded set $B \subset \mathbb{R}^N$. Suppose now that $\psi \in C_c^\infty(\mathbb{R}^N)$, $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ in a neighbourhood of $\bar{\Omega}$. Denote by B an open ball such that $\operatorname{supp} \psi \subset B$. Clearly $\psi u_n \in H_0^1(B)$ and $(1-\psi)u_n \in H_0^1(U_n)$ for all $n \in \mathbb{N}$. By characterisation (4.1) of the spectral bound

$$\begin{aligned}
\|u_n\|_2 &\leqslant \|\psi u_n\|_2 + \|(1-\psi)u_n\|_2 \\
&\leqslant \lambda_1(B)^{-1/2}\|\nabla(\psi u_n)\|_2 + \lambda_1(U_n)^{-1/2}\|\nabla((1-\psi)u_n)\|_2 \\
&\leqslant \max\{\lambda_1(B)^{-1/2}, \lambda_1(U_n)^{-1/2}\}(\|\nabla u_n\|_2 + \|u_n\|_{2,B}),
\end{aligned}$$

where we used in the last step that $\operatorname{supp}(\nabla\psi) \subset B$. By assumption there exists a constant $c > 0$ such that $\max\{\lambda_1(B)^{-1/2}, \lambda_1(U_n)^{-1/2}\} \geqslant c$ for all $n \in \mathbb{N}$. As $\|\nabla u_n\|_2 +$

$\|u_n\|_{2,B} \to 0$ it follows that $u_n \to 0$ in $L_2(\mathbb{R}^N)$, contradicting the assumption that $\|u_n\|_2 = 1$ for all $n \in \mathbb{N}$. $\square$

8. Examples

In this section we provide some examples of converging domains. The main purpose is to illustrate by simple examples the various conditions discussed, and to show that they are optimal. To show that we do not gain anything by working with connected sets, all examples given involve connected sets Ω_n. We first show that convergence of resolvents does not need to be uniform.

Example 8.1. Consider a sequence of dumbbell-shaped domains as depicted in Fig. 1, where B_n is a ball of radius $r_n > 0$ and C_n a strip of length ℓ_n. We claim that $\Omega_n = B_0 \cup C_n \cup B_n \to \Omega := B_0$ if $\ell_n \to \infty$ and the width of C_n goes to zero. Clearly $|B \cap \Omega_n| \to 0$ for every bounded open set $B \subset \mathbb{R}^N \backslash \bar{\Omega}$, so by Proposition 7.6 we have $\lambda_1(\Omega_n \cap B) \to \infty$. As $\partial\Omega$ is smooth and $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$ it follows from Theorem 7.5 that $\Omega_n \to \Omega$. We now assume that $r_n = r$ is fixed. By (4.1) we have $\lambda_1(\Omega_n \backslash \bar{\Omega}) \leqslant \lambda_1(B_n)$ for all $n \in \mathbb{N}$. As B_n is a ball of fixed radius $\lambda_1(\Omega_n \backslash \bar{\Omega})$ is bounded. Hence by Theorem 4.4 we do not have uniform convergence of resolvents.

Next, we show that (3.2) is not automatically satisfied even if $\lambda \in \varrho(-A_{\Omega_n}) \cap \varrho(-A_\Omega)$ for all $n \in \mathbb{N}$.

Example 8.2. Consider a similar sequence of dumbbell-shaped domains as in Example 8.1, and assume that $\ell_n \to \infty$ and $r_n \to \infty$. Moreover let $\mathscr{A} := -\Delta$ and $\lambda = 0$. As $\lambda_1(\Omega_n), \lambda_1(\Omega) > 0$ for all $n \in \mathbb{N}$ we have $0 \in \varrho(-A_{\Omega_n}) \cap \varrho(A_\Omega)$ for all $n \in \mathbb{N}$. Assuming that $r_n \to \infty$ we have $\lambda_1(\Omega_n) \to 0$ as $n \to \infty$. On the other hand, we know that $\|\mathscr{R}_n(0)\| \geqslant \lambda_1(\Omega_n)^{-1} \to \infty$, so (3.2) cannot be true. By the uniform boundedness principle (see [47, Section II.1]), condition (3.2) is necessary for $\mathscr{R}_n(\lambda)f$ to converge. Hence we cannot do without (3.2) in Theorem 3.3.

As a variant of the above, we construct an example where Ω_n has uniformly bounded measure. To do so we first look at a dumbbell with two fixed balls B_1 and

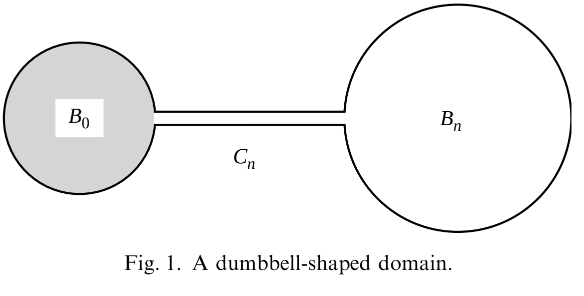

Fig. 1. A dumbbell-shaped domain.

B_2 and handle C_n. Letting the width of C_n go to zero the dumbbells converge to $B_1 \cup B_2$. The domains are contained in a fixed bounded set, so by Corollary 4.7 we have uniform convergence, and thus by Remark 4.3 the first eigenvalue converges to the first eigenvalue of the larger of the two balls. Now go back to the original situation. We let B_n be a ball of fixed radius larger than the radius of B_0 and set $\lambda := \lambda_1(B_n)$ (which is independent of n). Note that $\lambda < \lambda_1(B_0)$. By the above considerations we can choose C_n such that $|\lambda_1(\Omega_n) - \lambda| < 1/n$ and $\Omega_n \to B_0$. Hence, even if $\ell_n \to \infty$, we get that $\lambda_1(\Omega_n) \to \lambda < \lambda_1(B_0)$. If we choose $\mathscr{A} := -\Delta$ then $\|\mathscr{R}(\lambda)\| \geqslant (\lambda(\Omega_n) - \lambda)^{-1} \to \infty$ as $n \to \infty$. Hence, $\mathscr{R}_n(\lambda)f$ does not converge in general if (3.2) does not hold.

Next, we give examples showing that the spectrum does not in general depend continuously on the domain. The above example shows that the limit points of the spectrum do not need to be in the spectrum of the limit problem. Next, we show that a point in the resolvent of the limit problem can be in the spectrum of all perturbed problems, so the assertions of Theorem 4.1 are not true in general.

Example 8.3. Here we show that a point in the resolvent set of the limit problem does not need to be in the resolvent set of the corresponding problem on Ω_n even for large n. Consider Ω_n as depicted in Fig. 2 with the angle $\beta_n \to 0$ as $n \to \infty$.

If Ω is the ball then it is obvious that (1.4) holds. Moreover $|B \cap \Omega_n| \to 0$ as $n \to \infty$ for every bounded open set $B \subset \mathbb{R}^N \setminus \bar{\Omega}$. As Ω is smooth Proposition 7.6 and Theorem 7.5 imply that $\Omega_n \to \Omega$. Now let $\mathscr{A} := -\Delta$ and $\lambda := 0$. As Ω is a bounded domain clearly $0 \in \varrho(-A_\Omega)$, but $0 \in \sigma(-A_{\Omega_n})$ for all $n \in \mathbb{N}$ as Ω_n contains arbitrarily large balls. Hence $\mathscr{R}_n(0)$ does not exist for all $n \in \mathbb{N}$.

Next, we give some examples showing that the part of Ω_n outside the limit domain may have large, even increasing or infinite measure, and still $\Omega_n \to \Omega$ with resolvents converging uniformly.

Example 8.4. Let us discuss simple cases where $\Omega_n \to \Omega$ but the measure of $\Omega_n \cap \complement\bar{\Omega}$ does not converge to zero. In all the examples we make use of Friedrich's inequality (see [39, Theorem III.5.3]) which implies that $\lambda_1(S_n) \to \infty$ if S_n is an open set lying between two parallel hyper-planes whose distance approaches zero as $n \to \infty$.

In a first example we let Ω be an open cube and add "fingers" to one of the sides as shown in Fig. 3. If we increase the number of fingers such that the volume is

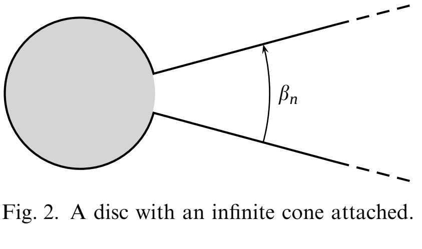

Fig. 2. A disc with an infinite cone attached.

Fig. 3. A cube with fingers attached.

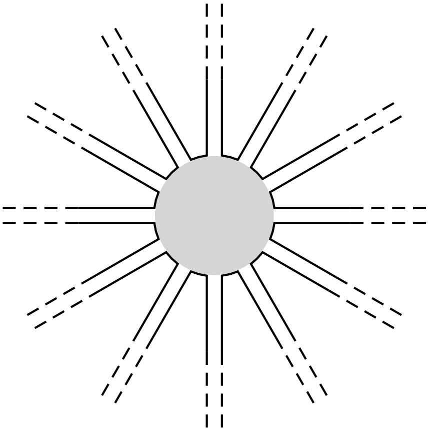

Fig. 4. A sun-like domain.

preserved, letting their width go to zero, then by the above remark and Proposition 7.6 we have $\lambda_1(\Omega_n \backslash \bar{\Omega}) \to \infty$. As $\Omega \subset \Omega_n$ and Ω is Lipschitz all conditions of Theorem 7.5 are satisfied, so $\Omega_n \to \Omega$. If we extend the fingers to infinity then still $\Omega_n \to \Omega$, but $|\Omega_n \backslash \Omega| = \infty$. Moreover, as $\lambda_1(\Omega_n \backslash \bar{\Omega}) \to \infty$ Theorem 4.4 applies so we have uniform convergence of resolvents.

Similar arguments apply to the sun-like domain shown in Fig. 4, where we increase the number of rays but make them thinner. We arrive at the same conclusion as above. Note that every open set in $\mathbb{R}^N$ intersects Ω_n if n is large enough.

Next, we show that (3) in Theorem 7.5 is not necessary for convergence of solutions.

Example 8.5. We use an example in [38, p. 46] to show that (3) in Theorem 7.5 is not necessary for convergence. We let $U \subset \mathbb{R}^3$ be an open bounded set, and $S \subset U$ a compact smooth surface. Let $K_n := \bigcup_{j=1}^n B_{n,j}$, where $B_{n,j}$ are n balls of radius r_n centred at the evenly spaced points $x_{n,j} \in S$. Moreover, assume that $nr_n \to \infty$, but $nr_n^2 \to 0$. Hence we can make sure that the balls do not intersect. Finally we let $\Omega_n := U \backslash K_n$ and $\Omega := U \backslash S$. It is then shown in [38, p. 46] that $\Omega_n \to \Omega$. On the other hand, if we take $x \in S \subset \partial\Omega$ and a neighbourhood V of x then clearly $\Omega_n \cap \partial\Omega \cap V \neq \emptyset$

for all $n\in\mathbb{N}$. Hence $S=\Gamma$ as defined in (7.2). Note however, that $H_0^1(U)\neq H_0^1(U\backslash\Gamma)=H_0^1(\Omega)$ but $\Omega_n\to\Omega$, showing that (3) in Theorem 7.5 is not necessary for convergence of solutions.

9. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3. We will use the assumptions and the framework introduced in Section 2. In particular, we will make extensive use of the formally adjoint problem and its properties. Moreover, λ_0 is always given by (2.5). We start by a lemma allowing us to prove strong convergence of solutions in $H^1(\mathbb{R}^N)$.

Lemma 9.1. *Suppose that $\lambda\geqslant\lambda_0$, that $f_n, f\in H^{-1}(\mathbb{R}^N)$ and that $\mathscr{R}_n(\lambda)f_n\rightharpoonup\mathscr{R}(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$. If $f_n\to f$ in $H^{-1}(\mathbb{R}^N)$ then $\mathscr{R}_n(\lambda)f_n\to\mathscr{R}(\lambda)f$ in $H^1(\mathbb{R}^N)$. Moreover, if there exists an open set $B\subset\mathbb{R}^N$ such that $f_n\to f$ in $H^{-1}(B)$ then $\mathscr{R}_n(\lambda)f_n\to\mathscr{R}(\lambda)f$ in $H^1_{\mathrm{loc}}(B)$.*

Proof. Suppose that $\psi\in C^\infty(\mathbb{R}^N)\cap L_\infty(\mathbb{R}^N)$ with $|\nabla\psi|\in L_\infty(\mathbb{R}^N)$. Then $\psi u\in H^1(\mathbb{R}^N)$ and by an elementary calculation

$$a(\psi u,\psi u)=a(u,\psi^2u)+\int_{\mathbb{R}^N}u^2\left(\sum_{i=1}^N\left(\sum_{j=1}^N a_{i,j}\partial_j\psi\right)+(b_i-a_i)\psi\right)\partial_i\psi\,dx$$
$$+\int_{\mathbb{R}^N}\psi u\sum_{i=1}^N\sum_{j=1}^N(a_{i,j}-a_{j,i})\partial_j\psi\partial_i u\,dx \tag{9.1}$$

for all $u\in H^1(\mathbb{R}^N)$. Suppose now that $f_n\rightharpoonup f$ in $H^{-1}(\mathbb{R}^N)$, and that $u_n:=\mathscr{R}_n(\lambda)f_n\rightharpoonup u:=\mathscr{R}(\lambda)f$. As $\lambda\geqslant\lambda_0$ we conclude from (2.6) that

$$\begin{aligned}\frac{\alpha}{2}\|\psi(u_n-u)\|^2_{H^1(\mathbb{R}^N)}&\leqslant a(\psi u_n-\psi u,\psi u_n-\psi u)+\lambda\|\psi(u_n-u)\|_2^2\\&=a(\psi u_n,\psi u_n)+\lambda\|\psi u_n\|_2^2+a(\psi u,\psi u)+\lambda\|\psi u\|_2^2\\&\quad-a(\psi u_n,\psi u)-a(\psi u,\psi u_n)-2\lambda\langle\psi u,\psi u_n\rangle\end{aligned} \tag{9.2}$$

for all $n\in\mathbb{N}$. As $u_n\rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ we have

$$\lim_{n\to\infty}(a(\psi u_n,\psi u)+a(\psi u,\psi u_n)+2\lambda\langle\psi u,\psi u_n\rangle)=2a(\psi u,\psi u)+2\lambda\|\psi u\|_2^2. \tag{9.3}$$

Since u_n is the unique weak solution of (1.1) we get from (9.1)

$$a(\psi u_n, \psi u_n) + \lambda ||\psi u_n||_2^2 = \langle f_n, \psi^2 u_n \rangle + \int_{\mathbb{R}^N} u_n^2 \left(\sum_{i=1}^N \left(\sum_{j=1}^N a_{i,j} \partial_j \psi \right) + (b_i - a_i)\psi \right) \partial_i \psi \, dx + \int_{\mathbb{R}^N} \psi u_n \sum_{i=1}^N \sum_{j=1}^N (a_{i,j} - a_{j,i}) \partial_j \psi \partial_i u_n \, dx. \tag{9.4}$$

Assume now that $f_n \to f$ strongly, and let $\psi \equiv 1$. As $u_n \rightharpoonup u$ weakly and $f_n \to f$ strongly we get from (9.4) that

$$\lim_{n\to\infty} (a(u_n, u_n) + \lambda ||u_n||_2^2) = \lim_{n\to\infty} \langle f, u_n \rangle = \langle f, u \rangle = a(u, u) + \lambda ||u||_2^2,$$

where we used that u is the weak solution of (1.2). Together with (9.2) and (9.3) it follows that $||u_n - u||_{H^1(\mathbb{R}^N)} \to 0$, that is, $u_n \to u$ in $H^1(\mathbb{R}^N)$. This proves the first assertion of the lemma. Now consider the case where $f_n \to f$ in $H^{-1}(B)$ for some open set $B \subset \mathbb{R}^N$. Let $U \subset\subset B$ be open and choose $\psi \in C_c^\infty(B)$ such that $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on U. As $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ it follows from Rellich's theorem that $u_n \to u$ in $L_2(\operatorname{supp} \psi)$. Hence in every term in (9.4) there is a weakly and a strongly converging sequence. Using (9.1) and that u is the weak solution of (1.2) we therefore get

$$\lim_{n\to\infty} (a(\psi u_n, \psi u_n) + \lambda ||\psi u_n||_2^2) = a(\psi u, \psi u) + \lambda ||\psi u||_2^2.$$

Together with (9.2) and (9.3) we see that $||\psi(u_n - u)||_{H^1(\mathbb{R}^N)} \to 0$ as $n \to \infty$. As $\psi = 1$ on U it follows that $u_n \to u$ in $H^1(U)$, showing that $u_n \to u$ in $H^1_{\text{loc}}(B)$. $\square$

We next prove a lemma about strong convergence without assuming that $\lambda \geqslant \lambda_0$.

Lemma 9.2. *Suppose that $\mathscr{R}_n(\lambda) f_n \rightharpoonup \mathscr{R}(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. If $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$ then $\mathscr{R}_n^\sharp(\lambda) f_n \to \mathscr{R}^\sharp(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L_2(\mathbb{R}^N)$.*

Proof. Suppose that $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$. As $\mathscr{R}_n(\lambda) g \rightharpoonup \mathscr{R}(\lambda) g$ weakly in $H^1(\mathbb{R}^N)$ by assumption and (2.15)

$$\lim_{n\to\infty} \langle g, \mathscr{R}_n^\sharp(\lambda) f_n \rangle = \lim_{n\to\infty} \langle f_n, \mathscr{R}_n(\lambda) g \rangle = \langle f, \mathscr{R}(\lambda) g \rangle = \langle g, \mathscr{R}^\sharp(\lambda) f \rangle \tag{9.5}$$

for all $g \in H^{-1}(\mathbb{R}^N)$, showing that $\mathscr{R}_n^\sharp(\lambda) f_n \rightharpoonup \mathscr{R}^\sharp(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$. As $f_n \to f$ strongly in $H^{-1}(\mathbb{R}^N)$ it follows from (2.15) that

$$\lim_{n\to\infty} ||\mathscr{R}_n(\lambda) f_n||_2^2 = \lim_{n\to\infty} \langle \mathscr{R}_n(\lambda) f_n, \mathscr{R}_n(\lambda) f_n \rangle = \lim_{n\to\infty} \langle f_n, \mathscr{R}_n^\sharp(\lambda) \mathscr{R}_n(\lambda) f_n \rangle = \langle f, \mathscr{R}^\sharp(\lambda) \mathscr{R}(\lambda) f \rangle = ||\mathscr{R}(\lambda) f||_2^2.$$

As we know already that $\mathscr{R}_n^\sharp(\lambda)f_n$ converges weakly it follows that it converges strongly in $L_2(\mathbb{R}^N)$, completing the proof of the lemma. □

Proposition 9.3. *Suppose that* $\lambda\in\mathbb{R}$ *is such that* $\mathscr{R}_n(\lambda)$ *and* $\mathscr{R}(\lambda)$ *exist for all* $n\in\mathbb{N}$. *Then the following assertions are equivalent*:

(1) $\mathscr{R}_n(\lambda)f_n\to\mathscr{R}(\lambda)f$ *in* $H^1(\mathbb{R}^N)$ *whenever* $f_n\to f$ *in* $H^1(\mathbb{R}^N)$.
(2) $\mathscr{R}_n^\sharp(\lambda)f_n\rightharpoonup\mathscr{R}^\sharp(\lambda)f$ *weakly in* $H^1(\mathbb{R}^N)$ *whenever* $f_n\rightharpoonup f$ *weakly in* $H^1(\mathbb{R}^N)$.

Proof. We first prove that (1) ⇒ (2). Suppose that $f_n\rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$, and that $g\in H^{-1}(\mathbb{R}^N)$. Then by our assumptions we know that $\mathscr{R}_n(\lambda)g\to\mathscr{R}(\lambda)g$ in $H^1(\mathbb{R}^N)$. Hence (9.5) applies for all $g\in H^{-1}(\mathbb{R}^N)$, showing that $\mathscr{R}_n^\sharp(\lambda)f_n\rightharpoonup\mathscr{R}^\sharp(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$. To prove that (2) ⇒ (1) we suppose that $f_n\to f$ in $H^{-1}(\mathbb{R}^N)$. From Lemma 9.2 we know that $u_n:=\mathscr{R}_n^\sharp(\lambda)f_n\to\mathscr{R}^\sharp(\lambda)f=:u$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L_2(\mathbb{R}^N)$. In particular $f_n+(\lambda_0-\lambda)u_n\to f+(\lambda_0-\lambda)u$ in $H^{-1}(\mathbb{R}^N)$. Hence by Lemma 9.1 and the resolvent equation

$$\mathscr{R}_n^\sharp(\lambda)f_n=\mathscr{R}_n^\sharp(\lambda_0)(f_n+(\lambda_0-\lambda)u_n)\to\mathscr{R}^\sharp(\lambda_0)(f+(\lambda_0-\lambda)u)=\mathscr{R}^\sharp(\lambda)f$$

in $H^1(\mathbb{R}^N)$ as $n\to\infty$, completing the proof of the proposition. □

Proposition 9.4. *If one of the equivalent statements in Proposition* 9.3 *are true for some operator* $\mathscr{A}$, *then* $\Omega_n\to\Omega$, *that is*, (1.3) *and* (1.4) *hold.*

Proof. We prove that the first statement in Proposition 9.3 implies (1.4), and that the second implies (1.3). As both are equivalent the assertion of the proposition follows. Suppose now that $\mathscr{R}_n(\lambda)f_n\to\mathscr{R}(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n\to f$ in $H^1(\mathbb{R}^N)$. Fix $\varphi\in H_0^1(\Omega)$ and set $f:=(\lambda+A_{\mathbb{R}^N})\varphi$. Then by Lemma 2.1 we know that $\varphi=\mathscr{R}(\lambda)$ is the unique weak solution of (1.2). Set $\varphi_n:=\mathscr{R}_n(\lambda)f$. Then by assumption $\varphi_n\to\varphi$ in $H^1(\mathbb{R}^N)$. As $\varphi_n\in H_0^1(\Omega_n)$ for all $n\in\mathbb{N}$, this proves (1.4). Suppose now that $\mathscr{R}_n^\sharp(\lambda)f_n\rightharpoonup\mathscr{R}^\sharp(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n\rightharpoonup f$ weakly in $H^1(\mathbb{R}^N)$. Let $u_n\in H_0^1(\Omega_n)$ be such that $u_{n_k}\rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ for some subsequence $(u_{n_k})_{k\in\mathbb{N}}$. To show that $u\in H_0^1(\mathbb{R}^N)$ we set $f_n:=(\lambda+A_{\mathbb{R}^N}^\sharp)u_n$. By Lemma 2.1 we know that u_n is the unique weak solution of (1.1) with $\mathscr{A}$ replaced by $\mathscr{A}^\sharp$. As $A_{\mathbb{R}^N}^\sharp\in\mathscr{L}(H^1(\mathbb{R}^N),H^{-1}(\mathbb{R}^N))$, and every bounded operator is weakly continuous, it follows that f_{n_k} converges to $f:=(A_{\mathbb{R}^N}^\sharp+\lambda)u$ weakly in $H^1(\mathbb{R}^N)$. By our assumptions $u_n=\mathscr{R}_{n_k}^\sharp(\lambda)f_{n_k}\rightharpoonup\mathscr{R}^\sharp(\lambda)f=u\in H_0^1(\Omega)$, proving (1.3). Hence $\Omega_n\to\Omega$ as claimed. □

Corollary 9.5. *Assertions* (1)–(3) *of Theorem* 3.3 *are equivalent.*

Proof. Suppose that (1) of Theorem 3.3 holds and that $f_n \rightharpoonup f$ weakly in $H^{-1}(\Omega)$. By (3.2) the sequence $u_n := \mathscr{R}_n(\lambda)f_n$ is bounded in $H^1(\mathbb{R}^N)$, and therefore has a weak limit point $u \in H^1(\mathbb{R}^N)$. By (1.3) we have $u \in H_0^1(\Omega)$. As $\lambda \in \varrho(-A_\Omega)$ it follows that u is the unique solution of (1.2). Recall that (u_n) is bounded, so by Corollary 3.2 we have $u_n \rightharpoonup u = \mathscr{R}(\lambda)f$. As the spectra of the formal adjoint problems is the same as the one of the original problems the above procedure also works for the formally adjoint operator $\mathscr{A}^\sharp$. Combining this with Proposition 9.3 shows that (1) $\Rightarrow$ (3). Suppose that (2) or (3) of Theorem 3.3 hold. Then $\mathscr{R}_n(\mu)f$ exists and is a weakly convergent in $H^1(\mathbb{R}^N)$ for all $f \in H^{-1}(\mathbb{R}^N)$. In particular $\mathscr{R}_n(\mu)f$ is bounded for all $f \in H^{-1}(\mathbb{R}^N)$, and thus by the uniform boundedness principle (3.2) follows. Using Proposition 9.4 we conclude that (1) holds. Hence (1)–(3) in Theorem 3.3 are equivalent. $\square$

Next, we consider the case $\lambda \geqslant \lambda_0$ and prove the remaining part of Theorem 3.3.

Proposition 9.6. *If $\lambda \geqslant \lambda_0$ then* (3) *and* (4) *of Theorem* 3.3 *are equivalent.*

Proof. It is obvious that (3) $\Rightarrow$ (4). Assume that (4) is true, and that V is the dense subset of $H^{-1}(\mathbb{R}^N)$ for which $R_n(\lambda)g \rightharpoonup \mathscr{R}(\lambda)g$ for all $g \in V$. Let $(f_n)_{n\in\mathbb{N}}$ be an arbitrary sequence converging to some f in $H^{-1}(\mathbb{R}^N)$. Given $\varepsilon > 0$ we find $g \in V$ such that $\|f - g\|_{H^{-1}(\mathbb{R}^N)} \leqslant \alpha\varepsilon/8$. As $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$ there exists $n_0 \in \mathbb{N}$ such that $\|f_n - g\|_{H^{-1}(\mathbb{R}^N)} \leqslant \alpha\varepsilon/4$ for all $n \geqslant n_0$. Set $u_n := \mathscr{R}_n(\lambda)f_n$ and $v_n := \mathscr{R}_n(\lambda)g$. Similarly define $u := \mathscr{R}(\lambda)f$ and $v := \mathscr{R}(\lambda)g$. As $\lambda \geqslant \lambda_0$ we have from (2.7) and the choice of g that

$$\begin{aligned}\|u_n - u\|_{H^1(\mathbb{R}^N)} &\leqslant \|u_n - v_n\|_{H^1(\mathbb{R}^N)} + \|v_n - v\|_{H^1(\mathbb{R}^N)} + \|v - u\|_{H^1(\mathbb{R}^N)} \\ &\leqslant \frac{2}{\alpha}\|f_n - g\|_{H^{-1}(\mathbb{R}^N)} + \|v_n - v\|_{H^1(\mathbb{R}^N)} + \frac{2}{\alpha}\|f - g\|_{H^{-1}(\mathbb{R}^N)} \\ &\leqslant \frac{\varepsilon}{4} + \|v_n - v\|_{H^1(\mathbb{R}^N)} + \frac{\varepsilon}{2}\end{aligned}$$

for all $n \geqslant n_0$. By assumption (4) we have $v_n \rightharpoonup v$ weakly, and thus strongly in $H^1(\mathbb{R}^N)$ by Lemma 9.1 an $\lambda \geqslant \lambda_0$. Therefore there exists $n_1 \in \mathbb{N}$ such that $\|v_n - v\|_{H^1(\mathbb{R}^N)} \leqslant \varepsilon/4$ for all $n \geqslant n_1$. Hence, $\|u_n - u\|_{H^1(\mathbb{R}^N)} \leqslant \varepsilon$ for all $n \geqslant \max\{n_0, n_1\}$. As $\varepsilon > 0$ was arbitrary $u_n \to u$ in $H^1(\mathbb{R}^N)$, proving (3). $\square$

To complete the proof of Theorem 3.3 it remains to establish the equivalence of (5) to the other assertions if $|\Omega_n|$ is uniformly bounded. The difficulties in the proof come for two reasons. The first is that we do not assume that all Ω_n are contained in a fixed bounded set and thus $\mathscr{R}_n(\lambda)1$ may only converge weakly. The second is that $\mathscr{A}$ has only bounded and measurable coefficients, so we cannot assume that $\mathscr{R}(\lambda)1 \in C^1(\Omega)$.

Proposition 9.7. *Suppose that $(\Omega_n)_{n\in\mathbb{N}}$ has uniformly bounded measure, and that $\lambda \geqslant \lambda_0$. Then* (5) *of Theorem* 3.3 *is equivalent to* (1)–(4).

Proof. By assumption there exists $M>0$ such that $|\Omega_n| \leqslant M$ for all $n \in \mathbb{N}$. To show that (1) implies (5) we set $u_n := \mathscr{R}_n(\lambda)1$. As $|\Omega_n| \leqslant M$ it follows from (2.7) that $\|u_n\|_{H^1(\mathbb{R}^N)} \leqslant 2\alpha^{-1}\|1\|_{H^{-1}(\Omega_n)} = 2\alpha^{-1}|\Omega|^{1/2} \leqslant 2\alpha^{-1}M^{1/2}$. Hence, (u_n) is relatively weakly sequentially compact in $H^1(\mathbb{R}^N)$. Now fix $\varphi \in H_0^1(\Omega)$. By (1.4) there exist $\varphi_n \in H_0^1(\Omega_n)$ such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Note that $\|\varphi_n - \varphi_m\|_1 \leqslant |\Omega_n \cup \Omega_m|^{1/2}\|\varphi_n - \varphi_m\|_2 \leqslant (2M)^{1/2}\|\varphi_n - \varphi_m\|_2$ for all $n, m \in \mathbb{N}$, showing that $(\varphi_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L_1(\mathbb{R}^N)$. As $u_n = \mathscr{R}_n(\lambda)1$ we have

$$a(u_n, \varphi_n) + \lambda\langle u_n, \varphi_n\rangle = \int_{\mathbb{R}^N} \varphi_n \, dx.$$

As $\varphi_n \to \varphi$ in $L_1(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ it follows that every weak limit u point of (u_n) satisfies the above identity with n deleted. By (1.3) and since φ was arbitrary we have $u = \mathscr{R}(\lambda)1$. Hence the relatively weakly compact sequence (u_n) has a unique limit point, and must therefore weakly converge to $\mathscr{R}(\lambda)1$ as claimed.

We next prove that (5) implies (4). Set $v_n := \mathscr{R}_n(\lambda)1$ and $v := \mathscr{R}(\lambda)1$. Then by assumption $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. Fix $\varphi \in C_c^\infty(\Omega)$ and an open balls B and B_1 such that $\operatorname{supp}\varphi \subset B \subset\subset B_1$. We want to show that $v_n \to v$ in $H^1(B)$. We set $f_n(x) := 1$ for $x \in B_1 \cup \Omega_n$ and $f_n(x) := 0$ otherwise. Similarly define f by deleting n in the definition of f_n. Then clearly $v_n = \mathscr{R}_n(\lambda)f_n$ and $v = \mathscr{R}(\lambda)f$. Moreover, $f_n \to f$ in $L_2(B)$, and thus by Lemma 9.1 we have $v_n \to v$ in $H^1(B)$. We now derive some properties of v. By standard regularity theory $v \in C(\Omega)$ (see [26, Theorem 8.24]). Moreover, by Lemma 2.2 we know that $v \geqslant 0$. Hence we can apply Harnack's inequality for super-solutions (see [26, Theorem 8.18]) to conclude that $v(x) > 0$ for all $x \in \Omega$. As $\operatorname{supp} v \subset\subset \Omega$

$$\varphi_n := \frac{\varphi}{v} v_n$$

is well defined for all $n \in \mathbb{N}$, and $\|\varphi/v\|_\infty < \infty$. We next show that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. First note that

$$\|\varphi_n - \varphi\|_2 = \left\|\frac{\varphi}{v}(v_n - v)\right\|_2 \leqslant \left\|\frac{\varphi}{v}\right\|_\infty \|v_n - v\|_{2,B},$$

showing that $\varphi_n \to \varphi$ in $L_2(\mathbb{R}^N)$. To prove that the gradients also converge we first show that $\varphi/v \in H^1(\mathbb{R}^N)$. To do so set $w_\varepsilon := \varphi(v^2+\varepsilon^2)^{-1/2}$. Clearly $w_\varepsilon \to \varphi/v$ in $L_2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Using the chain rule for Sobolev functions (see [26, Theorem 7.8]) and then passing to the limit we get

$$\nabla w_\varepsilon = \frac{1}{(v^2+\varepsilon^2)^{1/2}}\nabla\varphi - \frac{v\varphi}{(v^2+\varepsilon^2)^{3/2}}\nabla v \xrightarrow{\varepsilon\to 0} \frac{1}{v}\nabla\varphi - \frac{\varphi}{v^2}\nabla v$$

in $L_2(\mathbb{R}^N)$. By definition of weak derivatives $\nabla(\varphi/v) = v^{-1}\nabla\varphi + v^{-2}\varphi\nabla v$. Hence

$$\nabla\varphi_n = \frac{\varphi}{v}\nabla v_n + \frac{\nabla\varphi}{v}v_n - \frac{\varphi}{v^2}v_n\nabla v$$

for $n \in \mathbb{N}$, and so $\varphi_n \in H^1(\mathbb{R}^N)$. Moreover,

$$\|\nabla\varphi_n - \nabla\varphi\|_2 \leqslant \|(\nabla\varphi)/v\|_\infty \|v_n - v\|_{2,B} + \|\varphi/v\|_\infty \|\nabla(v_n - v)\|_{2,B} + \|\varphi/v^2\|_\infty \||\nabla v|(v_n - v)\|_{2,B}.$$

As $v_n \to v$ in $H^1(B)$ the first and the second terms on the right-hand side of the above inequality converge to zero. It remains to show that also the third converges to zero. To do so we first observe that there exists a constant $M \geqslant 1$ such that $\max\{\|v_n\|_\infty, \|v\|_\infty\} \leqslant M$ for all $n \in \mathbb{N}$ (see [26, Theorem 8.15] or [21]). Hence,

$$\begin{aligned}
&\||\nabla v|(v_n - v)\|_{2,B}^2 \\
&\quad = \int_{[|\nabla v| \leqslant k] \cap B} |\nabla v|^2 (v_n - v)^2 \, dx + \int_{[|\nabla v| > k]} |\nabla v|^2 (v_n - v)^2 \, dx \\
&\quad \leqslant k \|v_n - v\|_{2,B}^2 + 4M^2 \int_{[|\nabla v| > k]} |\nabla v|^2 \, dx.
\end{aligned}$$

Given $\varepsilon > 0$ we can fix k such that the second term on the right-hand side is smaller than $\varepsilon/2$. As $v_n \to v$ in $L_2(B)$ we can then choose $n_0 \in \mathbb{N}$ such that the first term is smaller than $\varepsilon/2$ for $n \geqslant n_0$. Hence $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Suppose now that $f \in C_c^\infty(\mathbb{R}^N)$, and that $u_n := \mathscr{R}(\lambda)f$. Then by (2.7) the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, and thus relatively weakly sequentially compact. For $\varphi \in C_c^\infty(\Omega)$ we let $\varphi_n \in H_0^1(\Omega_n)$ be the functions constructed above. Then $a(u_n, \varphi_n) + \lambda\langle u_n, \varphi_n\rangle = \langle f, \varphi_n\rangle$ for all $n \in \mathbb{N}$. As $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ it follows that $a(u, \varphi) + \lambda\langle u, \varphi\rangle = \langle f, \varphi\rangle$ for every weak limit point u of $(u_n)_{n\in\mathbb{N}}$. We next show that every weak limit point of $(u_n)_{n\in\mathbb{N}}$ is in $H_0^1(\Omega)$. Suppose that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Clearly $(A_{\Omega_n} + \lambda)(\|f\|_\infty v_n \pm u_n) = \|f\|_\infty \pm f_n \geqslant 0$, and thus from Lemma 2.2 we have $\|f\|_\infty v_n \pm u_n \geqslant 0$. As the sequences converge weakly in $L_2(\mathbb{R}^N)$ we have $\langle \|f\|_\infty v_n \pm u_n, \varphi\rangle \to \langle \|f\|_\infty v \pm u, \varphi\rangle \geqslant 0$ for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ non-negative. Hence $|v| \leqslant \|f\|_\infty u$ almost everywhere. As $H_0^1(\Omega)$ is an order ideal in $H^1(\mathbb{R}^N)$ (see [4, Lemma 1.3]) it follows that $v \in H_0^1(\Omega)$. Therefore the only weak limit point of $(u_n)_{n\in\mathbb{N}}$ is $u = \mathscr{R}(\lambda)f$, and as the sequence is relatively weakly sequentially compact $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. Finally, recall that $C_c^\infty(\mathbb{R}^N)$ is dense in $H^{-1}(\mathbb{R}^N)$, so we have proved (4) of Theorem 3.3. By Proposition 9.6 the assertion of the proposition follows. $\square$

Appendix A. A spectral mapping theorem

Suppose that E, F are Banach spaces, and that A is a closed densely defined operator on F with domain $D(A)$. Moreover, suppose that there exist $i \in \mathscr{L}(F, E)$

and $r \in \mathscr{L}(E, F)$ such that $r \circ i = \mathrm{id}_F$. For all $\lambda \in \varrho(A)$ we set

$$\mathscr{R}(\lambda) := i \circ (A - \lambda)^{-1} \circ r.$$

We then have the following spectral mapping theorem.

Proposition A.1. *Suppose that $\lambda \in \varrho(A)$, and that $\mu \neq \lambda$. Then $\mu \in \varrho(A)$ if and only if $(\mu - \lambda)^{-1} \in \varrho(\mathscr{R}(\lambda))$. If that is the case then*

$$\mathscr{R}(\mu) = -(\mu - \lambda)^{-1} \mathscr{R}(\lambda)(\mathscr{R}(\lambda) - (\mu - \lambda)^{-1})^{-1}. \tag{A.1}$$

Proof. By replacing A by $A - \lambda$ we can assume without loss of generality that $\lambda = 0$. Hence let us assume that $A^{-1} \in \mathscr{L}(F)$. It is well known that $0 \neq \mu \in \varrho(A)$ if and only if $\mu^{-1} \in \varrho(A^{-1})$ (see [31, Theorem III.6.15]). Hence we only need to show that $\mu^{-1} \in \varrho(\mathscr{R}(0))$ if and only if $\mu^{-1} \in \varrho(A^{-1})$. To do so we first split equation

$$\mathscr{R}(0)u - \mu^{-1}u = f \tag{A.2}$$

into an equivalent system of equations. Observe that $P := i \circ r$ is a projection on E onto some subspace. If we set $E_1 := P(E)$ and $E_2 := (\mathrm{id} - P)(E)$ then $E = E_1 \oplus E_2$ is a direct sum. Clearly the image of $\mathscr{R}(0)$ is in E_1. As $r = r \circ P$ we have $P \circ \mathscr{R}(0) = \mathscr{R}(0) \circ P$ and thus (A.2) is equivalent to

$$(\mathscr{R}(0) - \mu^{-1})Pu = Pf, \tag{A.3}$$

$$-\mu^{-1}(\mathrm{id} - P)u = (\mathrm{id} - P)f. \tag{A.4}$$

Assume now that $\mu \in \varrho(A^{-1})$, and fix $f \in E$ arbitrary. It follows that $v := (A^{-1} - \mu^{-1}) \circ Pf$ is uniquely determined. We set $u_1 := i(v)$ and note that $u_1 = Pu_1$ is the unique solution of the first of (A.3). Clearly $u_2 := -\mu(\mathrm{id} - P)f$ is the unique solution of (A.4) in E_2. Hence $u := u_1 + u_2$ is the unique solution of (A.2), showing that $\mu^{-1} \in \varrho(\mathscr{R}(0))$. Next assume that $\mu^{-1} \in \varrho(\mathscr{R}(0))$, and that $g \in F$ is arbitrary. Set $f := i(g)$ and note that $Pf = f$ in that case. By assumption (A.3) has a unique solution u_1. As $(\mathrm{id} - P)f = 0$ the solution of (A.4) is zero. Hence $r(u_1)$ is the unique solution of $(\mu^{-1} - A^{-1})u = g$, showing that $\mu^{-1} \in \varrho(A^{-1})$. We finally prove identity (A.1), provided $\lambda, \mu \in \varrho(A)$. By the resolvent equation

$$(A - \lambda)^{-1} = (A - \mu)^{-1}(\mathrm{id}_F - (\mu - \lambda)(A - \lambda)^{-1}).$$

Using that $r \circ i = \mathrm{id}_F$ this yields

$$\begin{aligned}\mathscr{R}(\lambda) &= i \circ (A-\mu)^{-1}(\mathrm{id}_F \circ r - (\mu-\lambda) r \circ i \circ (A-\lambda)^{-1} \circ r) \\ &= i \circ (A-\mu)^{-1} \circ r(\mathrm{id}_E - (\mu-\lambda) i \circ (A-\lambda)^{-1} \circ r) \\ &= \mathscr{R}(\mu)(\mathrm{id}_E - (\mu-\lambda)\mathscr{R}(\lambda)).\end{aligned}$$

Rearranging we get $\mathscr{R}(\mu)(\mathscr{R}(\lambda) - (\mu-\lambda)^{-1}) = -(\mu-\lambda)^{-1}\mathscr{R}(\lambda)$. As we know that $(\mu-\lambda)^{-1} \in \varrho(\mathscr{R}(\lambda))$ identity (A.1) follows, completing the proof of the proposition. □

Appendix B. Uniform convergence of operators

We prove a convergence theorem useful in the context of domain convergence. Note that we do not assume that T_n below be compact.

Proposition B.1. *Suppose H is a Hilbert space and $T_n, T \in \mathscr{L}(H)$. Then the following assertions are equivalent*:

(1) *T is compact and $T_n \to T$ in $\mathscr{L}(H)$;*
(2) *$T_n f_n \to Tf$ in H whenever $f_n \rightharpoonup f$ weakly in H;*
(3) *$T_n \to T$ strongly and $T_n f_n \to 0$ in H whenever $f_n \rightharpoonup 0$ weakly in H.*

Proof. We first prove that (1) ⇒ (2). Assuming that $f_n \rightharpoonup f$ weakly in H we have

$$\|T_n f_n - Tf\| \leqslant \|T_n - T\| \|f_n\| + \|T(f_n - f)\|.$$

The first term on the right-hand side converges to zero as $T_n \to T$ uniformly by assumption. The second term converges to zero as $f_n - f \rightharpoonup 0$ weakly in H and T is compact. Hence (2) holds. It is clear that (2) ⇒ (3), so we prove that (3) ⇒ (1). We start by showing that T is compact. To do so it is sufficient to show that $Tf_n \to 0$ in H whenever $f_n \rightharpoonup 0$ weakly in H. From (2) it is clear that T_n converges strongly to T. Hence $T_k f_n \to Tf_n$ as $k \to \infty$ for every $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there exists $k_n \geqslant n$ such that $\|T_{k_n} f_n - Tf_n\| \leqslant 1/n$. Therefore

$$\begin{aligned}\|Tf_n\| &\leqslant \|Tf_n - T_{k_n} f_n\| + \|T_{k_n}(f_n - f_{k_n})\| + \|T_{k_n} f_{k_n}\| \\ &\leqslant \frac{1}{n} + \|T_{k_n}(f_n - f_{k_n})\| + \|T_{k_n} f_{k_n}\|.\end{aligned}$$

As $f_n \rightharpoonup 0$ weakly if follows from the assumptions that $\|T_{k_n} f_{k_n}\| \to 0$ as $n \to \infty$. Setting $g_{k_n} := f_n - f_{k_n}$ it follows again from the assumptions that $\|T_{k_n} g_{k_n}\| = \|T_{k_n}(f_n -$

$f_{k_n})||\to 0$ as $n\to\infty$. Hence the right-hand side of the above inequality converges to zero as $n\to\infty$, so $Tf_n\to 0$ in H. This shows that T is compact. Now we prove uniform convergence of T_n. Assume to the contrary that T_n does not converge uniformly. Then there exists $\varepsilon>0$ and $f_n\in H$ with $||f_n||=1$ such that $\varepsilon||T_nf_n - Tf_n||$ for all $n\in\mathbb{N}$. As bounded sets in a Hilbert space are weakly sequentially compact we can assume that $f_n\rightharpoonup f$ weakly in H by possibly passing to a subsequence. Therefore

$$0<\varepsilon\leqslant||T_nf_n - Tf_n||\leqslant||T_n(f_n - f)|| + ||T_nf - Tf|| + ||T(f - f_n)||. \qquad \text{(B.1)}$$

The first term converges to zero by assumption as $f_n - f\rightharpoonup 0$ weakly in H. The second term converges to zero as $T_n\to T$ strongly. The last term converges to zero as T is compact and $f - f_n\rightharpoonup 0$ weakly in H. Hence we get a contradiction to (B.1), showing that T_n must converge in $\mathscr{L}(H)$. Hence (3) holds, completing the proof of the proposition. □

Note that in the above proposition we could replace the Hilbert space by an arbitrary reflexive Banach space.

Note added in proof

Part (2) of Proposition 7.3 is wrong. The statement is taken from [25]. However, [25] cannot be right since in [32, p. 55] there is an example of a Wrener regular domain which is not stable!

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